From M&Ms to Mathematics, or, How I learned to answer questions and help my kids love math.

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Maine-Québec Number Theory Conference
October 2015
The M&M Game
Motivating Question

Cam (4 years): If you’re born on the same day, do you die on the same day?
**M&M Game Rules**

**Cam (4 years):** If you’re born on the same day, do you die on the same day?

(1) Everyone starts off with $k$ M&Ms (we did 5).
(2) All toss fair coins, eat an M&M if and only if head.
Be active – ask questions!

What are natural questions to ask?

Question 1: How likely is a tie (as a function of $k$)?

Question 2: How long until one dies?

Question 3: Generalize the game: More people? Biased coin?

Important to ask questions – curiosity is good and to be encouraged! Value to the journey and not knowing the answer.

Let’s gather some data!
Takeaways

Probability of a tie in the M&M game (2 players)

\[
\text{Prob(tie)} \approx 33\% \text{ (1 M&M)}, \ 19\% \text{ (2 M&Ms)}, \ 14\% \text{ (3 M&Ms)}, \ 10\% \text{ (4 M&Ms)}.
\]
Probability of a tie in the M&M game (2 players)

Gave at a 110th anniversary talk....
... asked them: what will the next 110 bring us?
Never too early to lay foundations for future classes.
Welcome to Statistics and Inference!

◊ **Goal**: Gather data, see pattern, extrapolate.

◊ **Methods**: Simulation, analysis of special cases.

◊ **Presentation**: It matters how we show data, and which data we show.
Viewing M&M Plots

Hard to predict what comes next.
Viewing M&M Plots: Log-Log Plot

Not *just* sadistic teachers: logarithms useful!
Viewing M&M Plots: Log-Log Plot

Best fit line:
\[
\log (\text{Prob(tie)}) = -1.42022 - 0.545568 \log(\#\text{M&Ms}) \text{ or } \text{Prob}(k) \approx 0.2412/k^{0.5456}.
\]

Predicts probability of a tie when \( k = 220 \) is 0.01274, but answer is 0.0137. What gives?
Small values can mislead / distort. Let’s go from $k = 50$ to 110.

Best fit line:

$$\log(\text{Prob(tie)}) = -1.58261 - 0.50553 \log(\#\text{M&Ms})$$

or

$$\text{Prob}(k) \approx 0.205437/k^{0.50553} \text{ (had } 0.241662/k^{0.5456}).$$

Get 0.01344 for $k = 220$ (answer 0.01347); much better!
The M&M Game
Solving the M&M Game

Overpower with algebra: Assume $k$ M&Ms, two people, fair coins:

$$\text{Prob(tie)} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2},$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is a binomial coefficient.

“Simplifies” to $4^{-k} \, _2F_1(k, k, 1, 1/4)$, a special value of a hypergeometric function!
Solving the M&M Game (cont)

Where did formula come from? Each turn one of four equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is $1/4$ or 25%.
Solving the M&M Game (cont)

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Each person has exactly $k - 1$ heads in first $n - 1$ tosses, then ends with a head.

$$\text{Prob(tie)} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}.$$
Solving the M&M Game (cont)

Use the lesson from the Hoops Game: Memoryless process!

If neither eat, as if toss didn’t happen. Now game is finite.
Solving the M&M Game (cont)

Use the lesson from the Hoops Game: Memoryless process!

If neither eat, as if toss didn’t happen. Now game is finite.

Much better perspective: each “turn” one of **three equally likely** events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.

Probability of each event is $1/3$ or about $33\%$

$$
\sum_{n=0}^{k-1} \binom{2k-n-2}{n} \left( \frac{1}{3} \right)^n \binom{2k-2n-2}{k-n-1} \left( \frac{1}{3} \right)^{k-n-1} \left( \frac{1}{3} \right)^{k-n-1} \left( \frac{1}{3} \right) \left( \frac{1}{3} \right) \left( \frac{1}{3} \right).
$$
Interpretation: Let Cam have $c$ M&Ms and Kayla have $k$; write as $(c, k)$.

Then each of the following happens $1/3$ of the time after a ‘turn’:

- $(c, k) \rightarrow (c - 1, k - 1)$.
- $(c, k) \rightarrow (c - 1, k)$.
- $(c, k) \rightarrow (c, k - 1)$. 
Solving the M&M Game (cont): Assume $k = 4$

**Figure:** The M&M game when $k = 4$. Count the paths! Answer $1/3$ of probability hit $(1,1)$. 
Interpreting Proof: Connections to the Fibonacci Numbers!

Fibonacci: \( F_{n+2} = F_{n+1} + F_n \) with \( F_0 = 0, F_1 = 1 \).

Starts 0, 1, 1, 2, 3, 5, 8, 13, 21, ….

[http://www.youtube.com/watch?v=kkGeOWYOFoA](http://www.youtube.com/watch?v=kkGeOWYOFoA).

Binet’s Formula (can prove via ‘generating functions’):

\[
F_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.
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M&Ms: For \( c, k \geq 1 \): \( x_{c,0} = x_{0,k} = 0 \); \( x_{0,0} = 1 \), and if \( c, k \geq 1 \):

\[
x_{c,k} = \frac{1}{3} x_{c-1,k-1} + \frac{1}{3} x_{c-1,k} + \frac{1}{3} x_{c,k-1}.
\]

Reproduces the tree but a lot ‘cleaner’.
Solving the Recurrence

\[ x_{c,k} = \frac{1}{3} x_{c-1,k-1} + \frac{1}{3} x_{c-1,k} + \frac{1}{3} x_{c,k-1}. \]

- \( x_{0,0} = 1. \)
- \( x_{1,0} = x_{0,1} = 0. \)
- \( x_{1,1} = \frac{1}{3} x_{0,0} + \frac{1}{3} x_{0,1} + \frac{1}{3} x_{1,0} = \frac{1}{3} \approx 33.3\%. \)
- \( x_{2,0} = x_{0,2} = 0. \)
- \( x_{2,1} = \frac{1}{3} x_{1,0} + \frac{1}{3} x_{1,1} + \frac{1}{3} x_{2,0} = \frac{1}{9} = x_{1,2}. \)
- \( x_{2,2} = \frac{1}{3} x_{1,1} + \frac{1}{3} x_{1,2} + \frac{1}{3} x_{2,1} = \frac{1}{9} + \frac{1}{27} + \frac{1}{27} = \frac{5}{27} \approx 18.5\%. \)
Examining Probabilities of a Tie

When \( k = 1 \), \( \text{Prob(tie)} = 1/3 \).

When \( k = 2 \), \( \text{Prob(tie)} = 5/27 \).

When \( k = 3 \), \( \text{Prob(tie)} = 11/81 \).

When \( k = 4 \), \( \text{Prob(tie)} = 245/2187 \).

When \( k = 5 \), \( \text{Prob(tie)} = 1921/19683 \).

When \( k = 6 \), \( \text{Prob(tie)} = 575/6561 \).

When \( k = 7 \), \( \text{Prob(tie)} = 42635/531441 \).

When \( k = 8 \), \( \text{Prob(tie)} = 355975/4782969 \).
Examining Ties: Multiply by $3^{2k-1}$ to clear denominators.

When $k = 1$, get 1.

When $k = 2$, get 5.

When $k = 3$, get 33.

When $k = 4$, get 245.

When $k = 5$, get 1921.

When $k = 6$, get 15525.

When $k = 7$, get 127905.

When $k = 8$, get 1067925.
Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ....


Our sequence: [http://oeis.org/A084771](http://oeis.org/A084771).

The web exists! Use it to build conjectures, suggest proofs....
OEIS (continued)

A084771  Coefficients of 1/sqrt(1-10*x+9*x^2); also, a(n) is the central coefficient of (1+5*x+4*x^2)^n.

1, 5, 33, 245, 1921, 15525, 127905, 1067925, 9004545, 76499525, 653808675, 5614995675, 48416454529, 418895174885, 3634723102113, 3161693184725, 275621102802945, 2407331941640325, 21061836725455905, 184550106298084725 (list; graph; refs; listen; history; text; internal format)
OFFSET 0,2
COMMENTS Also number of paths from (0,0) to (n,0) using steps U=(1,1), H=(1,0) and D=(1,-1), the U steps come in four colors and the H steps come in five colors. - N-E. Fahssi, Mar 30 2008
Number of lattice paths from (0,0) to (n,n) using steps (1,0), (0,1), and three kinds of steps (1,1). [Joerg Arndt, Jul 01 2011]
Sums of squares of coefficients of (1+2*x)^n. [Joerg Arndt, Jul 06 2011]
The Hankel transform of this sequence gives A103488. - Philippe Deléham, Dec 02 2007
LINKS Table of n, a(n) for n=0..19.
FORMULA G.f.: 1/sqrt(1-10*x+9*x^2).
Binomial transform of A059304. G.f.: Sum_{k>=0} binomial(2*k, k)*
(2*x)^k/(1-x)^k (x+1). E.g.f.: exp(5*x)*BesselI(0, 4*x). - Vladeta Jovovic (vladeta(AT)eunet.rs), Aug 20 2003
a(n) = sum(k=0..n, sum(j=0..n-k, C(n,j)*C(n-j,k)*C(2*n-2*j,n-j) ) ). - Paul Barry, May 19 2006
a(n) = sum(k=0..n, 4^k*C(n,k)^2 ) [From heruneedollar (heruneedollar(AT)gmail.com), Mar 20 2010]
Asymptotic: a(n) ~ 3^(2*n+1)/(2*sqrt(2*Pi*n)). [Vaclav Kotesovec, Sep 11 2012]
Conjecture: n*a(n) +5*(-2*n+1)*a(n-1) +9*(n-1)*a(n-2)=0. - R. J. Mathar,
Takeaways
Lessons

◊ Always ask questions.

◊ Many ways to solve a problem.

◊ Experience is useful and a great guide.

◊ Need to look at the data the right way.

◊ Often don’t know where the math will take you.

◊ Value of continuing education: more math is better.

◊ Connections: My favorite quote: *If all you have is a hammer, pretty soon every problem looks like a nail.*