From M&Ms to Mathematics, or, How I learned to answer questions and help my kids love math.

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Topics in Recreational Math and Finite Geometry
AMS Eastern Sectional, Albany: 20 October 2024

Some Issues for the Future

- World rapidly changing – powerful computing cheap & available.
- What skills are we teaching? What should we?
- One of hardest skills: how to think / attack a new problem, how to see connections, what data to gather.

Goals of the Talk: Opportunities Everywhere!

- Ask Questions! Often simple questions lead to good math.
- Gather data: observe, program and simulate.
- Use games to get to mathematics.
- Discuss implementation: `Please interrupt!`

Joint work with Cameron (was 4) and Kayla (was 2) Miller:

The M&M Game: From Morsels to Modern Mathematics (Ivan Badinski, Nathan McCue, Cameron Miller, Kayla Miller, Michael Stone), Math. Magazine **90** (2017), no. 3, 197–207: https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/MMProblem10.pdf

Pre-requisite: Logarithms

The logarithm of x base b is the power we raise b to get x; it is the inverse of exponentiation.

$$x = b^y \quad \text{if and only if} \quad \log_b(x) = y.$$

$$\log_{10}(10^4) = 4, \quad \log_{10}(1/100) = -2.$$

Properties:

- $\log_{10} 1 = 0.$
- $10^{\log_{10} x} = x.$
- $\log_{10}(A \cdot B) = \log_{10} A + \log_{10} B.$

Pre-requisites: Combinatorics Review

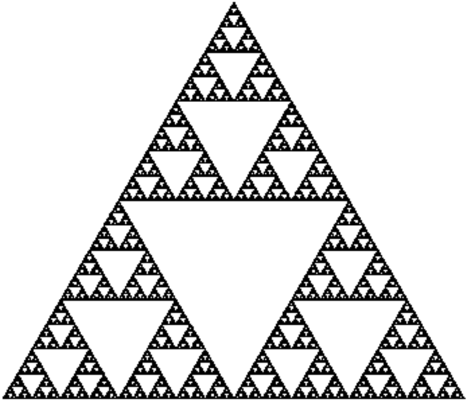
- $n!$: number of ways to order n people, order matters.

- $\frac{n!}{k!(n-k)!} = n\mathrm{C}k = \binom{n}{k}$: number of ways to choose k from n, order doesn't matter.

- Examples: $\binom{n}{1} = n$, $\binom{4}{2} = 6$, in general $\binom{n}{2} = \frac{n(n-1)}{2}$.

Pre-req: Pascal's Triangle: (4min26sec): Hidden Structure:
https://www.youtube.com/watch?v=_vkGakVt1RA&t=216s

Sierpinski's triangle: Look at Pascal's triangle modulo 2:
https://www.youtube.com/watch?v=tt4_4YajqRM.



Sierpinki's Gasket
http://www.math.sunysb.edu/~scott/Book331/Sierpinski_gasket.html

The M&M Game

Motivating Question

Cam (4 years): `If you're born on the same day, do you die on the same day?`

M&M Game Rules

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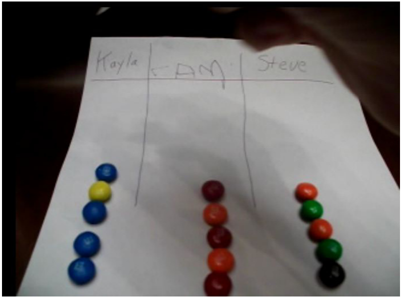


(1) Everyone starts off with k M&Ms (we did 5).
(2) All toss fair coins, eat an M&M if and only if head.

Be active – ask questions!

`What are natural questions to ask?`

Be active – ask questions!

What are natural questions to ask?

Question 1: How likely is a tie (as a function of k)?

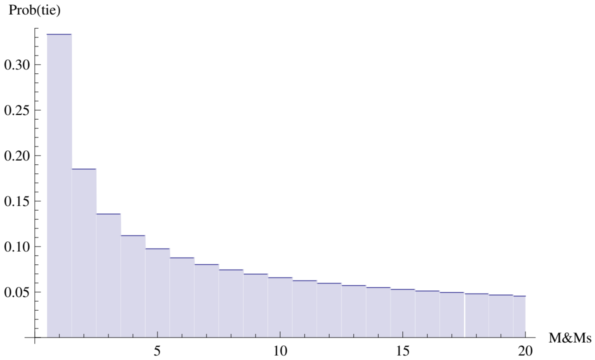
Question 2: How long until one dies?

Question 3: Generalize the game: More people? Biased coin?

Important to ask questions – curiousity is good and to be encouraged! Value to the journey and not knowing the answer.

Let's gather some data! Let's play!

Probability of a tie in the M&M game (2 players)



Prob(tie) $\approx$ 33% (1 M&M), 19% (2 M&Ms), 14% (3 M&Ms), 10% (4 M&Ms).

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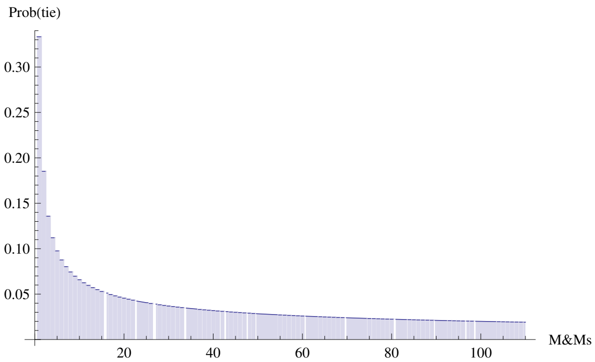
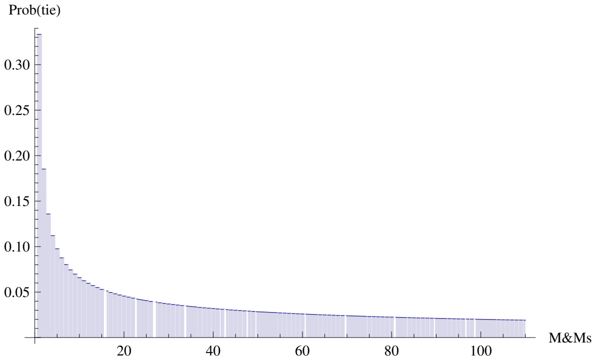


Gave at a 110th anniversary talk....

Probability of a tie in the M&M game (2 players)



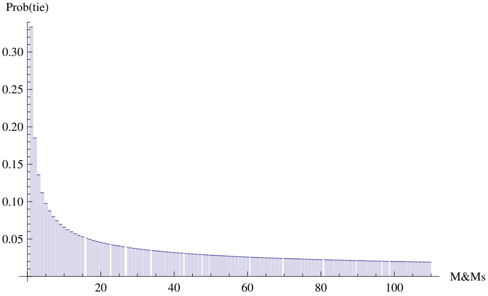
... asked them: what will the next 110 bring us?

Never too early to lay foundations for future classes.

Welcome to Statistics and Inference!

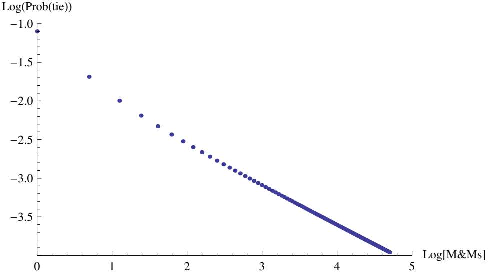
- ◇ Goal: Gather data, see pattern, extrapolate.
- ◇ Methods: Simulation, analysis of special cases.
- ◇ Presentation: It matters how we show data, and which data we show.

Viewing M&M Plots



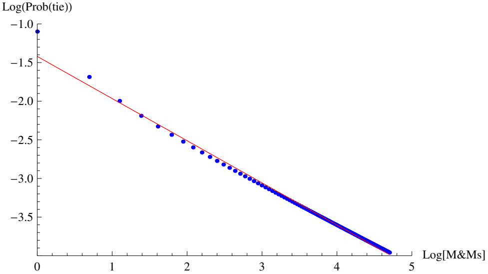
Hard to predict what comes next.

Viewing M&M Plots: Log-Log Plot



Not *just* sadistic teachers: logarithms useful!

Viewing M&M Plots: Log-Log Plot

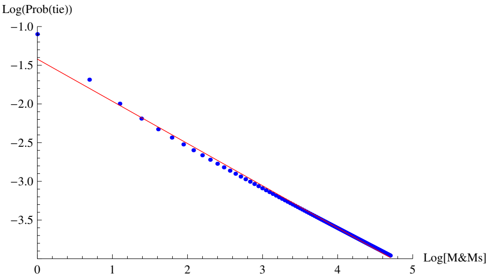


Best fit line:

$\log(\text{Prob(tie)}) = -1.42022 - 0.545568 \log(\#\text{M\&Ms})$ or

$\text{Prob}(k) \approx 0.2412/k^{.5456}$.

Viewing M&M Plots: Log-Log Plot



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Predicts probability of a tie when $k = 220$ is 0.01274, but answer is 0.01347. **What gives?**

Statistical Inference: Too Much Data Is Bad!

Small values can mislead / distort. Let's go from $k = 50$ to 110.

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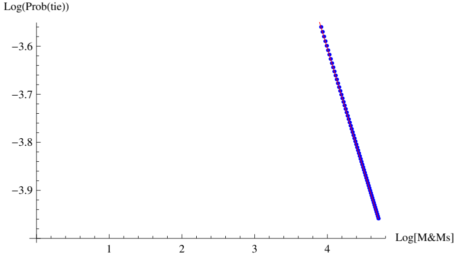

Best fit line:

$\log(\text{Prob(tie)}) = -1.58261 - 0.50553 \log(\#\text{M\&Ms})$ or
$\text{Prob}(k) \approx 0.205437/k^{.50553}$ (had $0.241662/k^{.5456}$).

Statistical Inference: Too Much Data Is Bad!

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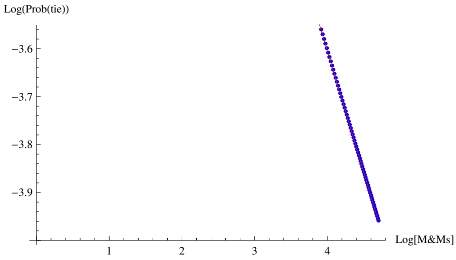

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$\text{Prob}(k) \approx 0.205437/k^{.50553}$ (had $0.241662/k^{.5456}$).

Get 0.01344 for $k = 220$ (answer 0.01347); **much better!**

From Shooting Hoops
to the Geometric Series Formula

Simpler Game: Hoops

Game of hoops: first basket wins, alternate shooting.

Simpler Game: Hoops: Mathematical Formulation

Bird and **Magic** (I'm old!) alternate shooting; first basket wins.

- **Bird** always gets basket with probability p.
- **Magic** always gets basket with probability q.

Let x be the probability **Bird** wins – what is x?

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p.
- **Bird** wins on 2nd shot: $(1-p)(1-q)\cdot p$.
- **Bird** wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.
- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Let $r = (1-p)(1-q)$. Then

$$\begin{aligned} x &= \text{Prob}(\textbf{Bird}\text{ wins}) \\ &= p + rp + r^2p + r^3p + \cdots \\ &= p\left(1 + r + r^2 + r^3 + \cdots\right), \end{aligned}$$

the geometric series.

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird}\text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Solving the Hoop Game: The Power of Perspective

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$$x = \text{Prob}(\textbf{Bird}\ \text{wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

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Have

$$x = \text{Prob}(\textbf{Bird}\ \text{wins}) = p + (1-p)(1-q)$$

Solving the Hoop Game: The Power of Perspective

Showed

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Have

$$x = \text{Prob}(\textbf{Bird}\ \text{wins}) = p + (1 - p)(1 - q)x$$

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

Have

$$x = \text{Prob}(\textbf{Bird} \text{ wins}) = p + (1-p)(1-q)x = p + rx.$$

Thus

$$(1-r)x = p \quad \text{or} \quad x = \frac{p}{1-r}.$$

As $x = p(1 + r + r^2 + r^3 + \cdots)$, find

$$1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}.$$

Lessons from Hoop Problem

- Power of Perspective: Memoryless process.
- Can circumvent algebra with deeper understanding! (Hard)
- Depth of a problem not always what expect.
- Importance of knowing more than the minimum: connections.
- Math is fun!

The M&M Game

Solving the M&M Game

Overpower with algebra: Assume k M&Ms, two people, fair coins:

$$\text{Prob(tie)} = \sum_{n=k}^{\infty} \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-1}\frac{1}{2} \cdot \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-1}\frac{1}{2},$$

where as always binomial coefficient

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

"Simplifies" to $4^{-k}\ {}_2F_1(k, k, 1, 1/4)$, a special value of a hypergeometric function!
If $k = 100,000$ about .063% (predict .061%).

A look at your future classes, but is there a better way?

Solving the M&M Game (cont)

Where did formula come from? Each turn one of four equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is 1/4 or 25%.

Solving the M&M Game (cont)

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- Neither eat.

Probability of each event is 1/4 or 25%.
Each person has exactly $k-1$ heads in first $n-1$ tosses, then ends with a head.

$$\text{Prob(tie)} = \sum_{n=k}^{\infty} \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-1}\frac{1}{2} \cdot \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-1}\frac{1}{2}.$$

Solving the M&M Game (cont)

Use the lesson from the Hoops Game: Memoryless process!

If neither eat, as if toss didn't happen. Now game is finite.

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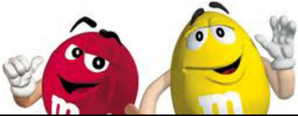
Much better perspective: each "turn" one of three equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.

Probability of each event is 1/3 or about 33%

$$\sum_{n=0}^{k-1} \binom{2k-n-2}{n} \left(\frac{1}{3}\right)^n \binom{2k-2n-2}{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \binom{1}{1} \frac{1}{3}.$$



Solving the M&M Game (cont)

Interpretation: Let Cam have c M&Ms and Kayla have k; write as (c, k).

Then each of the following happens 1/3 of the time after a ‘turn’:

- $(c, k) \longrightarrow (c-1, k-1)$.
- $(c, k) \longrightarrow (c-1, k)$.
- $(c, k) \longrightarrow (c, k-1)$.

Solving the M&M Game (cont): Assume $k = 4$

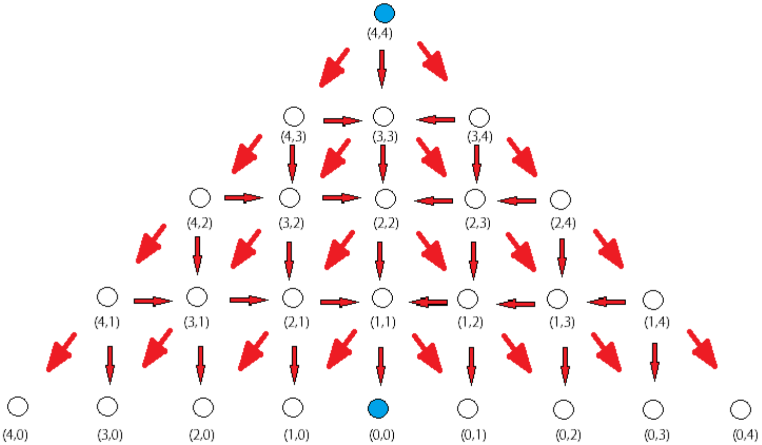


Figure: The M&M game when $k = 4$. Count the paths! Answer 1/3 of probability hit (1,1).

Solving the M&M Game (cont): Assume $k = 4$

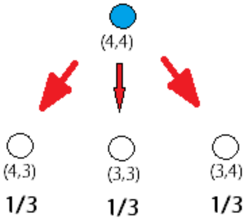


Figure: The M&M game when $k = 4$, going down one level.

Solving the M&M Game (cont): Assume $k = 4$

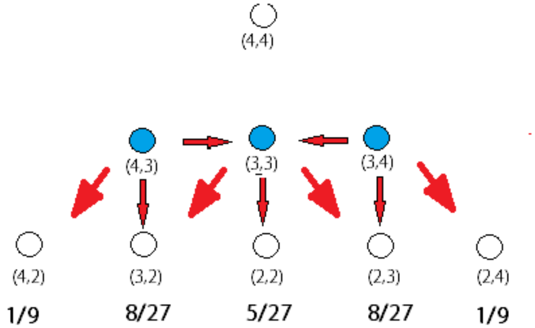


Figure: The M&M game when $k = 4$, removing probability from the second level.

Solving the M&M Game (cont): Assume $k = 4$

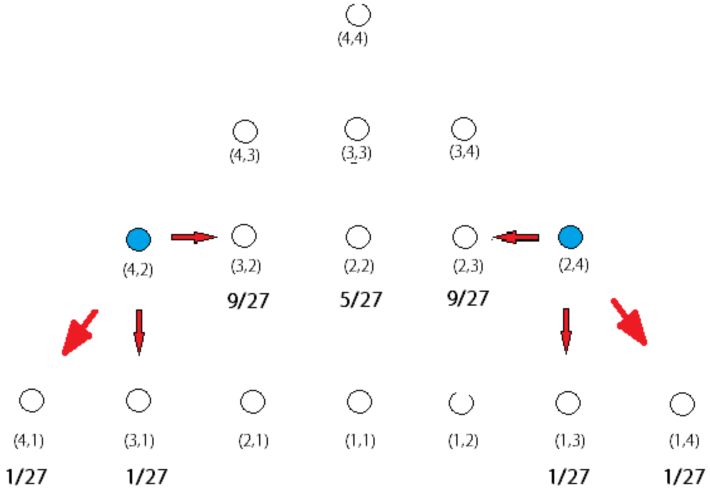


Figure: Removing probability from two outer on third level.

Solving the M&M Game (cont): Assume $k = 4$

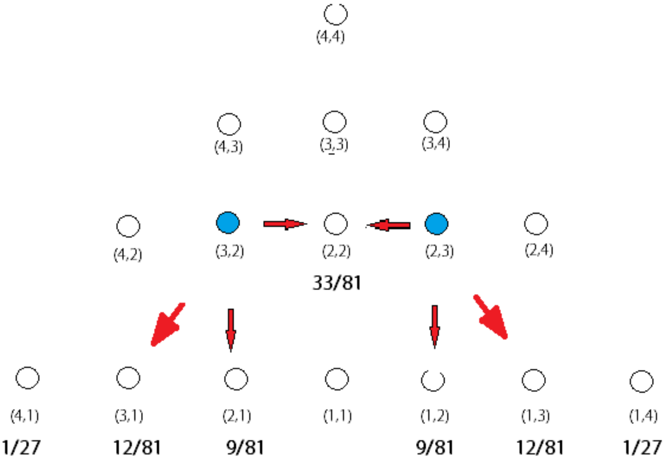


Figure: Removing probability from the (3,2) and (2,3) vertices.

Solving the M&M Game (cont): Assume $k = 4$

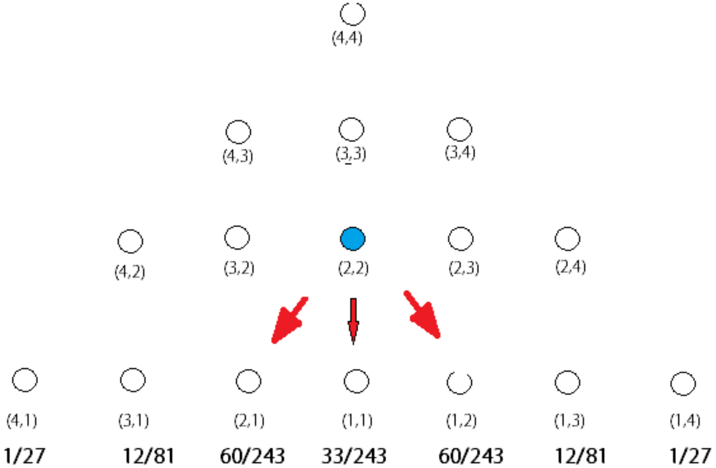


Figure: Removing probability from the (2,2) vertex.

Solving the M&M Game (cont): Assume $k = 4$

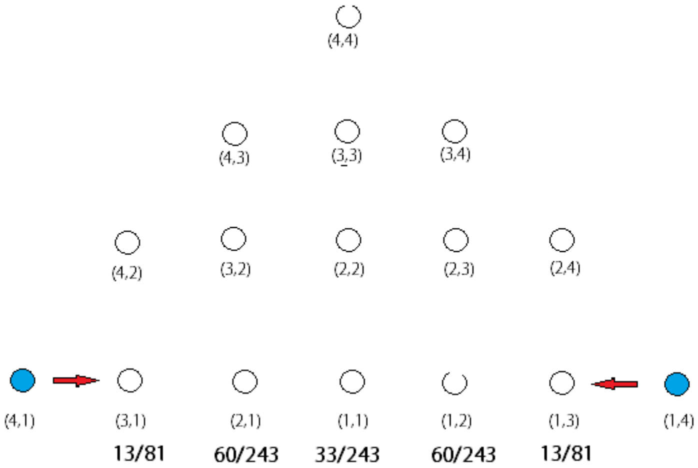


Figure: Removing probability from the (4,1) and (1,4) vertices.

Solving the M&M Game (cont): Assume $k = 4$

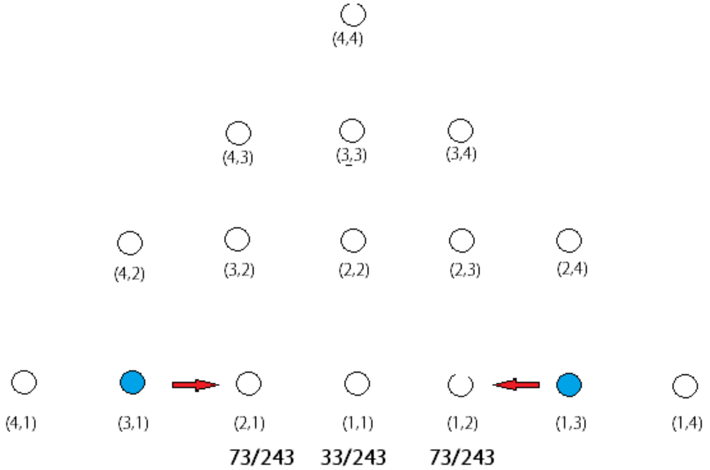


Figure: Removing probability from the (3,1) and (1,3) vertices.

Solving the M&M Game (cont): Assume $k = 4$

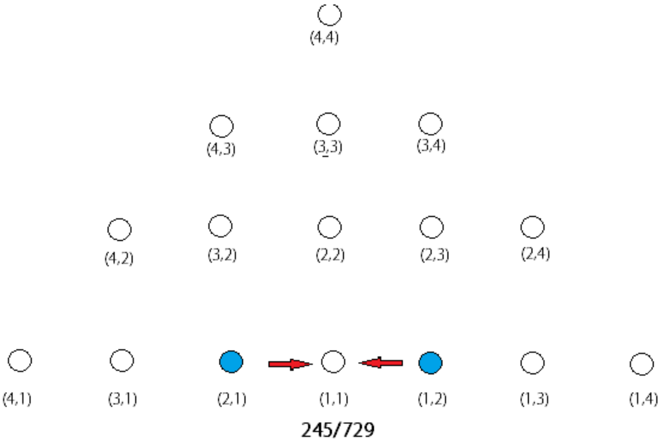


Figure: Removing probability from (2,1) and (1,2) vertices. Answer is 1/3 of (1,1) vertex, or 245/2187 (about 11%).

Interpreting Proof: Connections to the Fibonacci Numbers!

Fibonaccis: $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0, F_1 = 1$.

Starts 0, 1, 1, 2, 3, 5, 8, 13, 21, ...
http://www.youtube.com/watch?v=kkGeOWYOFoA.

Binet's Formula (can prove via 'generating functions'):

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

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M&Ms: For $c, k \geq 1$: $x_{c,0} = x_{0,k} = 0$; $x_{0,0} = 1$, and if $c, k \geq 1$:

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

Reproduces the tree but a lot 'cleaner'.

Interpreting Proof: Finding the Recurrence

What if we didn't see the 'simple' recurrence?

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

The following recurrence is 'natural':

$$x_{c,k} \;=\; \frac{1}{4}x_{c,k} + \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1}.$$

Obtain 'simple' recurrence by algebra: subtract $\frac{1}{4}x_{c,k}$:

$$\begin{aligned} \frac{3}{4}x_{c,k} &= \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1} \\ \text{therefore } x_{c,k} &= \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}. \end{aligned}$$

Solving the Recurrence

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

- $x_{0,0} = 1.$

- $x_{1,0} = x_{0,1} = 0.$
- $x_{1,1} = \frac{1}{3}x_{0,0} + \frac{1}{3}x_{0,1} + \frac{1}{3}x_{1,0} = \frac{1}{3} \approx 33.3\%.$

- $x_{2,0} = x_{0,2} = 0.$
- $x_{2,1} = \frac{1}{3}x_{1,0} + \frac{1}{3}x_{1,1} + \frac{1}{3}x_{2,0} = \frac{1}{9} \;=\; x_{1,2}.$
- $x_{2,2} = \frac{1}{3}x_{1,1} + \frac{1}{3}x_{1,2} + \frac{1}{3}x_{2,1} = \frac{1}{9} + \frac{1}{27} + \frac{1}{27} \;=\; \frac{5}{27} \approx 18.5\%.$

Try Simpler Cases!!!

Try and find an easier problem and build intuition.

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Walking from (0,0) to (k, k) with allowable steps (1,0), (0,1) and (1,1), hit (k, k) before hit top or right sides.

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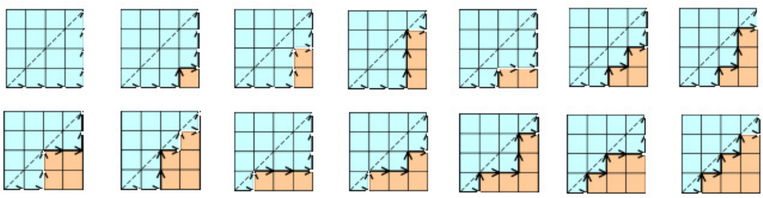

Interpretation: Catalan numbers are valid placings of (and).

Examining Probabilities of a Tie

When $k = 1$, Prob(tie) = 1/3.

When $k = 2$, Prob(tie) = 5/27.

When $k = 3$, Prob(tie) = 11/81.

When $k = 4$, Prob(tie) = 245/2187.

When $k = 5$, Prob(tie) = 1921/19683.

When $k = 6$, Prob(tie) = 575/6561.

When $k = 7$, Prob(tie) = 42635/531441.

When $k = 8$, Prob(tie) = 355975/4782969.

Examining Ties: Multiply by 3^{2k-1} to clear denominators.

When $k = 1$, get 1.

When $k = 2$, get 5.

When $k = 3$, get 33.

When $k = 4$, get 245.

When $k = 5$, get 1921.

When $k = 6$, get 15525.

When $k = 7$, get 127905.

When $k = 8$, get 1067925.

OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ...

OEIS

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OEIS: http://oeis.org/.

OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ...

OEIS: http://oeis.org/.

Our sequence: http://oeis.org/A084771.

The web exists! Use it to build conjectures, suggest proofs....

OEIS (continued)

A084771 Coefficients of 1/sqrt(1-10*x+9*x^2); also, a(n) is the central coefficient of (1+5*x+4*x^2)^n. 5

1, 5, 33, 245, 1921, 15525, 127905, 1067925, 9004545, 76499525, 653808673, 5614995765, 48416454529, 418895174885, 3634723102113, 31616937184725, 275621102802945, 2407331941640325, 21061836725455905, 184550106298084725 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Also number of paths from (0,0) to (n,0) using steps U=(1,1), H=(1,0) and D=(1,-1), the U steps come in four colors and the H steps come in five colors. - N-E. Fahssi, Mar 30 2008
Number of lattice paths from (0,0) to (n,n) using steps (1,0), (0,1), and three kinds of steps (1,1). [Joerg Arndt, Jul 01 2011]
Sums of squares of coefficients of (1+2*x)^n. [Joerg Arndt, Jul 06 2011]
The Hankel transform of this sequence gives A103488 . - Philippe DELEHAM, Dec 02 2007

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Michael Z. Spivey and Laura L. Steil, The k-Binomial Transforms and the Hankel Transform, Journal of Integer Sequences, Vol. 9 (2006), Article 06.1.1.

LINKS Table of n, a(n) for n=0..19.
Tony D. Noe, On the Divisibility of Generalized Central Trinomial Coefficients, Journal of Integer Sequences, Vol. 9 (2006), Article 06.2.7.

FORMULA G.f.: 1/sqrt(1-10*x+9*x^2).
Binomial transform of A059304. G.f.: Sum_{k>=0} binomial(2*k, k)*(2*x)^k/(1-x)^(k+1). E.g.f.: exp(5*x)*BesselI(0, 4*x). - Vladeta Jovovic (vladeta(AT)eunet.rs), Aug 20 2003
a(n) = sum(k=0..n, sum(j=0..n-k, C(n,j)*C(n-j,k)*C(2*n-2*j,n-j))). - Paul Barry, May 19 2006
a(n) = sum(k=0..n, 4^k*(C(n,k))^2) [From heruneedollar (heruneedollar(AT)gmail.com), Mar 20 2010]
Asymptotic: a(n) ~ 3^(2*n+1)/(2*sqrt(2*Pi*n)). [Vaclav Kotesovec, Sep 11 2012]
Conjecture: n*a(n) +5*(-2*n+1)*a(n-1) +9*(n-1)*a(n-2)=0. - R. J. Mathar,

Future Work and Takeaways

Current and Future Work

Current and Future Projects (inspired by conversations at Texas State with April Yang), done in 2024 Polymath Jr REU[1]:

- What if each person tosses several fair coins simultaneously of different denominations (some positive, some negative, sum perhaps is zero)?
- What if probability of a head tends to 1 as the number of M&M's left decreases?

[1] https://geometrynyc.wixsite.com/polymathreu

Lessons

- Always ask questions.
- Many ways to solve a problem.
- Experience is useful and a great guide.
- Need to look at the data the right way.
- Often don't know where the math will take you.
- Value of continuing education: more math is better.
- Connections: My favorite quote: `If all you have is a hammer, pretty soon every problem looks like a nail.`