Finding Patterns in Counting Schreier Sets

Y. Geng[1] J. King [2] Z. L. Vasseur [3]
Mentors: H. V. Chu[4], S. J. Miller[5], G.Tresch[6]

[1]geng0114@umn.edu [2]mg168@rice.edu

[3]johnsonj93@lakeforest.edu [4]jhk12@geneseo.edu [5]zachary.l.v@tamu.edu

[6]hchu@wlu.edu [7]sjm1@williams.edu [8]treschgd@tamu.edu

April 22, 2026

What is a Schreier Set?

A finite set $F \subset \mathbb{N}$ is said to be Schreier if $F = \varnothing$ or $\min F \geqslant |F|$.

Schreier: $\{2, 3\}, \{4, 5, 10\}$

Not Schreier: $\{3, 7, 10, 13\}$

What is a Schreier Set?

A finite set $F \subset \mathbb{N}$ is said to be Schreier if $F = \varnothing$ or $\min F \geqslant |F|$.

Schreier: $\{2,3\}, \{4,5,10\}$

Not Schreier: $\{3,7,10,13\}$

$$\mathcal{A}_n \;=\; \{F \subset \{1,2,\ldots,n\} \;:\; n \in F \text{ and } F \text{ is Schreier}\}$$

$$\mathcal{A}_n \;=\; \{F \subset \{1, 2, \ldots, n\} \;:\; n \in F \text{ and } F \text{ is Schreier}\}$$

$$\mathcal{A}_n = \{F \subset \{1, 2, \ldots, n\} : n \in F \text{ and } F \text{ is Schreier}\}$$

$$\mathcal{A}_1 = \{\{1\}\} \implies |\mathcal{A}_1| = 1$$

$$\mathcal{A}_n = \{F \subset \{1, 2, \ldots, n\} : n \in F \text{ and } F \text{ is Schreier}\}$$

$$\mathcal{A}_1 = \{\{1\}\} \implies |\mathcal{A}_1| = 1$$

$$\mathcal{A}_2 = \{\{2\}\} \implies |\mathcal{A}_2| = 1$$

What does $\mathcal{A}_n$ look like?

$$\mathcal{A}_n = \{F \subset \{1, 2, \ldots, n\} : n \in F \text{ and } F \text{ is Schreier}\}$$

$$\mathcal{A}_1 = \{\{1\}\} \implies |\mathcal{A}_1| = 1$$

$$\mathcal{A}_2 = \{\{2\}\} \implies |\mathcal{A}_2| = 1$$

$$\mathcal{A}_3 = \{\{2, 3\}, \{3\}\} \implies |\mathcal{A}_3| = 2$$

$$\mathcal{A}_n = \{F \subset \{1, 2, \ldots, n\} : n \in F \text{ and } F \text{ is Schreier}\}$$

$$\mathcal{A}_1 = \{\{1\}\} \implies |\mathcal{A}_1| = 1$$

$$\mathcal{A}_2 = \{\{2\}\} \implies |\mathcal{A}_2| = 1$$

$$\mathcal{A}_3 = \{\{2, 3\}, \{3\}\} \implies |\mathcal{A}_3| = 2$$

$$\mathcal{A}_4 = \{\{2, 4\}, \{3, 4\}, \{4\}\} \implies |\mathcal{A}_4| = 3$$

$$\mathcal{A}_n = \{F \subset \{1, 2, \ldots, n\} : n \in F \text{ and } F \text{ is Schreier}\}$$

$$\mathcal{A}_1 = \{\{1\}\} \implies |\mathcal{A}_1| = 1$$

$$\mathcal{A}_2 = \{\{2\}\} \implies |\mathcal{A}_2| = 1$$

$$\mathcal{A}_3 = \{\{2, 3\}, \{3\}\} \implies |\mathcal{A}_3| = 2$$

$$\mathcal{A}_4 = \{\{2, 4\}, \{3, 4\}, \{4\}\} \implies |\mathcal{A}_4| = 3$$

$$\mathcal{A}_5 = \{\{2, 5\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}, \{5\}\} \implies |\mathcal{A}_5| = 5$$

A. Bird (2012)

For $n \in \mathbb{N}$,

$$|\{F \subseteq \{1,2,\ldots,n\} : n \in F \text{ and } F \text{ is Schreier}\}| = F_n,$$

where $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geqslant 3$.

A proof Idea

$n=5 \quad F \subset \{1,2,3,4,5\}$

$1 \quad 2 \quad 3 \quad 4 \quad \cancel{5}$

$n \in F$

$|F| \leq \min F$

	choose from
$\lvert F\rvert = 1$	$\{\ \}$
$\lvert F\rvert = 2$	$\{2,3,4\}$
$\lvert F\rvert = 3$	$\{3,4\}$
$\lvert F\rvert = 4$	$\{\ \}$

Figure: Counting Schreier Sets for n = 5.

Proof.

Suppose $F \subset \{1, 2, 3, \cdots, n\}$.

We need $n \in F$, so there are $n - 1$ possible choices from $\{1, 2, 3, \cdots, n-1\}$

Proof.

Suppose $F \subset \{1, 2, 3, \cdots, n\}$.

We need $n \in F$, so there are $n-1$ possible choices from $\{1, 2, 3, \cdots, n-1\}$

We also need $|F| \leqslant \min F$.

Let $k = |F|$. Then, we want to choose $k-1$ elements from

$$\{k, k+1, k+2, \cdots, n-1\}$$

Proof.

Suppose $F \subset \{1, 2, 3, \cdots, n\}$.

We need $n \in F$, so there are $n-1$ possible choices from $\{1, 2, 3, \cdots, n-1\}$

We also need $|F| \leqslant \min F$.

Let $k = |F|$. Then, we want to choose $k-1$ elements from

$$\{k, k+1, k+2, \cdots, n-1\}$$

$\binom{n-k}{k-1}$ for all possible Schreier Sets of size k.

$k \leqslant \lfloor \frac{n+1}{2} \rfloor$.

A proof Idea

Proof.

Suppose $F \subset \{1, 2, 3, \cdots, n\}$.

We need $n \in F$, so there are $n-1$ possible choices from $\{1, 2, 3, \cdots, n-1\}$

We also need $|F| \leqslant \min F$.

Let $k = |F|$. Then, we want to choose $k-1$ elements from

$$\{k, k+1, k+2, \cdots, n-1\}$$

$\binom{n-k}{k-1}$ for all possible Schreier Sets of size k.

$k \leqslant \lfloor \frac{n+1}{2} \rfloor$.

So, to count all of the Schreier Sets we need,

$$\mathcal{A}_n = \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n-k}{k-1} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} = F_n$$

$\square$

For $(s, p, n) \in \mathbb{N}^3$,

$$\mathcal{A}_{p,n}^{(s)} := \{F \subset \{\underbrace{1, \ldots, 1}_{s}, \ldots, \underbrace{n-1, \ldots, n-1}_{s}, n\} : n \in F \text{ and } \min F \geqslant p|F|\}$$

For $(s, p, n) \in \mathbb{N}^3$,

$$\mathcal{A}^{(s)}_{p,n} := \{F \subset \{\underbrace{1, \ldots, 1}_{s}, \ldots, \underbrace{n-1, \ldots, n-1}_{s}, n\} : n \in F \text{ and } \min F \geqslant p|F|\}$$

Polymath Jr. 25 Schreier Group

For $(s, p) \in \mathbb{N}^2$ and $n \geqslant sp + 2$,

$$|\mathcal{A}^{(s)}_{p,n}| = |\mathcal{A}^{(s)}_{p,n-1}| + |\mathcal{A}^{(s)}_{p,n-1-p}| + |\mathcal{A}^{(s)}_{p,n-1-2p}| + \cdots + |\mathcal{A}^{(s)}_{p,n-1-sp}|.$$

Theorem

Let k_i denote the integer k with color i.

$$B_n^{(s)} = \{1_1, 1_2, \ldots, 1_s, \ldots, (n-1)_1, \ldots, (n-1)_s,\ n\}$$

Let k_i denote the integer k with color i.

$$B_n^{(s)} \;=\; \{1_1, 1_2, \ldots, 1_s, \ldots, (n-1)_1, \ldots, (n-1)_s,\; n\}$$

$$\mathcal{B}_{p,n}^{(s)} \;:=\; \{F \subset B_n^{(s)} \;:\; n \in F \text{ and } \min F \geqslant p|F|\}.$$

Let k_i denote the integer k with color i.

$$B_n^{(s)} = \{1_1, 1_2, \ldots, 1_s, \ldots, (n-1)_1, \ldots, (n-1)_s,\ n\}$$

$$\mathcal{B}_{p,n}^{(s)} := \{F \subset B_n^{(s)} : n \in F \text{ and } \min F \geqslant p|F|\}.$$

Polymath Jr. 25 Schreier Group

For $(s, p) \in \mathbb{N}^2$ and $n \geqslant sp + 2$,

$$|\mathcal{B}_{p,n}^{(s)}| = \binom{s}{0}|\mathcal{B}_{p,n-1}^{(s)}| + \binom{s}{1}|\mathcal{B}_{p,n-1-p}^{(s)}| + \cdots + \binom{s}{s}|\mathcal{B}_{p,n-1-sp}^{(s)}|.$$

For $(s, n) \in \mathbb{N}^2$,

$$\mathcal{A}_n^{(s)} := \{F \subset \underbrace{\{1, 2 \ldots, n\} \times \ldots \times \{1, 2 \ldots, n\}}_{s} : n \in F \text{ and } \min F \geqslant |F|\}$$

For $(s, n) \in \mathbb{N}^2$,

$$\mathcal{A}_n^{(s)} := \{F \subset \underbrace{\{1, 2 \ldots, n\} \times \ldots \times \{1, 2 \ldots, n\}}_{s} : n \in F \text{ and } \min F \geqslant |F|\}$$

What is the sequence $(|\mathcal{A}_n^{(s)}|)_{n=1}^{\infty}$?

Take a finite non-empty set, $F \subset \mathbb{N} \times \mathbb{N}$.

How is the minimum defined?

Special Case: s = 2

Take a finite non-empty set, $F \subset \mathbb{N} \times \mathbb{N}$.

How is the minimum defined?

Coordinate-wise Minimum: Define the min $F = \min\{a, b\}$ for all $(a, b) \in F$.

Schreier: $\{(4, 3), (3, 3), (6, 7)\}$

Not Schreier: $\{(3, 7), (10, 13), (4, 2)\}$

Take a finite non-empty set, $F \subset \mathbb{N} \times \mathbb{N}$.

How is the minimum defined?

Coordinate-wise Minimum: Define the $\min F = \min\{a, b\}$ for all $(a, b) \in F$.

Schreier: $\{(4,3),(3,3),(6,7)\}$

Not Schreier: $\{(3,7),(10,13),(4,2)\}$

$$\mathcal{A}_n^2 = \{F \subset \{1,2\ldots,n\} \times \{1,2\ldots,n\} : n \in F \text{ and } \min F \geqslant |F|\}$$

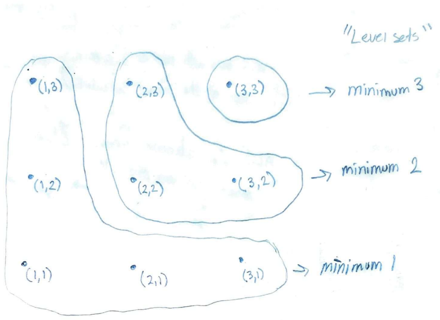
"Level sets"
(1,3)
(2,3)
(3,3)
minimum 3
(1,2)
(2,2)
(3,2)
minimum 2
(1,1)
(2,1)
(3,1)
minimum 1

Figure: $n = 3$.

Special Case: s = 2

Theorem

For $n \in \mathbb{N}$,

$$|\{A \subset \{1, 2, \ldots, n\} \times \{1, 2, \ldots, n\} : \min A \geqslant |A|\}| = \sum_{i=1}^{n} \binom{(n-i+1)^2}{i}$$

where the $\min F = \min\{a, b\}$ for all $(a, b) \in F$.

PROOF.

$|A| = i$, then the condition $\min A \geqslant i$ implies that for every $(a, b) \in A$ we have $a \geqslant i$ and $b \geqslant i$. Hence

$$A \subset \{i, i+1, \ldots, n\} \times \{i, i+1, \ldots, n\},$$

which contains exactly $(n - i + 1)^2$ elements.

Proof.

$|A| = i$, then the condition $\min A \geqslant i$ implies that for every $(a, b) \in A$ we have $a \geqslant i$ and $b \geqslant i$. Hence

$$A \subset \{i, i+1, \dots, n\} \times \{i, i+1, \dots, n\},$$

which contains exactly $(n - i + 1)^2$ elements.

Every i element subset from this set satisfies $\min A \geqslant |A|$.

PROOF.

$|A| = i$, then the condition $\min A \geqslant i$ implies that for every $(a, b) \in A$ we have $a \geqslant i$ and $b \geqslant i$. Hence

$$A \subset \{i, i+1, \ldots, n\} \times \{i, i+1, \ldots, n\},$$

which contains exactly $(n-i+1)^2$ elements.

Every i element subset from this set satisfies $\min A \geqslant |A|$.

Therefore, the number of possible subsets for this i is $\binom{(n-i+1)^2}{i}$.

PROOF.

$|A| = i$, then the condition $\min A \geqslant i$ implies that for every $(a, b) \in A$ we have $a \geqslant i$ and $b \geqslant i$. Hence

$$A \subset \{i, i+1, \ldots, n\} \times \{i, i+1, \ldots, n\},$$

which contains exactly $(n-i+1)^2$ elements.

Every i element subset from this set satisfies $\min A \geqslant |A|$.

Therefore, the number of possible subsets for this i is $\binom{(n-i+1)^2}{i}$.

Summing over all possible i,

$$a_n = \sum_{i=1}^{n} \binom{(n-i+1)^2}{i}$$

$\square$

Take a finite non-empty set, $F \subset \mathbb{N} \times \mathbb{N}$.

Let's define a new minimum

Take a finite non-empty set, $F \subset \mathbb{N} \times \mathbb{N}$.

Let's define a new minimum

Minimum of a+b: Define the min $F = \min\{a+b\}$ for all $(a, b) \in F$.

Schreier: $\{(1,2),(3,3),(6,7)\}$

Not Schreier: $\{(3,7),(10,13),(2,1),(6,5)\}$

Take a finite non-empty set, $F \subset \mathbb{N} \times \mathbb{N}$.

Let's define a new minimum

Minimum of a+b: Define the min $F = \min\{a + b\}$ for all $(a, b) \in F$.

Schreier: $\{(1, 2), (3, 3), (6, 7)\}$

Not Schreier: $\{(3, 7), (10, 13), (2, 1), (6, 5)\}$

$$\mathcal{A}_n^2 = \{F \subset \{1, 2 \ldots, n\} \times \{1, 2 \ldots, n\} : n \in F \text{ and } \min F \geqslant |F|\}$$

An interesting idea.

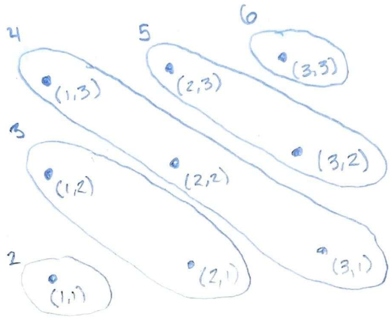

Figure: n = 3.

Conjecture for min of a+b

For $n \in \mathbb{N}$,

$$|\{A \subset \{1,2,\ldots,n\} \times \{1,2,\ldots,n\} : \min A \geqslant |A|\}| = \sum_{i=1}^{\lfloor -\frac{3}{2}+\sqrt{\frac{9}{4}+2n^2} \rfloor} \binom{n^2 - \sum_{k=1}^{i} k}{i}$$

where the min $F = \min\{a+b\}$ for all $(a,b) \in F$

Find a restriction to make linear recurrences

Find a restriction to make linear recurrences

Study the more general condition $q \min F \geqslant p|F|$

Find a restriction to make linear recurrences

Study the more general condition $q \min F \geqslant p|F|$

Create asymptotic growth formulas

Thank you!