From M&Ms to Mathematics, or, How I learned to answer questions and help my kids love math.

Cameron, Kayla and Steven J. Miller, Williams College
sjm1@williams.edu,
Steven.Miller.MC.96@aya.yale.edu
http://web.williams.edu/Mathematics/sjmiller/public_html

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The M&M Game

Motivating Question

Cam (4 years): `If you're born on the same day, do you die on the same day?`

M&M Game Rules

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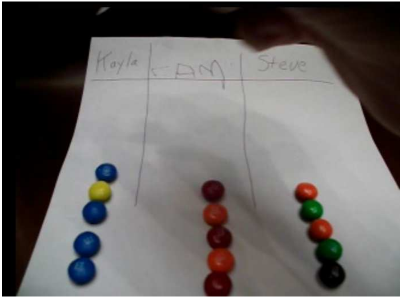


(1) Everyone starts off with k M&Ms (we did 5).
(2) All toss fair coins, eat an M&M if and only if head.

Be active – ask questions!

What are natural questions to ask?

Question 1: How likely is a tie (as a function of k)?

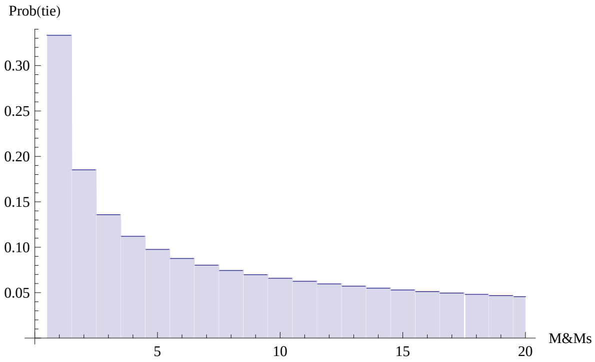
Question 2: How long until one dies?

Question 3: Generalize the game: More people? Biased coin?

Important to ask questions – curiousity is good and to be encouraged! Value to the journey and not knowing the answer.

Let's gather some data!

Probability of a tie in the M&M game (2 players)



Prob(tie) $\approx$ 33% (1 M&M), 19% (2 M&Ms), 14% (3 M&Ms), 10% (4 M&Ms).

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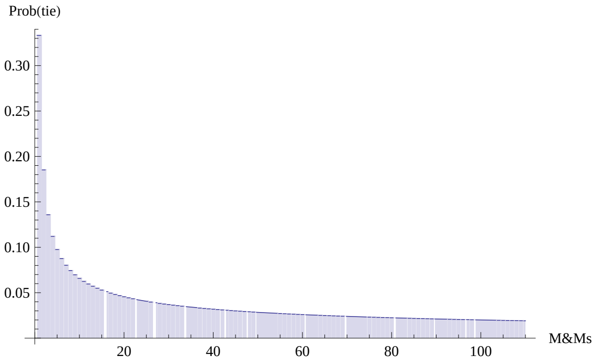
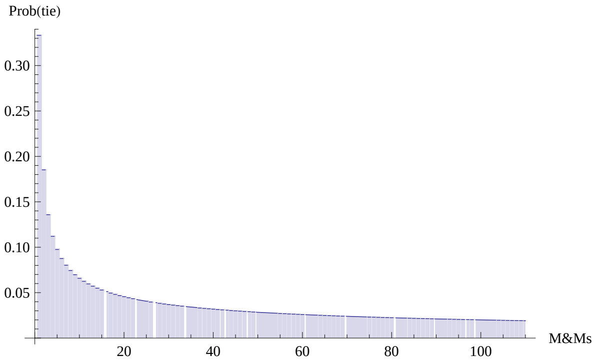


Gave at a 110th anniversary talk....

Probability of a tie in the M&M game (2 players)



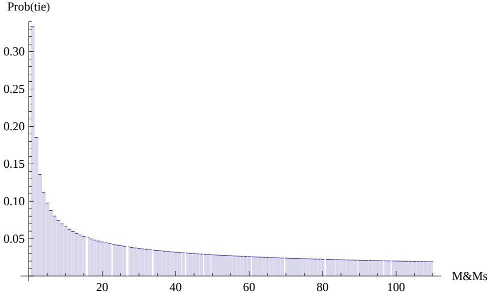
... asked them: what will the next 110 bring us?

Never too early to lay foundations for future classes.

Welcome to Statistics and Inference!

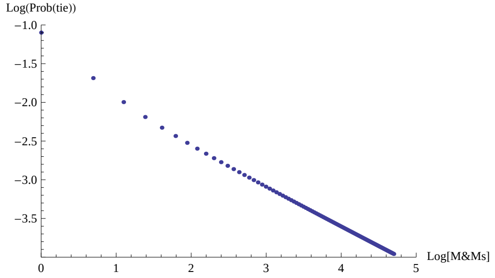
- ◇ Goal: Gather data, see pattern, extrapolate.
- ◇ Methods: Simulation, analysis of special cases.
- ◇ Presentation: It matters how we show data, and which data we show.

Viewing M&M Plots



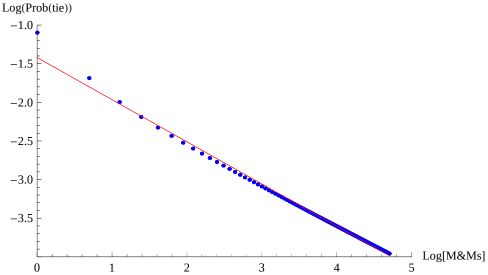
Hard to predict what comes next.

Viewing M&M Plots: Log-Log Plot



Not *just* sadistic teachers: logarithms useful!

Viewing M&M Plots: Log-Log Plot

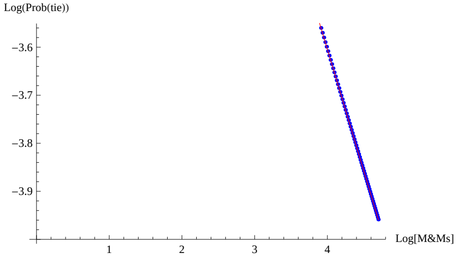


Best fit line:
$\log(\text{Prob(tie)}) = -1.42022 - 0.545568 \log(\#\text{M\&Ms})$ or $\text{Prob}(k) \approx 0.2412/k^{.5456}$.

Predicts probability of a tie when $k = 220$ is 0.01274, but answer is 0.0137. **What gives?**

Statistical Inference: Too Much Data Is Bad!

Small values can mislead / distort. Let's go from $k = 50$ to 110.



Best fit line:

$\log(\text{Prob(tie)}) = -1.58261 - 0.50553 \log(\#\text{M\&Ms})$ or
$\text{Prob}(k) \approx 0.205437/k^{.50553}$ (had $0.241662/k^{.5456}$).

Get 0.01344 for $k = 220$ (answer 0.01347); **much better!**

The M&M Game

Solving the M&M Game

Overpower with algebra: Assume k M&Ms, two people, fair coins:

$$\text{Prob(tie)} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2},$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is a binomial coefficient.

"Simplifies" to $4^{-k}\ {}_2F_1(k, k, 1, 1/4)$, a special value of a hypergeometric function!

Solving the M&M Game (cont)

Where did formula come from? Each turn one of four equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is 1/4 or 25%.

Solving the M&M Game (cont)

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Each person has exactly $k-1$ heads in first $n-1$ tosses, then ends with a head.

$$\mathrm{Prob(tie)} = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}.$$

Solving the M&M Game (cont)

Use the lesson from the Hoops Game: Memoryless process!

If neither eat, as if toss didn't happen. Now game is finite.

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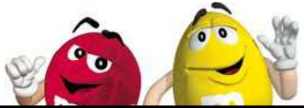
Much better perspective: each "turn" one of three equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.

Probability of each event is 1/3 or about 33%

$$\sum_{n=0}^{k-1} \binom{2k-n-2}{n} \left(\frac{1}{3}\right)^n \binom{2k-2n-2}{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \binom{1}{1} \frac{1}{3}.$$



Solving the M&M Game (cont)

Interpretation: Let Cam have c M&Ms and Kayla have k; write as (c, k).

Then each of the following happens 1/3 of the time after a 'turn':

- $(c, k) \longrightarrow (c-1, k-1)$.
- $(c, k) \longrightarrow (c-1, k)$.
- $(c, k) \longrightarrow (c, k-1)$.

Solving the M&M Game (cont): Assume $k = 4$

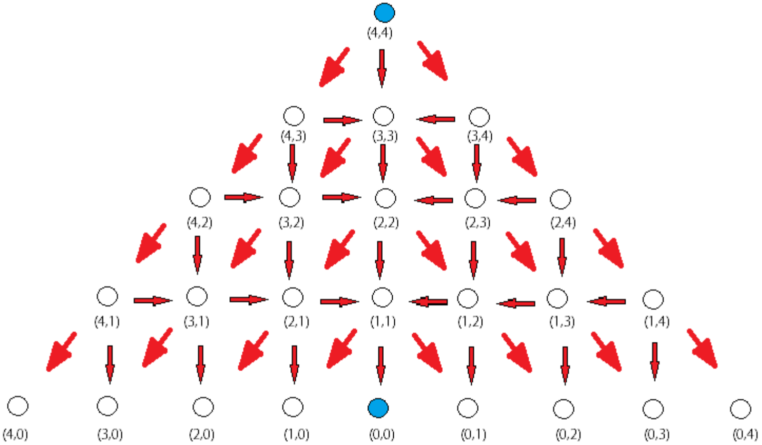


Figure: The M&M game when $k = 4$. Count the paths! Answer 1/3 of probability hit (1,1).

Interpreting Proof: Connections to the Fibonacci Numbers!

Fibonaccis: $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0, F_1 = 1$.

Starts 0, 1, 1, 2, 3, 5, 8, 13, 21, ...
http://www.youtube.com/watch?v=kkGeOWYOFoA.

Binet's Formula (can prove via 'generating functions'):

$$F_n \;=\; \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

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M&Ms: For $c, k \geq 1$: $x_{c,0} = x_{0,k} = 0$; $x_{0,0} = 1$, and if $c, k \geq 1$:

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

Reproduces the tree but a lot 'cleaner'.

Solving the Recurrence

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

- $x_{0,0} = 1.$

- $x_{1,0} = x_{0,1} = 0.$
- $x_{1,1} = \frac{1}{3}x_{0,0} + \frac{1}{3}x_{0,1} + \frac{1}{3}x_{1,0} = \frac{1}{3} \approx 33.3\%.$

- $x_{2,0} = x_{0,2} = 0.$
- $x_{2,1} = \frac{1}{3}x_{1,0} + \frac{1}{3}x_{1,1} + \frac{1}{3}x_{2,0} = \frac{1}{9} \;=\; x_{1,2}.$
- $x_{2,2} = \frac{1}{3}x_{1,1} + \frac{1}{3}x_{1,2} + \frac{1}{3}x_{2,1} = \frac{1}{9} + \frac{1}{27} + \frac{1}{27} \;=\; \frac{5}{27} \approx 18.5\%.$

Examining Probabilities of a Tie

When $k = 1$, Prob(tie) = 1/3.

When $k = 2$, Prob(tie) = 5/27.

When $k = 3$, Prob(tie) = 11/81.

When $k = 4$, Prob(tie) = 245/2187.

When $k = 5$, Prob(tie) = 1921/19683.

When $k = 6$, Prob(tie) = 575/6561.

When $k = 7$, Prob(tie) = 42635/531441.

When $k = 8$, Prob(tie) = 355975/4782969.

Examining Ties: Multiply by 3^{2k-1} to clear denominators.

When $k = 1$, get 1.

When $k = 2$, get 5.

When $k = 3$, get 33.

When $k = 4$, get 245.

When $k = 5$, get 1921.

When $k = 6$, get 15525.

When $k = 7$, get 127905.

When $k = 8$, get 1067925.

OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ...

OEIS: http://oeis.org/.

Our sequence: http://oeis.org/A084771.

The web exists! Use it to build conjectures, suggest proofs....

OEIS (continued)

A084771 Coefficients of 1/sqrt(1-10*x+9*x^2); also, a(n) is the central coefficient of (1+5*x+4*x^2)^n. 5

1, 5, 33, 245, 1921, 15525, 127905, 1067925, 9004545, 76499525, 653808673, 5614995765, 48416454529, 418895174885, 3634723102113, 31616937184725, 275621102802945, 2407331941640325, 21061836725455905, 184550106298084725 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS
Also number of paths from (0,0) to (n,0) using steps U=(1,1), H=(1,0) and D=(1,-1), the U steps come in four colors and the H steps come in five colors. - N-E. Fahssi, Mar 30 2008
Number of lattice paths from (0,0) to (n,n) using steps (1,0), (0,1), and three kinds of steps (1,1). [Joerg Arndt, Jul 01 2011]
Sums of squares of coefficients of (1+2*x)^n. [Joerg Arndt, Jul 06 2011]
The Hankel transform of this sequence gives A103488 . - Philippe DELEHAM, Dec 02 2007

REFERENCES
Paul Barry and Aoife Hennessy, Generalized Narayana Polynomials, Riordan Arrays, and Lattice Paths, Journal of Integer Sequences, Vol. 15, 2012, #12.4.8.- From N. J. A. Sloane, Oct 08 2012
Michael Z. Spivey and Laura L. Steil, The k-Binomial Transforms and the Hankel Transform, Journal of Integer Sequences, Vol. 9 (2006), Article 06.1.1.

LINKS
Table of n, a(n) for n=0..19.
Tony D. Noe, On the Divisibility of Generalized Central Trinomial Coefficients, Journal of Integer Sequences, Vol. 9 (2006), Article 06.2.7.

FORMULA
G.f.: 1/sqrt(1-10*x+9*x^2).
Binomial transform of A059304. G.f.: Sum_{k>=0} binomial(2*k, k)*(2*x)^k/(1-x)^(k+1). E.g.f.: exp(5*x)*BesselI(0, 4*x). - Vladeta Jovovic (vladeta(AT)eunet.rs), Aug 20 2003
a(n) = sum(k=0..n, sum(j=0..n-k, C(n,j)*C(n-j,k)*C(2*n-2*j,n-j))). - Paul Barry, May 19 2006
a(n) = sum(k=0..n, 4^k*(C(n,k))^2) [From heruneedollar (heruneedollar(AT)gmail.com), Mar 20 2010]
Asymptotic: a(n) ~ 3^(2*n+1)/(2*sqrt(2*Pi*n)). [Vaclav Kotesovec, Sep 11 2012]
Conjecture: n*a(n) +5*(-2*n+1)*a(n-1) +9*(n-1)*a(n-2)=0. - R. J. Mathar,

Takeaways

Lessons

- Always ask questions.
- Many ways to solve a problem.
- Experience is useful and a great guide.
- Need to look at the data the right way.
- Often don't know where the math will take you.
- Value of continuing education: more math is better.
- Connections: My favorite quote: `If all you have is a hammer, pretty soon every problem looks like a nail.`