Sensor sensitivity in several settings

Christopher Housholder
Mentors: Alex Iosevich, Steven J. Miller, Eyvindur Pálsson

Missouri State University
christopherlancehousholder@gmail.com

July 28, 2026

Presented at the Young Mathematicians Conference

Supported by NSF, Williams College, and Missouri State University

Compression and Reconstruction

Underlying arbitrary image

- Suppose we are given an arbitrary image generated by an arbitrary function.

Compression and Reconstruction

Underlying arbitrary image

Incomplete discretized data

- Suppose we are given an arbitrary image generated by an arbitrary function.
- We observe a discretized version of this signal with some measurements missing. The white squares represent unknown pixel values.

Compression and Reconstruction

Underlying arbitrary image

Incomplete discretized data

Reconstructed image

- Suppose we are given an arbitrary image generated by an arbitrary function.
- We observe a discretized version of this signal with some measurements missing. The white squares represent unknown pixel values.
- A standard reconstruction algorithm uses the observed measurements to estimate the missing values.

Compression and Reconstruction

Underlying arbitrary image

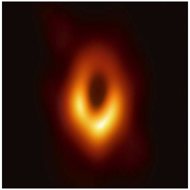
Incomplete discretized data

Reconstructed image

Complete discretized image

- Suppose we are given an arbitrary image generated by an arbitrary function.
- We observe a discretized version of this signal with some measurements missing. The white squares represent unknown pixel values.
- A standard reconstruction algorithm uses the observed measurements to estimate the missing values.
- The reconstruction does not look good because the initial image is not spectrally compressible.

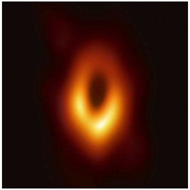
Compression and Reconstruction



Underlying smooth signal

- Now what if we assume that the image is generated by an underlying smooth continuous signal.

Compression and Reconstruction

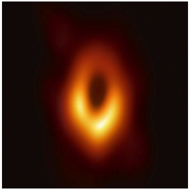


Underlying smooth signal

Incomplete discretized data

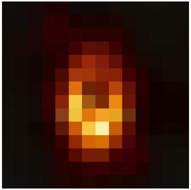
- Now what if we assume that the image is generated by an underlying smooth continuous signal.
- We observe only a discretized version of the signal, and some sampled values are missing. The white squares represent unknown measurements.

Compression and Reconstruction



Underlying smooth signal

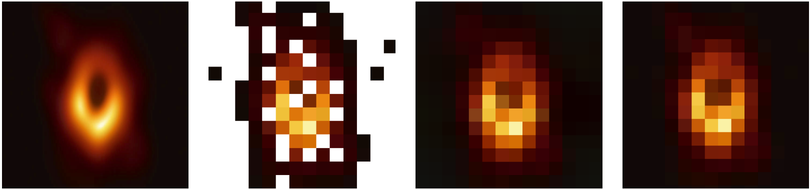
Incomplete discretized data



Reconstructed image

- Now what if we assume that the image is generated by an underlying smooth continuous signal.
- We observe only a discretized version of the signal, and some sampled values are missing. The white squares represent unknown measurements.
- A standard reconstruction algorithm uses the known measurements to estimate the missing values.

Compression and Reconstruction



Underlying smooth signal | Incomplete discretized data | Reconstructed image | Complete discretized image

- Now what if we assume that the image is generated by an underlying smooth continuous signal.
- We observe only a discretized version of the signal, and some sampled values are missing. The white squares represent unknown measurements.
- A standard reconstruction algorithm uses the known measurements to estimate the missing values.
- The reconstruction closely resembles the complete discretized image that would have been observed if no measurements were missing because this intial signal is indeed spectrally compressible.

Definitions

- For a function $g : \mathbb{Z}_N^2 \to \mathbb{C}$, define its Fourier ratio by

$$\mathrm{FR}(g) \;=\; \frac{\|\widehat{g}\|_{\ell^1}}{\|\widehat{g}\|_{\ell^2}}.$$

Definitions

- For a function $g : \mathbb{Z}_N^2 \to \mathbb{C}$, define its Fourier ratio by

$$\mathrm{FR}(g) \;=\; \frac{\|\widehat{g}\|_{\ell^1}}{\|\widehat{g}\|_{\ell^2}}.$$

- The finite Fourier transform of g is

$$\widehat{g}(m) \;=\; \frac{1}{N} \sum_{x \in \mathbb{Z}_N^2} g(x) e^{-2\pi i m \cdot x / N}, \qquad m \in \mathbb{Z}_N^2.$$

Definitions

- For a function $g : \mathbb{Z}_N^2 \to \mathbb{C}$, define its Fourier ratio by

$$\mathrm{FR}(g) \;=\; \frac{\|\widehat{g}\|_{\ell^1}}{\|\widehat{g}\|_{\ell^2}}.$$

- The finite Fourier transform of g is

$$\widehat{g}(m) \;=\; \frac{1}{N} \sum_{x \in \mathbb{Z}_N^2} g(x) e^{-2\pi i m \cdot x / N}, \qquad m \in \mathbb{Z}_N^2.$$

- The corresponding norms are

$$\|\widehat{g}\|_{\ell^1} \;=\; \sum_{m \in \mathbb{Z}_N^2} |\widehat{g}(m)| \quad \text{and} \quad \|\widehat{g}\|_{\ell^2} \;=\; \left(\sum_{m \in \mathbb{Z}_N^2} |\widehat{g}(m)|^2 \right)^{1/2}.$$

The Fourier Ratio

- For a function $g : \mathbb{Z}_N^2 \to \mathbb{C}$, define its Fourier ratio by

$$\mathrm{FR}(g) \;=\; \frac{\|\widehat{g}\|_{\ell^1}}{\|\widehat{g}\|_{\ell^2}}.$$

The Fourier Ratio

- For a function $g : \mathbb{Z}_N^2 \to \mathbb{C}$, define its Fourier ratio by

$$\mathrm{FR}(g) \;=\; \frac{\|\widehat{g}\|_{\ell^1}}{\|\widehat{g}\|_{\ell^2}}.$$

- A small ratio means the Fourier mass behaves as if it is concentrated.

The Fourier Ratio

- For a function $g : \mathbb{Z}_N^2 \to \mathbb{C}$, define its Fourier ratio by

$$\mathrm{FR}(g) \;=\; \frac{\|\widehat{g}\|_{\ell^1}}{\|\widehat{g}\|_{\ell^2}}.$$

- A small ratio means the Fourier mass behaves as if it is concentrated.
- A large ratio means the spectrum is spread out and harder to recover from few samples.

The Fourier Ratio

- For a function $g : \mathbb{Z}_N^2 \to \mathbb{C}$, define its Fourier ratio by

$$\mathrm{FR}(g) \;=\; \frac{\|\widehat{g}\|_{\ell^1}}{\|\widehat{g}\|_{\ell^2}}.$$

- A small ratio means the Fourier mass behaves as if it is concentrated.
- A large ratio means the spectrum is spread out and harder to recover from few samples.
- We find bounds for certain families of functions across several geometries.

Three geometric settings

Unit Square	**Unit Sphere**	**Manifold**
Fourier modes	spherical harmonics	Laplace–Beltrami eigenfunctions
uniform grid cells	equal-area cells	equal-volume cells
Euclidean perturbations	geodesic perturbations	geodesic perturbations

Bounding the distribution of data

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold then,

$$\mathrm{FR}_R(f) \lesssim_X \frac{\left|\int_X f\right|}{\|f\|_{L^2(X)}} + \sqrt{1+\log R}\,\frac{\|\nabla_X f\|_{L^2(X)}}{\|f\|_{L^2(X)}}.$$

Bounding the distribution of data

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold then,

$$\mathrm{FR}_R(f) \lesssim_X \frac{\left|\int_X f\right|}{\|f\|_{L^2(X)}} + \sqrt{1+\log R}\,\frac{\|\nabla_X f\|_{L^2(X)}}{\|f\|_{L^2(X)}}.$$

1. Split the ℓ^1 norm into zero and nonzero frequencies.

Bounding the distribution of data

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold then,

$$\mathrm{FR}_R(f) \;\lesssim_X\; \frac{\left|\int_X f\right|}{\|f\|_{L^2(X)}} + \sqrt{1+\log R}\,\frac{\|\nabla_X f\|_{L^2(X)}}{\|f\|_{L^2(X)}}.$$

1. Split the ℓ^1 norm into zero and nonzero frequencies.
2. Bound the zero frequency by the integral of f.

Bounding the distribution of data

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold then,

$$\mathrm{FR}_R(f) \lesssim_X \frac{\left|\int_X f\right|}{\|f\|_{L^2(X)}} + \sqrt{1+\log R}\,\frac{\|\nabla_X f\|_{L^2(X)}}{\|f\|_{L^2(X)}}.$$

1. Split the ℓ^1 norm into zero and nonzero frequencies.
2. Bound the zero frequency by the integral of f.
3. Bound the nonzero frequencies by Cauchy–Schwarz.

Bounding the distribution of data

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold then,

$$\mathrm{FR}_R(f) \;\lesssim_X\; \frac{\left|\int_X f\right|}{\|f\|_{L^2(X)}} + \sqrt{1+\log R}\,\frac{\|\nabla_X f\|_{L^2(X)}}{\|f\|_{L^2(X)}}.$$

1. Split the ℓ^1 norm into zero and nonzero frequencies.
2. Bound the zero frequency by the integral of f.
3. Bound the nonzero frequencies by Cauchy–Schwarz.
4. Convert the weighted ℓ^2 term into a derivative or Laplacian norm.

Bounding the distribution of data

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold then,

$$\mathrm{FR}_R(f) \ \lesssim_X \ \frac{\left|\int_X f\right|}{\|f\|_{L^2(X)}} + \sqrt{1+\log R}\,\frac{\|\nabla_X f\|_{L^2(X)}}{\|f\|_{L^2(X)}}.$$

1. Split the ℓ^1 norm into zero and nonzero frequencies.
2. Bound the zero frequency by the integral of f.
3. Bound the nonzero frequencies by Cauchy–Schwarz.
4. Convert the weighted ℓ^2 term into a derivative or Laplacian norm.
5. Count the frequencies in each setting so as to estimate the remaining spectral sum.

Perturbations in sensor placement

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold. If $d_X(x_i, y_i) \leq \delta$, then

$$\left(\frac{|X|}{q} \sum_{i=1}^{q} |f(y_i) - f(x_i)|^2 \right)^{1/2} \lesssim_X \delta \|\nabla_X f\|_{L^\infty(X)}.$$

Perturbations in sensor placement

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold. If $d_X(x_i, y_i) \leq \delta$, then

$$\left(\frac{|X|}{q} \sum_{i=1}^{q} |f(y_i) - f(x_i)|^2 \right)^{1/2} \lesssim_X \delta \|\nabla_X f\|_{L^\infty(X)}.$$

1. Join each sensor location x_i to its perturbation y_i by a minimizing geodesic.

Perturbations in sensor placement

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold. If $d_X(x_i, y_i) \leq \delta$, then

$$\left(\frac{|X|}{q} \sum_{i=1}^{q} |f(y_i) - f(x_i)|^2 \right)^{1/2} \lesssim_X \delta \, \|\nabla_X f\|_{L^\infty(X)}.$$

1. Join each sensor location x_i to its perturbation y_i by a minimizing geodesic.
2. Apply the Fundamental Theorem of Calculus along the geodesic.

Perturbations in sensor placement

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold. If $d_X(x_i, y_i) \leq \delta$, then

$$\left(\frac{|X|}{q} \sum_{i=1}^{q} |f(y_i) - f(x_i)|^2 \right)^{1/2} \lesssim_X \delta \|\nabla_X f\|_{L^\infty(X)}.$$

1. Join each sensor location x_i to its perturbation y_i by a minimizing geodesic.
2. Apply the Fundamental Theorem of Calculus along the geodesic.
3. Bound the integrand by the L^∞ norm of the gradient.

Perturbations in sensor placement

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold. If $d_X(x_i, y_i) \leq \delta$, then

$$\left(\frac{|X|}{q} \sum_{i=1}^{q} |f(y_i) - f(x_i)|^2 \right)^{1/2} \lesssim_X \delta \|\nabla_X f\|_{L^\infty(X)}.$$

1. Join each sensor location x_i to its perturbation y_i by a minimizing geodesic.
2. Apply the Fundamental Theorem of Calculus along the geodesic.
3. Bound the integrand by the L^∞ norm of the gradient.
4. Use the distance bound $d_X(x_i, y_i) \leq \delta$.

Perturbations in sensor placement

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold. If $d_X(x_i, y_i) \leq \delta$, then

$$\left(\frac{|X|}{q} \sum_{i=1}^{q} |f(y_i) - f(x_i)|^2 \right)^{1/2} \lesssim_X \delta \, \|\nabla_X f\|_{L^\infty(X)}.$$

1. Join each sensor location x_i to its perturbation y_i by a minimizing geodesic.
2. Apply the Fundamental Theorem of Calculus along the geodesic.
3. Bound the integrand by the L^∞ norm of the gradient.
4. Use the distance bound $d_X(x_i, y_i) \leq \delta$.
5. Square and average over the sensor locations to obtain the total placement error.

Random sensor sampling

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold partitioned into K equal-measure cells of diameter at most h_K. If sensors are placed uniformly at random, then

$$\mathbb{E}[T_q] \;=\; K\,(H_K - H_{K-q})\,, \qquad \mathcal{E}_{\text{place}} \;\lesssim_X\; h_K \|\nabla_X f\|_{L^\infty(X)}.$$

Random sensor sampling

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold partitioned into K equal-measure cells of diameter at most h_K. If sensors are placed uniformly at random, then

$$\mathbb{E}[T_q] \;=\; K\left(H_K - H_{K-q}\right), \qquad \mathcal{E}_{\text{place}} \;\lesssim_X\; h_K \|\nabla_X f\|_{L^\infty(X)}.$$

1. Partition X into K cells of equal measure.

Random sensor sampling

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold partitioned into K equal-measure cells of diameter at most h_K. If sensors are placed uniformly at random, then

$$\mathbb{E}[T_q] = K\left(H_K - H_{K-q}\right), \qquad \mathcal{E}_{\text{place}} \lesssim_X h_K \|\nabla_X f\|_{L^\infty(X)}.$$

1. Partition X into K cells of equal measure.
2. Associate each random sensor with the cell in which it lands.

Random sensor sampling

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold partitioned into K equal-measure cells of diameter at most h_K. If sensors are placed uniformly at random, then

$$\mathbb{E}[T_q] \;=\; K\left(H_K - H_{K-q}\right), \qquad \mathcal{E}_{\mathrm{place}} \;\lesssim_X\; h_K \|\nabla_X f\|_{L^\infty(X)}.$$

1. Partition X into K cells of equal measure.
2. Associate each random sensor with the cell in which it lands.
3. Compute the waiting time to hit each new cell.

Random sensor sampling

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold partitioned into K equal-measure cells of diameter at most h_K. If sensors are placed uniformly at random, then

$$\mathbb{E}[T_q] \;=\; K\,(H_K - H_{K-q})\,, \qquad \mathcal{E}_{\text{place}} \;\lesssim_X\; h_K \|\nabla_X f\|_{L^\infty(X)}.$$

1. Partition X into K cells of equal measure.
2. Associate each random sensor with the cell in which it lands.
3. Compute the waiting time to hit each new cell.
4. Sum these waiting times to obtain the coupon-collector estimate.

Random sensor sampling

Theorem

Let X be the unit square, the unit sphere, or a compact Riemannian manifold partitioned into K equal-measure cells of diameter at most h_K. If sensors are placed uniformly at random, then

$$\mathbb{E}[T_q] \;=\; K\left(H_K - H_{K-q}\right), \qquad \mathcal{E}_{\text{place}} \;\lesssim_X\; h_K \|\nabla_X f\|_{L^\infty(X)}.$$

1. Partition X into K cells of equal measure.
2. Associate each random sensor with the cell in which it lands.
3. Compute the waiting time to hit each new cell.
4. Sum these waiting times to obtain the coupon-collector estimate.
5. Use the cell diameter to bound the sensor-placement error.

Main takeaways

1. Smoothness controls the distribution of Fourier mass across several geometries.

Main takeaways

1. Smoothness controls the distribution of Fourier mass across several geometries.
2. Small errors in sensor placement produce proportionally small errors in the measured data.

Main takeaways

1. Smoothness controls the distribution of Fourier mass across several geometries.
2. Small errors in sensor placement produce proportionally small errors in the measured data.
3. Randomly placed sensors can be converted into approximately uniform samples by partitioning the space into equal-measure cells.

Main takeaways

1. Smoothness controls the distribution of Fourier mass across several geometries.
2. Small errors in sensor placement produce proportionally small errors in the measured data.
3. Randomly placed sensors can be converted into approximately uniform samples by partitioning the space into equal-measure cells.
4. Together, these results give a geometry-independent framework for stable signal recovery.

Future Work

Definition

An abstract spectral space consists of a probability space (X, μ) together with a non-negative, self-adjoint operator A having compact resolvent. The eigenfunctions of A form an orthonormal basis for $L^2(X, \mu)$.

1. We move to this abstract framework because it isolates the properties actually used in our proofs and allows the Fourier Ratio argument to apply beyond the square, sphere, and smooth manifolds.

Future Work

Conjecture

Let $A^{s/2}$ be a non-negative, self-adjoint generator with compact resolvent and let L be the spectral cutoff of the orthonormal basis. Then

$$FR(f) \leq \frac{\left|\int_X f \,\mathrm{d}\mu\right|}{\|f\|_2} + C\frac{\|A^{s/2}f\|_2}{\|f\|_2}\begin{cases} 1, & s > \alpha \\ \sqrt{1+\log L}, & s = \alpha \\ L^{(\alpha-s)/2}, & 0 < s < \alpha \end{cases}.$$

Where α describes the growth rate of the spectrum.

Future Work

Compact measured resistance spaces provide a natural setting in which to apply this abstract framework.

1. A compact metric space (X, R)

These spaces include many classical fractals where the usual differential-geometric framework is unavailable.

Future Work

Compact measured resistance spaces provide a natural setting in which to apply this abstract framework.

1. A compact metric space (X, R)
2. A probability measure μ

These spaces include many classical fractals where the usual differential-geometric framework is unavailable.

Future Work

Compact measured resistance spaces provide a natural setting in which to apply this abstract framework.

1. A compact metric space (X, R)
2. A probability measure μ
3. A resistance form $\mathcal{E}$

These spaces include many classical fractals where the usual differential-geometric framework is unavailable.

Future Work

Compact measured resistance spaces provide a natural setting in which to apply this abstract framework.

1. A compact metric space (X, R)
2. A probability measure μ
3. A resistance form $\mathcal{E}$
4. A function space $\mathcal{F}$ of finite-energy functions

These spaces include many classical fractals where the usual differential-geometric framework is unavailable.

Future Work

At first, the irregular structure of a fractal suggests that signals on it should be maximally non-compressible. However, many resistance spaces have low spectral dimension, which can produce the opposite behavior:

$$\mathrm{FR}(f) \lesssim \frac{\left|\int_X f \,\mathrm{d}\mu\right|}{\|f\|_2} + C_X \frac{\mathcal{E}(f,f)^{1/2}}{\|f\|_2}.$$

Even though the Fourier Ratio can remain small, localized eigenfunctions can cause standard point-sampling recovery algorithms to break down.

References

- A. Iosevich, E. Palsson, and A. Yavicoli, *Discretization, sampling, and the Fourier ratio*, 2026.
- W. Burstein, A. Iosevich, and H. Nathan, *The Fourier Ratio: A Unifying Measure of Complexity for Recovery, Localization, and Learning*, 2026.
- S. Musser, *Weyl's Law on Riemannian Manifolds*, University of Chicago Mathematics REU paper, (2016).
- J. Kigami, *Harmonic analysis for resistance forms*, Journal of Functional Analysis, 204, no. 2, (2003), pp. 399–444.
- E. J. Candès, J. Romberg, and T. Tao, *Robust uncertainty principles: exact signal reconstruction from highly incomplete frequency information*, IEEE Transactions on Information Theory, 52, no. 2, (2006), pp. 489–509.

Thanks ^_^