

Sensor sensitivity in several settings

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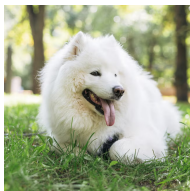
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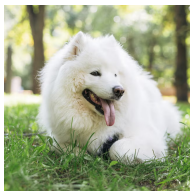
Compression and Reconstruction



Underlying arbitrary
image

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Compression and Reconstruction



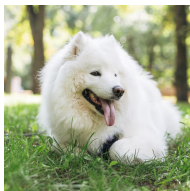
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Incomplete
discretized data

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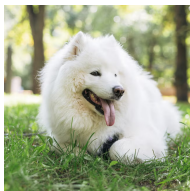
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Reconstructed
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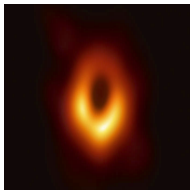
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- ▶ Suppose we are given an arbitrary image generated by an arbitrary function.
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- ▶ The reconstruction does not look good because the initial image is not spectrally compressible.

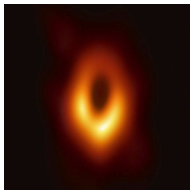
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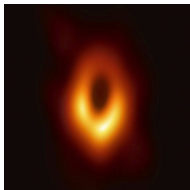
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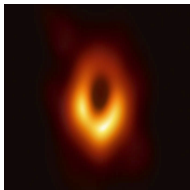
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- ▶ We observe only a discretized version of the signal, and some sampled values are missing. The white squares represent unknown measurements.
- ▶ A standard reconstruction algorithm uses the known measurements to estimate the missing values.
- ▶ The reconstruction closely resembles the complete discretized image that would have been observed if no measurements were missing because this initial signal is indeed spectrally compressible.

Definitions

- For a function $g : \mathbb{Z}_N^2 \rightarrow \mathbb{C}$, define its Fourier ratio by

$$\text{FR}(g) = \frac{\|\widehat{g}\|_{\ell^1}}{\|\widehat{g}\|_{\ell^2}}.$$

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- ▶ The corresponding norms are

$$\|\widehat{g}\|_{\ell^1} = \sum_{m \in \mathbb{Z}_N^2} |\widehat{g}(m)| \quad \text{and} \quad \|\widehat{g}\|_{\ell^2} = \left(\sum_{m \in \mathbb{Z}_N^2} |\widehat{g}(m)|^2 \right)^{1/2}.$$

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- ▶ A large ratio means the spectrum is spread out and harder to recover from few samples.
- ▶ We find bounds for certain families of functions across several geometries.

Three geometric settings

Unit Square	Unit Sphere	Manifold
Fourier modes	spherical harmonics	Laplace–Beltrami eigenfunctions
uniform grid cells	equal-area cells	equal-volume cells
Euclidean perturbations	geodesic perturbations	geodesic perturbations

Fractional Sobolev regularity on the square

Theorem

Let $0 < s < 1$, $1 < p < \infty$, $sp > 2$, and $\alpha = s - \frac{2}{p} > 0$. If $f \in W^{s,p}([0,1]^2)$ is nonzero and periodic, $g(x) = f(x/N)$, and

$$N^\alpha \|f\|_{L^2}^2 \geq K_{s,p} \|f\|_{W^{s,p}}^2,$$

then

$$\text{FR}(g) \leq \sqrt{2} \frac{\left| \int_{[0,1]^2} f \right|}{\|f\|_{L^2}} + C_{s,p} \frac{\|f\|_{W^{s,p}}}{\|f\|_{L^2}} \left[N^{1-s+2/p} \sqrt{1 + \log N} + N^{-s+2/p} \right].$$

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2. Any positive α improves on the trivial bound $\text{FR}(g) \leq N$.
3. The endpoint $\alpha = 1$ gives the critical $\sqrt{\log N}$ growth and already includes C^1 functions.

A translated grid

Theorem

Let $f \in C^1([0, 1]^2)$ be nonzero and periodic. For every $N \geq 2$, there exists $t \in [0, 1]^2$ such that, for

$$g_t(x) = f\left(\frac{x + t}{N}\right),$$

one has

$$\text{FR}(g_t) \leq 1 + \frac{1}{\sqrt{2}} \sqrt{1 + \log N} \frac{\|\nabla f\|_{L^2([0, 1]^2)}}{\|f\|_{L^2([0, 1]^2)}}.$$

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1. Average over a common translation of the entire grid.
2. Choose one translation with good sampled L^2 mass and discrete gradient energy.
3. This removes the fixed-grid mass hypothesis and replaces the pointwise C^1 term by the continuous H^1 energy.

Sobolev regimes on compact manifolds

Theorem

Let M be a compact d -dimensional Riemannian manifold and $f \in V_L$ be nonzero. For every $s > 0$,

$$\mathrm{FR}_L(f) \leq \frac{1}{\sqrt{|M|}} \frac{|\int_M f dV_g|}{\|f\|_{L^2(M)}} + C_{M,s} \frac{\|(-\Delta_g)^{s/2} f\|_{L^2(M)}}{\|f\|_{L^2(M)}} \Psi_{d,s}(L),$$
$$\Psi_{d,s}(L) = \begin{cases} 1, & s > \frac{d}{2}, \\ \sqrt{1 + \log L}, & s = \frac{d}{2}, \\ L^{\frac{d}{4} - \frac{s}{2}}, & 0 < s < \frac{d}{2}. \end{cases}$$

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1. Above the critical Sobolev index $s = d/2$, the bound is uniform in L .
2. At the critical index there is exactly a $\sqrt{\log L}$ loss.
3. Below the critical index the Fourier Ratio grows polynomially in L .
4. For surfaces, $d = 2$ and $s = 1$, so the gradient estimate is the critical case; the sphere is an explicit example.

The critical surface bound is sharp

Theorem

Let M be a compact Riemannian surface and define

$$f_L = \sum_{0 < \lambda_j \leq L} \frac{1}{\lambda_j} e_j.$$

Then

$$\mathrm{FR}_L(f_L) \asymp \log L, \quad \frac{\|\nabla_g f_L\|_{L^2(M)}}{\|f_L\|_{L^2(M)}} \asymp \sqrt{\log L},$$

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1. The logarithmic loss is caused by the eigenvalue distribution, not by an artifact of the proof.
2. Thus $\sqrt{\log L}$ cannot be replaced by any $o(\sqrt{\log L})$ factor in the critical energy estimate.

Perturbations in sensor placement

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold. If $d_X(x_i, y_i) \leq \delta$, then

$$\left(\frac{|X|}{q} \sum_{i=1}^q |f(y_i) - f(x_i)|^2 \right)^{1/2} \lesssim_X \delta \|\nabla_X f\|_{L^\infty(X)}.$$

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1. Join each sensor location x_i to its perturbation y_i by a minimizing geodesic.
2. Apply the Fundamental Theorem of Calculus along the geodesic.
3. Bound the integrand by the L^∞ norm of the gradient.
4. Use the distance bound $d_X(x_i, y_i) \leq \delta$.
5. Square and average over the sensor locations to obtain the total placement error.

Random sensor sampling

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold partitioned into K equal-measure cells of diameter at most h_K . If sensors are placed uniformly at random, then

$$\mathbb{E}[T_q] = K (H_K - H_{K-q}), \quad \mathcal{E}_{\text{place}} \lesssim_X h_K \|\nabla_X f\|_{L^\infty(X)}.$$

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4. Sum these waiting times to obtain the coupon-collector estimate.
5. Use the cell diameter to bound the sensor-placement error.

Stable ℓ^1 reconstruction

Theorem

Suppose $\text{FR}(f) \leq r$, let D be the spectral dimension, and set $s = \min\{D, \lceil r^2/\varepsilon^2 \rceil\}$. If

$$q \geq C_\gamma K^2 s \log^2(2s) \log(2D),$$

then, with high probability, ℓ^1 minimization gives

$$\|f^* - f\|_{L^2} \leq C_0 (\varepsilon \|f\|_{L^2} + \eta),$$

where η contains measurement error, including sensor displacement.

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1. Square: $D = N^2$ and $K = 1$.

Stable ℓ^1 reconstruction

Theorem

Suppose $\text{FR}(f) \leq r$, let D be the spectral dimension, and set $s = \min\{D, \lceil r^2/\varepsilon^2 \rceil\}$. If

$$q \geq C_\gamma K^2 s \log^2(2s) \log(2D),$$

then, with high probability, ℓ^1 minimization gives

$$\|f^* - f\|_{L^2} \leq C_0 (\varepsilon \|f\|_{L^2} + \eta),$$

where η contains measurement error, including sensor displacement.

1. Square: $D = N^2$ and $K = 1$.
2. Manifold: $D = D_L$ and $K = K_L$, the eigenfunction coherence parameter.

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1. Square: $D = N^2$ and $K = 1$.
2. Manifold: $D = D_L$ and $K = K_L$, the eigenfunction coherence parameter.
3. The Fourier Ratio and placement estimates therefore combine into the paper's end-to-end recovery theorems.

Main takeaways

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Main takeaways

1. Hölder and fractional Sobolev regularity force nontrivial Fourier Ratio bounds on finite grids.
2. Translating the grid gives a sharper critical estimate controlled by the continuous H^1 energy.
3. Compact manifolds exhibit the same three Sobolev regimes, and the $\sqrt{\log L}$ critical loss is sharp on surfaces.
4. Sensor perturbation and coupon-collector estimates convert these analytic bounds into stable random-sampling reconstruction results on the square and compact manifolds.

References

- ▶ A. Iosevich, E. Palsson, and A. Yavicoli, *Discretization, sampling, and the Fourier ratio*, 2026.
- ▶ W. Burstein, A. Iosevich, and H. Nathan, *The Fourier Ratio: A Unifying Measure of Complexity for Recovery, Localization, and Learning*, 2026.
- ▶ S. Musser, *Weyl's Law on Riemannian Manifolds*, University of Chicago Mathematics REU paper, (2016).
- ▶ R. A. Adams and J. J. F. Fournier, *Sobolev Spaces*, second edition, Academic Press, 2003.
- ▶ C. D. Sogge, *Fourier Integrals in Classical Analysis*, second edition, Cambridge University Press, 2017.
- ▶ S. Foucart and H. Rauhut, *A Mathematical Introduction to Compressive Sensing*, Birkhäuser, 2013.
- ▶ E. J. Candès, J. Romberg, and T. Tao, *Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information*, IEEE Transactions on Information Theory, 52, no. 2, (2006), pp. 489–509.

Thanks ^_^