

# From M&Ms to Mathematics, or, How I learned to answer questions and help my kids love math.

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[http://web.williams.edu/Mathematics/sjmillier/public\\_html/](http://web.williams.edu/Mathematics/sjmillier/public_html/)

HCSSiM, July 17++, 2013



## Some Issues for the Future

- World is rapidly changing – powerful computing cheaply and readily available.
- What skills are we teaching? What skills should we be teaching?
- One of hardest skills: how to think / attack a new problem, how to see connections, what data to gather.

## Goals of the Talk: Opportunities Everywhere!

- Ask Questions! Often simple questions lead to good math.
- Gather data: observe, program and simulate.
- Use games to get to mathematics.
- Discuss implementation: **Please interrupt!**

Joint work with Cameron (age 6) and Kayla (age 4) Miller

My math riddles page:

<http://mathriddles.williams.edu/>

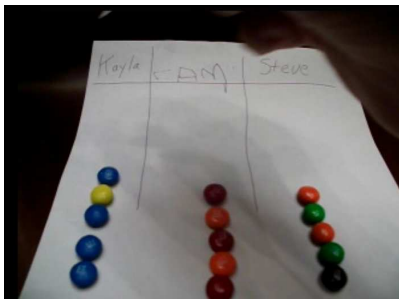
## The M&M Game

## Motivating Question

**Cam (4 years):** If you're born on the same day, do you die on the same day?

## M&M Game Rules

**Cam (4 years):** If you're born on the same day, do you die on the same day?



- (1) Everyone starts off with  $k$  M&Ms (we did 5).
- (2) All toss fair coins, eat an M&M if and only if head.



## Be active – ask questions!

**What are natural questions to ask?**

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### What are natural questions to ask?

**Question 1:** How likely is a tie (as a function of  $k$ )?

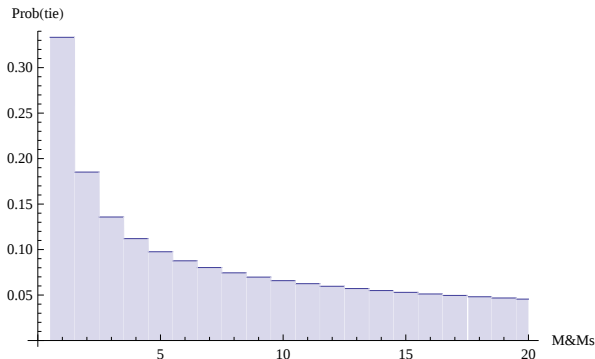
**Question 2:** How long until one dies?

**Question 3:** Generalize the game: More people? Biased coin?

Important to ask questions – curiosity is good and to be encouraged! Value to the journey and not knowing the answer.

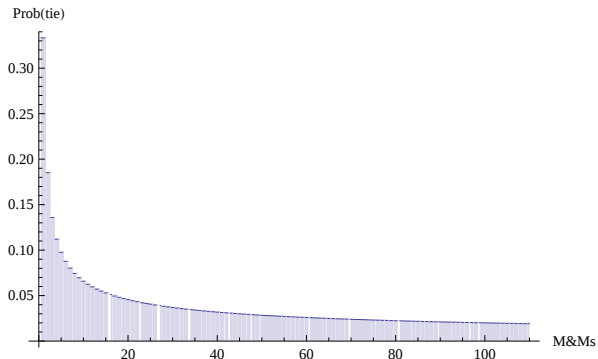
Let's gather some data!

## Probability of a tie in the M&M game (2 players)



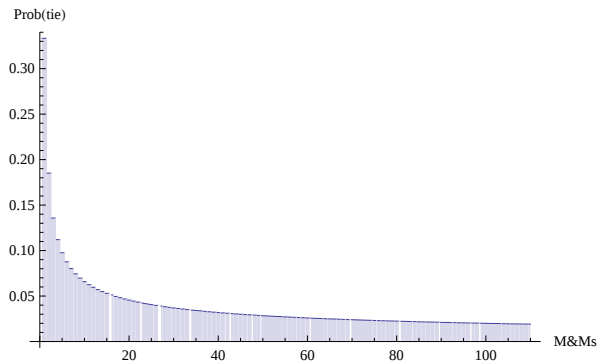
$\text{Prob}(\text{tie}) \approx 33\%$  (1 M&M),  $19\%$  (2 M&Ms),  $14\%$  (3 M&Ms),  $10\%$  (4 M&Ms).

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Gave at a 110th anniversary talk....

## Probability of a tie in the M&M game (2 players)

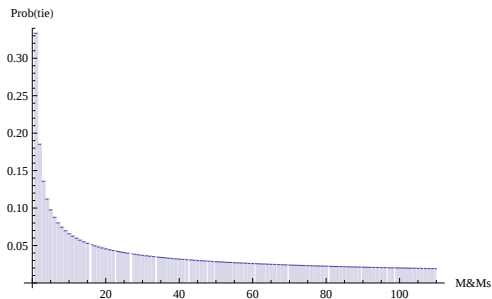


... asked them: what will the next 110 bring us?  
Never too early to lay foundations for future classes.

# Welcome to Statistics and Inference!

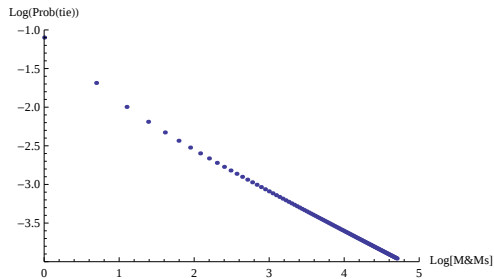
- ◇ **Goal:** Gather data, see pattern, extrapolate.
- ◇ **Methods:** Simulation, analysis of special cases.
- ◇ **Presentation:** It matters **how** we show data, and **which** data we show.

## Viewing M&M Plots



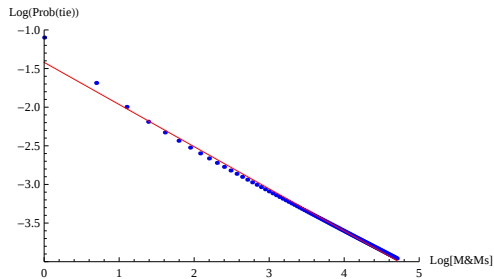
Hard to predict what comes next.

## Viewing M&M Plots: Log-Log Plot



Not *just* sadistic teachers: logarithms useful!

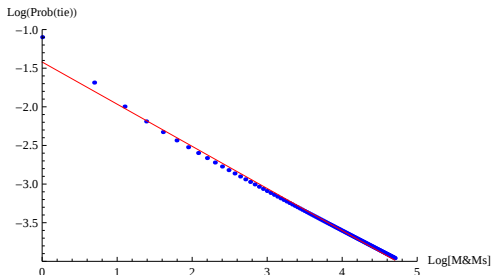
## Viewing M&M Plots: Log-Log Plot



**Best fit line:**

$$\log(\text{Prob}(\text{tie})) = -1.42022 - 0.545568 \log(\# \text{M\&Ms}) \text{ or}$$
$$\text{Prob}(k) \approx 0.2412/k^{.5456}.$$

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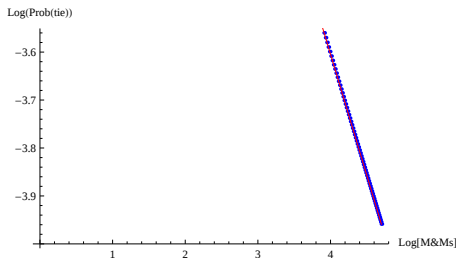
Predicts probability of a tie when  $k = 220$  is 0.01274, but answer is 0.0137. **What gives?**

## Statistical Inference: Too Much Data Is Bad!

Small values can mislead / distort. Let's go from  $k = 50$  to 110.

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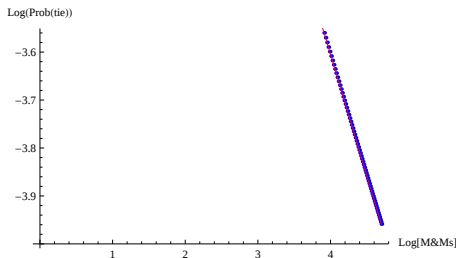


**Best fit line:**

$$\log(\text{Prob}(\text{tie})) = -1.58261 - 0.50553 \log(\# \text{M\&Ms}) \text{ or}$$
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Get 0.01344 for  $k = 220$  (answer 0.01347); **much better!**

From Shooting Hoops  
to the Geometric Series Formula

## Simpler Game: Hoops

Game of hoops: first basket wins, alternate shooting.



## Simpler Game: Hoops: Mathematical Formulation

**Bird** and **Magic** (I'm old!) alternate shooting; first basket wins.

- **Bird** always gets basket with probability  $p$ .
- **Magic** always gets basket with probability  $q$ .

Let  $x$  be the probability **Bird** wins – what is  $x$ ?

## Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

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Let  $r = (1 - p)(1 - q)$ . Then

$$\begin{aligned} \textcolor{red}{x} &= \text{Prob}(\textcolor{teal}{B} \text{ wins}) \\ &= p + rp + r^2p + r^3p + \cdots \\ &= p \left( 1 + r + r^2 + r^3 + \cdots \right), \end{aligned}$$

the geometric series.

## Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\text{Bird wins}) = p(1 + r + r^2 + r^3 + \dots);$$

will solve without the geometric series formula.

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$$(1 - r)x = p \quad \text{or} \quad x = \frac{p}{1 - r}.$$

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Thus

$$(1 - r)x = p \quad \text{or} \quad x = \frac{p}{1 - r}.$$

As  $x = p(1 + r + r^2 + r^3 + \dots)$ , find

$$1 + r + r^2 + r^3 + \dots = \frac{1}{1 - r}.$$

## Lessons from Hoop Problem

- ◇ Power of Perspective: Memoryless process.
- ◇ Can circumvent algebra with deeper understanding! (Hard)
- ◇ Depth of a problem not always what expect.
- ◇ Importance of knowing more than the minimum: [connections](#).
- ◇ Math is fun!

## The M&M Game

## Solving the M&M Game

**Overpower with algebra:** Assume  $k$  M&Ms, two people, fair coins:

$$\text{Prob}(\text{tie}) = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2},$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is a binomial coefficient.

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“Simplifies” to  $4^{-k} {}_2F_1(k, k, 1, 1/4)$ , a special value of a hypergeometric function! (Look up / write report.)

A look at your future classes, but is there a better way?

## Solving the M&M Game (cont)

Where did formula come from? Each turn one of four **equally likely** events happens:

- Both eat an M&M.
- Cam eats an M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is  $1/4$  or 25%.

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- Neither eat.

Probability of each event is **1/4** or **25%**.

Each person has exactly  $k - 1$  heads in first  $n - 1$  tosses, then ends with a head.

$$\text{Prob}(\text{tie}) = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}.$$



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Much better perspective: each "turn" one of **three equally likely** events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.

Probability of each event is **1/3** or about **33%**

$$\sum_{n=0}^{k-1} \binom{2k-n-2}{n} \left(\frac{1}{3}\right)^n \binom{2k-2n-2}{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \binom{1}{1} \frac{1}{3}.$$



## Solving the M&M Game (cont)

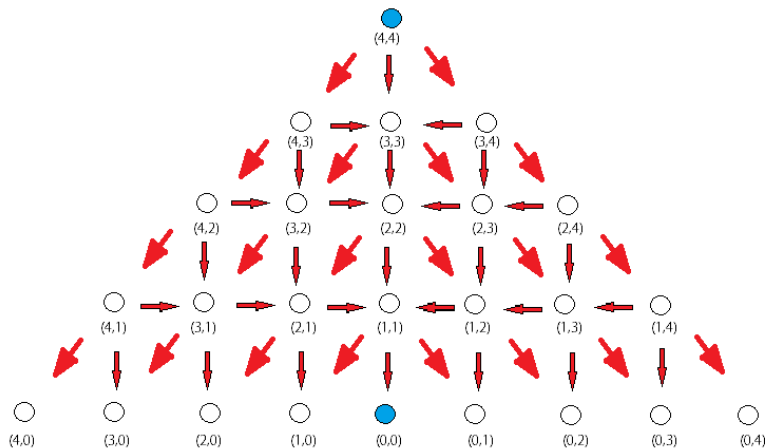
**Interpretation:** Let Cam have  $c$  M&Ms and Kayla have  $k$ ; write as  $(c, k)$ .

Then each of the following happens  $1/3$  of the time after a 'turn':

- $(c, k) \longrightarrow (c - 1, k - 1)$ .
- $(c, k) \longrightarrow (c - 1, k)$ .
- $(c, k) \longrightarrow (c, k - 1)$ .

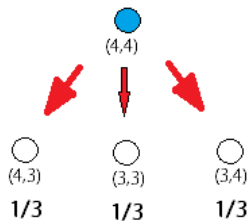


## Solving the M&M Game (cont): Assume $k = 4$



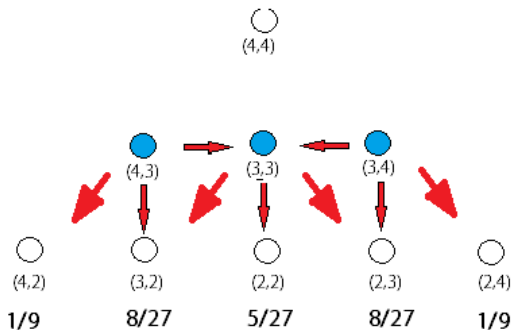
**Figure:** The M&M game when  $k = 4$ . Count the paths! Answer  $1/3$  of probability hit  $(1,1)$ .

## Solving the M&M Game (cont): Assume $k = 4$



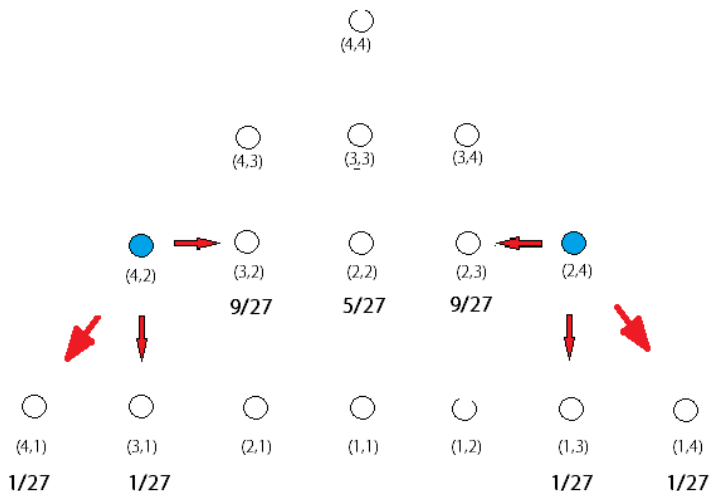
**Figure:** The M&M game when  $k = 4$ , going down one level.

## Solving the M&M Game (cont): Assume $k = 4$



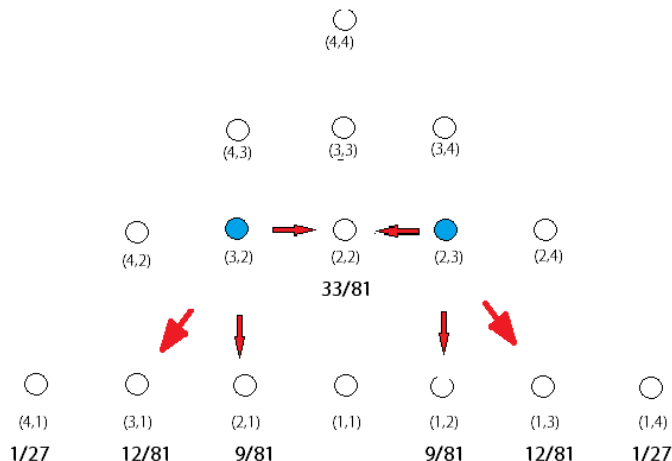
**Figure:** The M&M game when  $k = 4$ , removing probability from the second level.

## Solving the M&M Game (cont): Assume $k = 4$



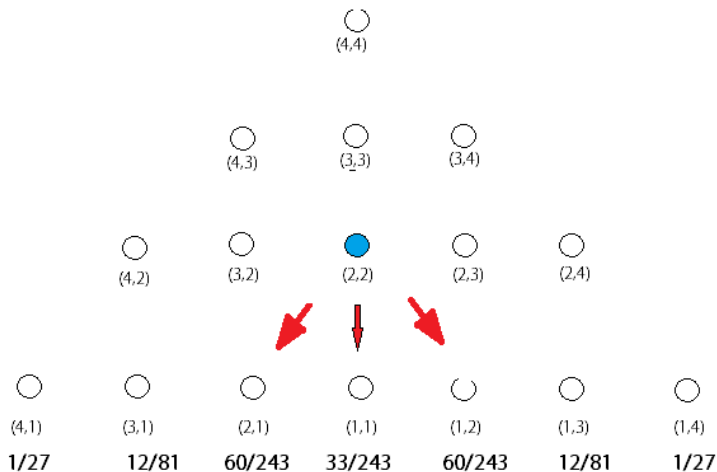
**Figure:** Removing probability from two outer on third level.

## Solving the M&M Game (cont): Assume $k = 4$



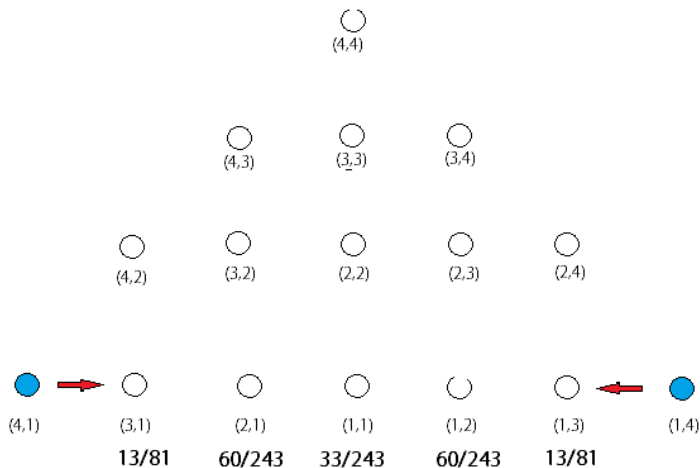
**Figure:** Removing probability from the  $(3,2)$  and  $(2,3)$  vertices.

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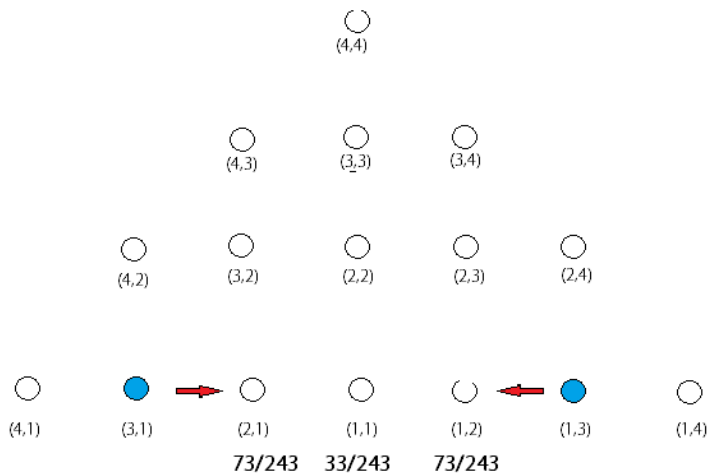
**Figure:** Removing probability from the (2,2) vertex.

## Solving the M&M Game (cont): Assume $k = 4$



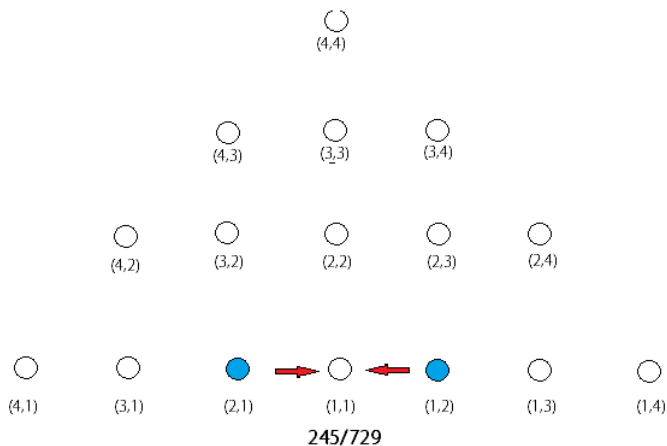
**Figure:** Removing probability from the  $(4, 1)$  and  $(1, 4)$  vertices.

## Solving the M&M Game (cont): Assume $k = 4$



**Figure:** Removing probability from the  $(3, 1)$  and  $(1, 3)$  vertices.

## Solving the M&M Game (cont): Assume $k = 4$



**Figure:** Removing probability from  $(2,1)$  and  $(1,2)$  vertices. Answer is  $1/3$  of  $(1,1)$  vertex, or  $245/2187$  (about 11%).

## Interpreting Proof: Connections to the Fibonacci Numbers!

**Fibonacci:**  $F_{n+2} = F_{n+1} + F_n$  with  $F_0 = 0, F_1 = 1$ .

Starts 0, 1, 1, 2, 3, 5, 8, 13, 21, ....

<http://www.youtube.com/watch?v=kkGe0WY0FoA>.

**Binet's Formula** (can prove via 'generating functions'):

$$F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$

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M&Ms: For  $c, k \geq 1$ :  $x_{c,0} = x_{0,k} = 0$ ;  $x_{0,0} = 1$ , and if  $c, k \geq 1$ :

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

Reproduces the tree but a lot 'cleaner'.

## Interpreting Proof: Finding the Recurrence

What if we didn't see the 'simple' recurrence?

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

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The following recurrence is 'natural':

$$x_{c,k} = \frac{1}{4}x_{c,k} + \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1}.$$

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Obtain 'simple' recurrence by algebra: subtract  $\frac{1}{4}x_{c,k}$ :

$$\begin{aligned} \frac{3}{4}x_{c,k} &= \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1} \\ \text{therefore } x_{c,k} &= \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}. \end{aligned}$$

## Solving the Recurrence

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

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- $x_{0,0} = 1.$
- $x_{1,0} = x_{0,1} = 0.$
- $x_{1,1} = \frac{1}{3}x_{0,0} + \frac{1}{3}x_{0,1} + \frac{1}{3}x_{1,0} = \frac{1}{3} \approx 33.3\%.$

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- $x_{2,0} = x_{0,2} = 0.$
- $x_{2,1} = \frac{1}{3}x_{1,0} + \frac{1}{3}x_{1,1} + \frac{1}{3}x_{2,0} = \frac{1}{9} = x_{1,2}.$
- $x_{2,2} = \frac{1}{3}x_{1,1} + \frac{1}{3}x_{1,2} + \frac{1}{3}x_{2,1} = \frac{1}{9} + \frac{1}{27} + \frac{1}{27} = \frac{5}{27} \approx 18.5\%.$

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Walking from  $(0,0)$  to  $(k,k)$  with allowable steps  $(1,0)$ ,  $(0,1)$  and  $(1,1)$ , hit  $(k,k)$  before hit top or right sides.

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Try and find an easier problem and build intuition.

Walking from  $(0,0)$  to  $(k,k)$  with allowable steps  $(1,0)$ ,  $(0,1)$  and  $(1,1)$ , hit  $(k,k)$  before hit top or right sides.

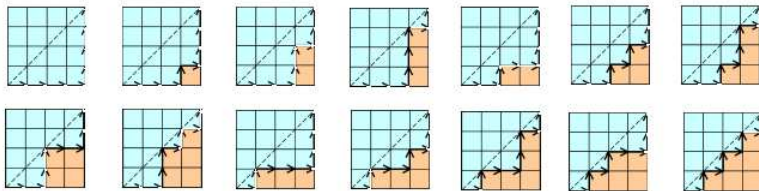
Generalization of the Catalan problem. There don't have  $(1,1)$  and stay on or below the main diagonal.

## Try Simpler Cases!!!

Try and find an easier problem and build intuition.

Walking from  $(0,0)$  to  $(k,k)$  with allowable steps  $(1,0)$ ,  $(0,1)$  and  $(1,1)$ , hit  $(k,k)$  before hit top or right sides.

Generalization of the Catalan problem. There don't have  $(1,1)$  and stay on or below the main diagonal.



Interpretation: Catalan numbers are valid placings of ( and ).

## Aside: Fun Riddle Related to Catalan Numbers

Young Saul, a budding mathematician and printer, is making himself a fake ID. He needs it to say he's 21. The problem is he's not using a computer, but rather he has some symbols he's bought from the store, and that's it. He has one 1, one 5, one 6, one 7, and an unlimited supply of  $+$   $-$   $*$   $/$  (the operations addition, subtraction, multiplication and division). Using each number exactly once (but you can use any number of  $+$ , any number of  $-$ , ...) how, oh how, can he get 21 from 1,5, 6,7? Note: you can't do things like  $15+6 = 21$ . You have to use the four operations as 'binary' operations:  $((1+5)*6) + 7$ . Problem submitted by [ohadbp@infolink.net.il](mailto:ohadbp@infolink.net.il), phrasing by yours truly.

Solution involves valid sentences:  $((w + x) + y) + z, w + ((x + y) + z), \dots$

For more riddles see my riddles page:  
<http://mathriddles.williams.edu/>.

## Examining Probabilities of a Tie

When  $k = 1$ ,  $\text{Prob}(\text{tie}) = 1/3$ .

When  $k = 2$ ,  $\text{Prob}(\text{tie}) = 5/27$ .

When  $k = 3$ ,  $\text{Prob}(\text{tie}) = 11/81$ .

When  $k = 4$ ,  $\text{Prob}(\text{tie}) = 245/2187$ .

When  $k = 5$ ,  $\text{Prob}(\text{tie}) = 1921/19683$ .

When  $k = 6$ ,  $\text{Prob}(\text{tie}) = 575/6561$ .

When  $k = 7$ ,  $\text{Prob}(\text{tie}) = 42635/531441$ .

When  $k = 8$ ,  $\text{Prob}(\text{tie}) = 355975/4782969$ .

## Examining Ties: Multiply by $3^{2k-1}$ to clear denominators.

When  $k = 1$ , get 1.

When  $k = 2$ , get 5.

When  $k = 3$ , get 33.

When  $k = 4$ , get 245.

When  $k = 5$ , get 1921.

When  $k = 6$ , get 15525.

When  $k = 7$ , get 127905.

When  $k = 8$ , get 1067925.

# OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ....

# OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ....

OEIS: <http://oeis.org/>.

# OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ....

OEIS: <http://oeis.org/>.

Our sequence: <http://oeis.org/A084771>.

**The web exists!** Use it to build conjectures, suggest proofs....

## OEIS (continued)

|            |                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |
|------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| A084771    | Coefficients of $1/\sqrt{1-10*x+9*x^2}$ ; also, $a(n)$ is the central coefficient of $(1+5*x+4*x^2)^n$ . <sup>5</sup>                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                 |
|            | 1, 5, 33, 245, 1921, 15525, 127905, 1067925, 9004545, 76499525, 653808673, 5614995765, 48416454529, 418895174885, 3634723102113, 31616937184725, 275621102802945, 2407331941640325, 21061836725455905, 184550106298084725 ( <a href="#">list</a> ; <a href="#">graph</a> ; <a href="#">refs</a> ; <a href="#">listen</a> ; <a href="#">history</a> ; <a href="#">text</a> ; <a href="#">internal format</a> )                                                                                                                                                                                                                                                                                                                         |
| OFFSET     | 0,2                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                   |
| COMMENTS   | Also number of paths from (0,0) to (n,0) using steps U=(1,1), H=(1,0) and D=(1,-1), the U steps come in four colors and the H steps come in five colors. - <a href="#">N-E. Fahssi</a> , Mar 30 2008<br>Number of lattice paths from (0,0) to (n,n) using steps (1,0), (0,1), and three kinds of steps (1,1). [Joerg Arndt, Jul 01 2011]<br>Sums of squares of coefficients of $(1+2*x)^n$ . [Joerg Arndt, Jul 06 2011]<br>The Hankel transform of this sequence gives <a href="#">A103488</a> . - <a href="#">Philippe DELEHAM</a> , Dec 02 2007                                                                                                                                                                                     |
| REFERENCES | Paul Barry and Aoife Hennessy, Generalized Narayana Polynomials, Riordan Arrays, and Lattice Paths, Journal of Integer Sequences, Vol. 15, 2012, #12.4.8.- From <a href="#">N. J. A. Sloane</a> , Oct 08 2012<br>Michael Z. Spivey and Laura L. Steil, The k-Binomial Transforms and the Hankel Transform, Journal of Integer Sequences, Vol. 9 (2006), Article 06.1.1.                                                                                                                                                                                                                                                                                                                                                               |
| LINKS      | <a href="#">Table of n, a(n) for n=0..19</a> .<br>Tony D. Noe, <a href="#">On the Divisibility of Generalized Central Trinomial Coefficients</a> , Journal of Integer Sequences, Vol. 9 (2006), Article 06.2.7.                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                                       |
| FORMULA    | G.f.: $1/\sqrt{1-10*x+9*x^2}$ .<br>Binomial transform of <a href="#">A059304</a> . G.f.: $\sum_{k=0}^{\infty} \text{binomial}(2*k, k)*(2*x)^k/(1-x)^{(k+1)}$ . E.g.f.: $\exp(5*x)*\text{BesselI}(0, 4*x)$ . - Vladeta Jovovic ( <a href="#">vladeta(AT)eunet.rs</a> ), Aug 20 2003<br>$a(n) = \sum_{k=0..n} \sum_{j=0..n-k} C(n,j)*C(n-j,k)*C(2*n-2*j,n-j) )$ . - Paul Barry, May 19 2006<br>$a(n) = \sum_{k=0..n} 4^k*(C(n,k))^2$ [From heruneedollar ( <a href="#">heruneedollar(AT)gmail.com</a> ), Mar 20 2010]<br>Asymptotic: $a(n) \sim 3^{(2*n+1)/(2*\sqrt{2*\pi*n})}$ . [ <a href="#">Vaclav Kotesovec</a> , Sep 11 2012]<br>Conjecture: $n*a(n) + 5*(-2*n+1)*a(n-1) + 9*(n-1)*a(n-2) = 0$ . - <a href="#">R. J. Mathar</a> , |

## Takeaways

## Lessons

- ◇ Always ask questions.
- ◇ Many ways to solve a problem.
- ◇ Experience is useful and a great guide.
- ◇ Need to look at the data the right way.
- ◇ Often don't know where the math will take you.
- ◇ Value of continuing education: more math is better.
- ◇ Connections: My favorite quote: If all you have is a hammer, pretty soon every problem looks like a nail.

## Generating Functions

## Generating Function (Example: Binet's Formula)

### Binet's Formula

$$F_1 = F_2 = 1; F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{-1+\sqrt{5}}{2} \right)^n \right].$$

- **Recurrence relation:**  $F_{n+1} = F_n + F_{n-1}$  (1)
- **Generating function:**  $g(x) = \sum_{n>0} F_n x^n$ .

$$(1) \Rightarrow \sum_{n \geq 2} F_{n+1} x^{n+1} = \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 2} F_{n-1} x^{n+1}$$

$$\Rightarrow \sum_{n \geq 3} F_n x^n = \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 1} F_n x^{n+2}$$

$$\Rightarrow \sum_{n \geq 3} F_n x^n = x \sum_{n \geq 2} F_n x^n + x^2 \sum_{n \geq 1} F_n x^n$$

$$\Rightarrow g(x) - F_1 x - F_2 x^2 = x(g(x) - F_1 x) + x^2 g(x)$$

$$\Rightarrow g(x) = x/(1 - x - x^2).$$

## Partial Fraction Expansion (Example: Binet's Formula)

- **Generating function:**  $g(x) = \sum_{n \geq 0} F_n x^n = \frac{x}{1-x-x^2}$ .
- **Partial fraction expansion:**

$$\Rightarrow g(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}} \left( \frac{\frac{1+\sqrt{5}}{2}x}{1 - \frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1 - \frac{-1+\sqrt{5}}{2}x} \right).$$

**Coefficient of  $x^n$**  (power series expansion):

$$F_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1+\sqrt{5}}{2} \right)^n - \left( \frac{-1+\sqrt{5}}{2} \right)^n \right] \text{ - Binet's Formula!}$$

(using geometric series:  $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \dots$ ).