

From M&Ms to Mathematics, or, How I learned to answer questions and help my kids love math.

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Prospective Days, Williams College, August 11, 2017



Some Issues for the Future

- World is rapidly changing – powerful computing cheaply and readily available.
- What skills are we teaching? What skills should we be teaching?
- One of hardest skills: how to think / attack a new problem, how to see connections, what data to gather.

Goals of the Talk: Opportunities Everywhere!

- Ask Questions! Often simple questions lead to good math.
- Gather data: observe, program and simulate.
- Use games to get to mathematics.
- Discuss implementation: **Please interrupt!**

Joint work with Cameron (age 10) and Kayla (age 8) Miller:
Erdős number 3

Problem Editor Pi Mu Epsilon Journal, and my math riddles
page: <http://mathriddles.williams.edu/>

Pre-requisite: Logarithms

The logarithm of x base b is the power we raise b to get x ; it is the inverse of exponentiation.

$$x = b^y \quad \text{if and only if} \quad \log_b(x) = y.$$

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Properties:

- $\log_{10} 1 = 0.$
- $10^{\log_{10} x} = x.$
- $\log_{10}(A \cdot B) = \log_{10} A + \log_{10} B.$

Pre-requisites: Combinatorics Review

- $n!$: number of ways to order n people, order matters.
- $\frac{n!}{k!(n-k)!} = nCk = \binom{n}{k}$: number of ways to choose k from n , order doesn't matter.
- Examples: $\binom{n}{1} = n$, $\binom{4}{2} = 6$, in general $\binom{n}{2} = \frac{n(n-1)}{2}$.

Pre-requisites: Pascal's Triangle

Pascal's Triangle: k^{th} entry in row n is $\binom{n}{k}$.

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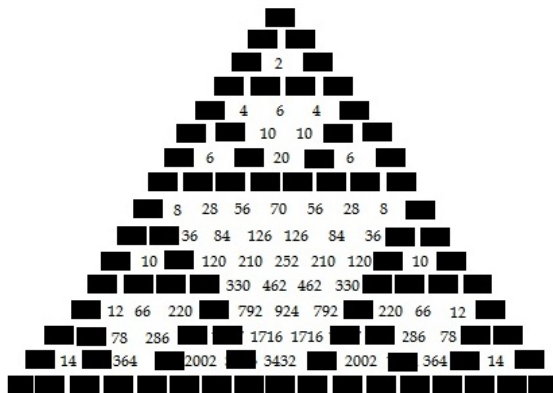
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Pascal's Triangle

Pre-requisites: Pascal's Triangle

Pascal's Triangle: k^{th} entry in row n is $\binom{n}{k}$.



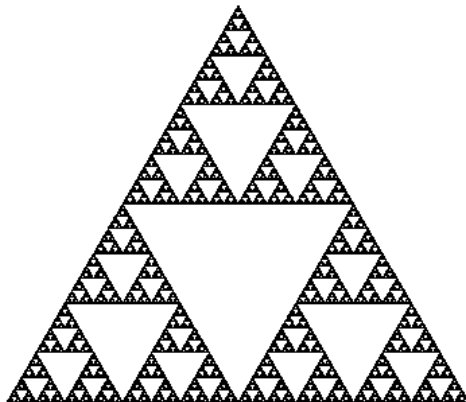
Blocking Out the Odd Entries
to create a Sierpinski Triangle

Pre-requisites: Pascal's Triangle

Pascal's Triangle: k^{th} entry in row n is $\binom{n}{k}$.

Sierpinski's triangle: Look at Pascal's triangle modulo 2:

https://www.youtube.com/watch?v=tt4_4YajqRM.



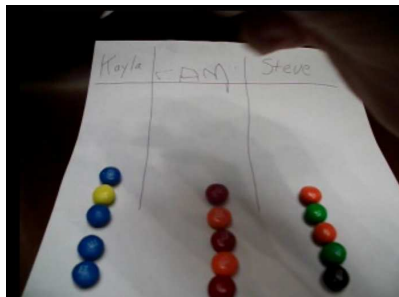
The M&M Game

Motivating Question

Cam (4 years): If you're born on the same day, do you die on the same day?

M&M Game Rules

Cam (4 years): If you're born on the same day, do you die on the same day?



- (1) Everyone starts off with k M&Ms (we did 5).
- (2) All toss fair coins, eat an M&M if and only if head.



Be active – ask questions!

What are natural questions to ask?

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What are natural questions to ask?

Question 1: How likely is a tie (as a function of k)?

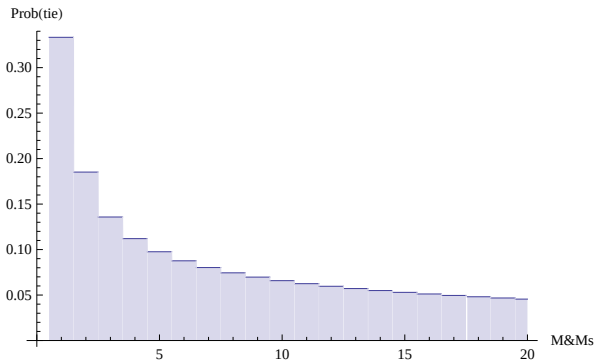
Question 2: How long until one dies?

Question 3: Generalize the game: More people? Biased coin?

Important to ask questions – curiosity is good and to be encouraged! Value to the journey and not knowing the answer.

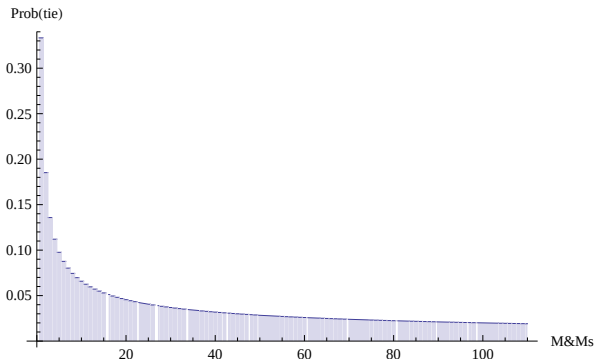
Let's gather some data! Let's play!

Probability of a tie in the M&M game (2 players)



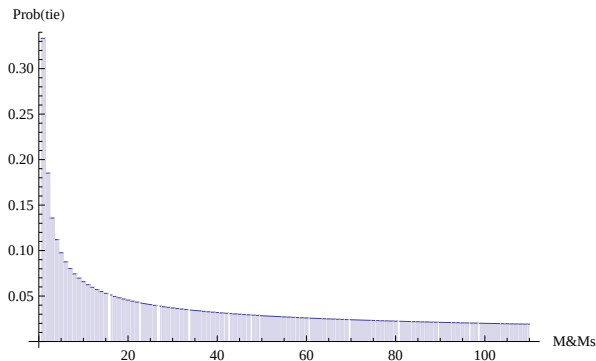
$\text{Prob}(\text{tie}) \approx 33\%$ (1 M&M), 19% (2 M&Ms), 14% (3 M&Ms), 10% (4 M&Ms).

Probability of a tie in the M&M game (2 players)



Gave at a 110th anniversary talk....

Probability of a tie in the M&M game (2 players)

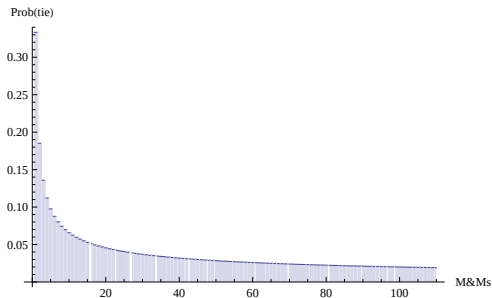


... asked them: what will the next 110 bring us?
Never too early to lay foundations for future classes.

Welcome to Statistics and Inference!

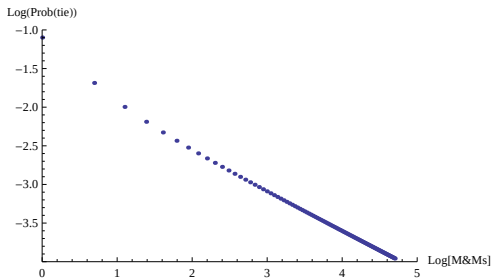
- ◇ **Goal:** Gather data, see pattern, extrapolate.
- ◇ **Methods:** Simulation, analysis of special cases.
- ◇ **Presentation:** It matters **how** we show data, and **which** data we show.

Viewing M&M Plots



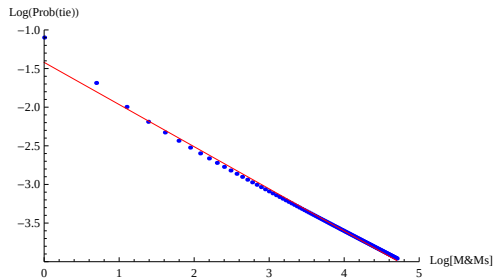
Hard to predict what comes next.

Viewing M&M Plots: Log-Log Plot



Not *just* sadistic teachers: logarithms useful!

Viewing M&M Plots: Log-Log Plot

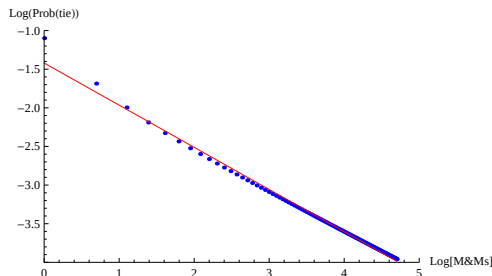


Best fit line:

$$\log(\text{Prob}(\text{tie})) = -1.42022 - 0.545568 \log(\# \text{M\&Ms}) \text{ or}$$

$$\text{Prob}(k) \approx 0.2412/k^{.5456}.$$

Viewing M&M Plots: Log-Log Plot



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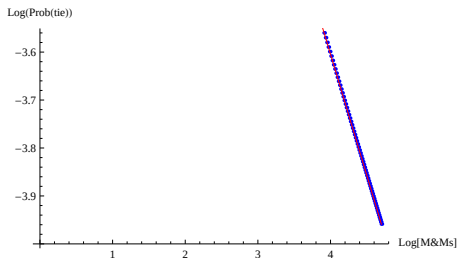
Predicts probability of a tie when $k = 220$ is 0.01274, but answer is 0.0137. **What gives?**

Statistical Inference: Too Much Data Is Bad!

Small values can mislead / distort. Let's go from $k = 50$ to 110.

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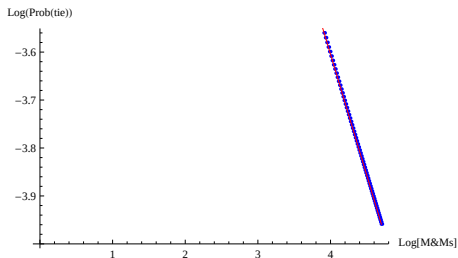
Best fit line:

$$\log(\text{Prob}(\text{tie})) = -1.58261 - 0.50553 \log(\# \text{M\&Ms}) \text{ or}$$

$$\text{Prob}(k) \approx 0.205437/k^{.50553} \text{ (had } 0.241662/k^{.5456}).$$

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Get 0.01344 for $k = 220$ (answer 0.01347); **much better!**

From Shooting Hoops to the Geometric Series Formula

Simpler Game: Hoops

Game of hoops: first basket wins, alternate shooting.



Simpler Game: Hoops: Mathematical Formulation

Bird and **Magic** (I'm old!) alternate shooting; first basket wins.

- **Bird** always gets basket with probability p .
- **Magic** always gets basket with probability q .

Let x be the probability **Bird** wins – what is x ?

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

Solving the Hoop Game

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- **Bird** wins on 1st shot: p .

Solving the Hoop Game

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Break into cases:

- **Bird** wins on 1st shot: p .
- **Bird** wins on 2nd shot: $(1 - p)(1 - q) \cdot p$.

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

- **Bird** wins on 1st shot: p .
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- **Bird** wins on 3rd shot: $(1 - p)(1 - q) \cdot (1 - p)(1 - q) \cdot p$.

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 $(1 - p)(1 - q) \cdot (1 - p)(1 - q) \cdots (1 - p)(1 - q) \cdot p$.

Let $r = (1 - p)(1 - q)$. Then

$$\begin{aligned} \text{X} &= \text{Prob}(\text{Bird wins}) \\ &= p + rp + r^2p + r^3p + \cdots \\ &= p \left(1 + r + r^2 + r^3 + \cdots \right), \end{aligned}$$

the geometric series.

Solving the Hoop Game: The Power of Perspective

Showed

$$x = \text{Prob}(\text{Bird wins}) = p(1 + r + r^2 + r^3 + \dots);$$

will solve without the geometric series formula.

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$$x = \text{Prob}(\text{Bird wins}) = p + (1 - p)(1 - q)x = p + rx.$$

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$$(1 - r)x = p \quad \text{or} \quad x = \frac{p}{1 - r}.$$

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Thus

$$(1 - r)x = p \quad \text{or} \quad x = \frac{p}{1 - r}.$$

As $x = p(1 + r + r^2 + r^3 + \dots)$, find

$$1 + r + r^2 + r^3 + \dots = \frac{1}{1 - r}.$$

Lessons from Hoop Problem

- ◇ Power of Perspective: Memoryless process.
- ◇ Can circumvent algebra with deeper understanding! (Hard)
- ◇ Depth of a problem not always what expect.
- ◇ Importance of knowing more than the minimum: [connections](#).
- ◇ Math is fun!

The M&M Game

Solving the M&M Game

Overpower with algebra: Assume k M&Ms, two people, fair coins:

$$\text{Prob}(\text{tie}) = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2},$$

where

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

is a binomial coefficient.

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“Simplifies” to $4^{-k} {}_2F_1(k, k, 1, 1/4)$, a special value of a hypergeometric function! (Look up / write report.)

A look at your future classes, but is there a better way?

Solving the M&M Game (cont)

Where did formula come from? Each turn one of four **equally likely** events happens:

- Both eat an M&M.
- Cam eats an M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is $1/4$ or 25%.

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- Neither eat.

Probability of each event is **1/4** or **25%**.

Each person has exactly $k - 1$ heads in first $n - 1$ tosses, then ends with a head.

$$\text{Prob}(\text{tie}) = \sum_{n=k}^{\infty} \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} \cdot \binom{n-1}{k-1} \left(\frac{1}{2}\right)^{n-1} \frac{1}{2}.$$



Solving the M&M Game (cont)

Use the lesson from the Hoops Game: Memoryless process!

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If neither eat, as if toss didn't happen. Now game is finite.

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Use the lesson from the Hoops Game: Memoryless process!

If neither eat, as if toss didn't happen. Now game is finite.

Much better perspective: each "turn" one of **three equally likely** events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.

Probability of each event is **1/3** or about **33%**

$$\sum_{n=0}^{k-1} \binom{2k-n-2}{n} \left(\frac{1}{3}\right)^n \binom{2k-2n-2}{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \binom{1}{1} \frac{1}{3}.$$



Solving the M&M Game (cont)

Interpretation: Let Cam have c M&Ms and Kayla have k ; write as (c, k) .

Then each of the following happens $1/3$ of the time after a 'turn':

- $(c, k) \longrightarrow (c - 1, k - 1)$.
- $(c, k) \longrightarrow (c - 1, k)$.
- $(c, k) \longrightarrow (c, k - 1)$.



Solving the M&M Game (cont): Assume $k = 4$

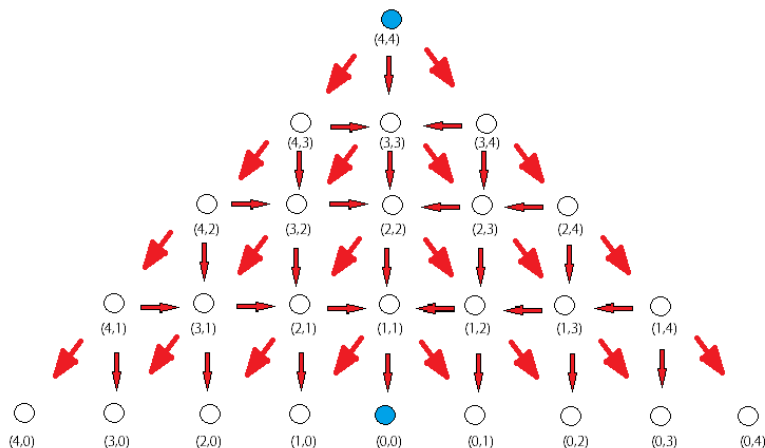


Figure: The M&M game when $k = 4$. Count the paths! Answer $1/3$ of probability hit $(1,1)$.

Solving the M&M Game (cont): Assume $k = 4$

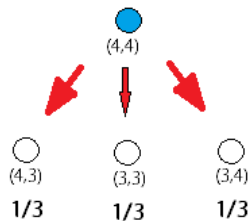


Figure: The M&M game when $k = 4$, going down one level.

Solving the M&M Game (cont): Assume $k = 4$

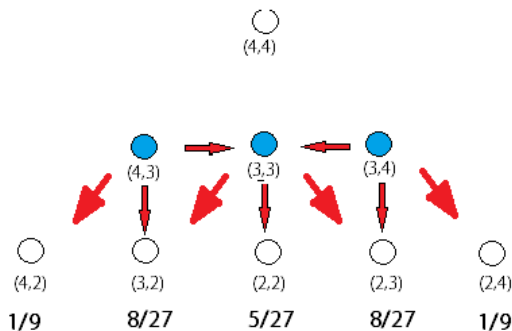


Figure: The M&M game when $k = 4$, removing probability from the second level.

Solving the M&M Game (cont): Assume $k = 4$

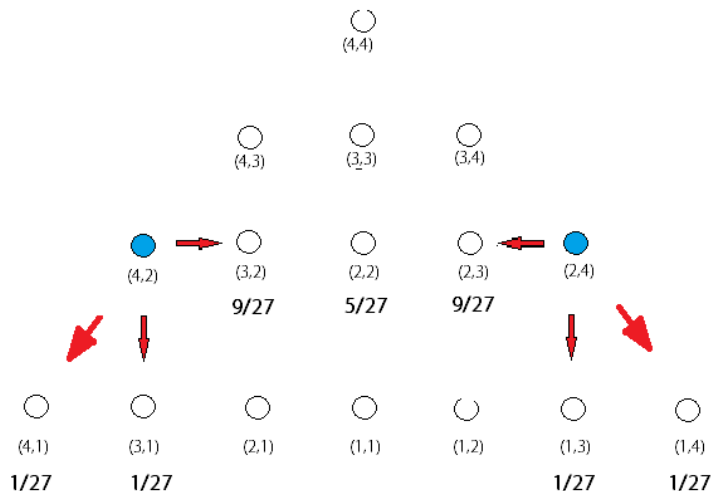


Figure: Removing probability from two outer on third level.

Solving the M&M Game (cont): Assume $k = 4$

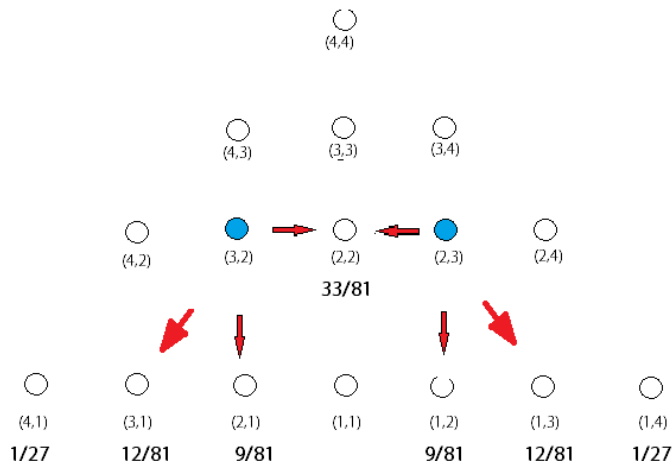


Figure: Removing probability from the $(3, 2)$ and $(2, 3)$ vertices.

Solving the M&M Game (cont): Assume $k = 4$

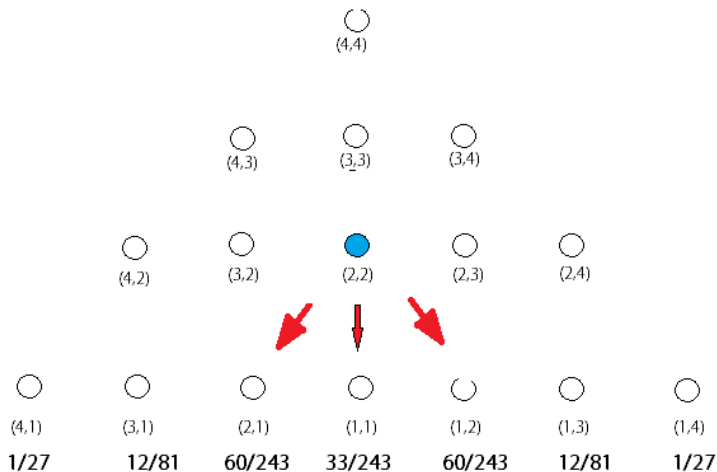


Figure: Removing probability from the $(2,2)$ vertex.

Solving the M&M Game (cont): Assume $k = 4$

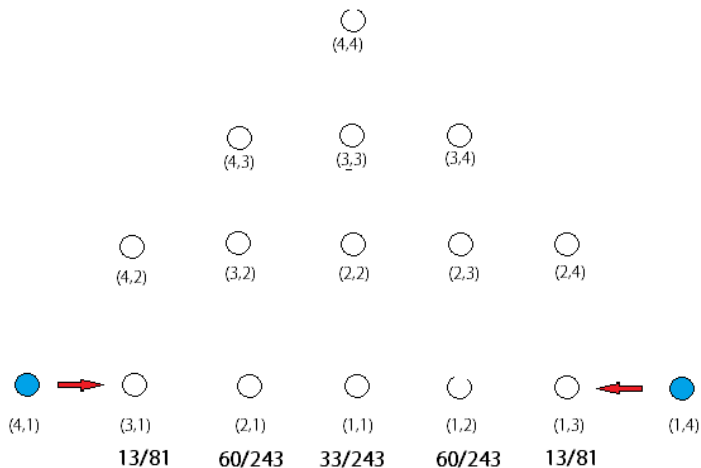


Figure: Removing probability from the $(4, 1)$ and $(1, 4)$ vertices.

Solving the M&M Game (cont): Assume $k = 4$

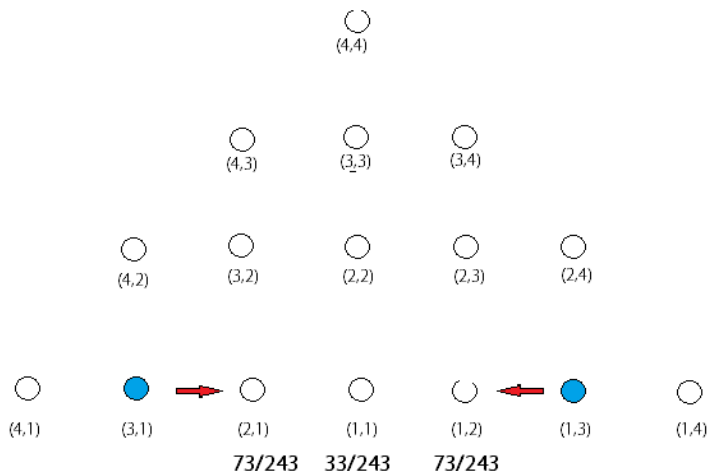


Figure: Removing probability from the $(3, 1)$ and $(1, 3)$ vertices.

Solving the M&M Game (cont): Assume $k = 4$

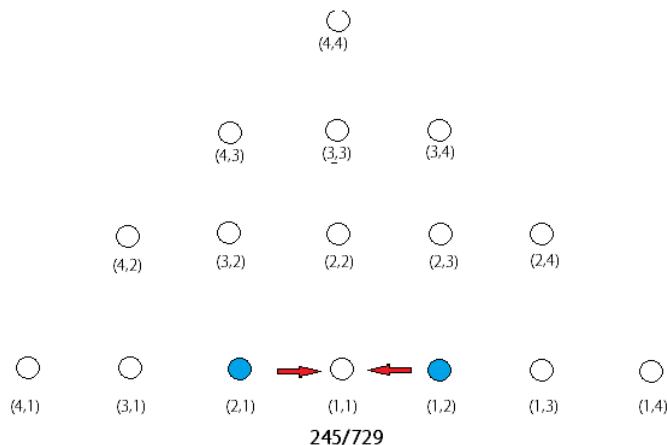


Figure: Removing probability from $(2, 1)$ and $(1, 2)$ vertices. Answer is $1/3$ of $(1, 1)$ vertex, or $245/2187$ (about 11%).

Interpreting Proof: Connections to the Fibonacci Numbers!

Fibonacci: $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0, F_1 = 1$.

Starts 0, 1, 1, 2, 3, 5, 8, 13, 21,

<http://www.youtube.com/watch?v=kkGe0WY0FoA>.

Binet's Formula (can prove via 'generating functions'):

$$F_n = \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2} \right)^n.$$

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M&Ms: For $c, k \geq 1$: $x_{c,0} = x_{0,k} = 0$; $x_{0,0} = 1$, and if $c, k \geq 1$:

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

Reproduces the tree but a lot 'cleaner'.

Interpreting Proof: Finding the Recurrence

What if we didn't see the 'simple' recurrence?

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

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The following recurrence is 'natural':

$$x_{c,k} = \frac{1}{4}x_{c,k} + \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1}.$$

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Obtain 'simple' recurrence by algebra: subtract $\frac{1}{4}x_{c,k}$:

$$\begin{aligned} \frac{3}{4}x_{c,k} &= \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1} \\ \text{therefore } x_{c,k} &= \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}. \end{aligned}$$

Solving the Recurrence

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

Solving the Recurrence

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

- $x_{0,0} = 1.$

Solving the Recurrence

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

- $x_{0,0} = 1.$
- $x_{1,0} = x_{0,1} = 0.$
- $x_{1,1} = \frac{1}{3}x_{0,0} + \frac{1}{3}x_{0,1} + \frac{1}{3}x_{1,0} = \frac{1}{3} \approx 33.3\%.$

Solving the Recurrence

$$x_{c,k} = \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

- $x_{0,0} = 1.$
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- $x_{2,0} = x_{0,2} = 0.$
- $x_{2,1} = \frac{1}{3}x_{1,0} + \frac{1}{3}x_{1,1} + \frac{1}{3}x_{2,0} = \frac{1}{9} = x_{1,2}.$
- $x_{2,2} = \frac{1}{3}x_{1,1} + \frac{1}{3}x_{1,2} + \frac{1}{3}x_{2,1} = \frac{1}{9} + \frac{1}{27} + \frac{1}{27} = \frac{5}{27} \approx 18.5\%.$

Try Simpler Cases!!!

Try and find an easier problem and build intuition.

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Walking from $(0,0)$ to (k,k) with allowable steps $(1,0)$, $(0,1)$ and $(1,1)$, hit (k,k) before hit top or right sides.

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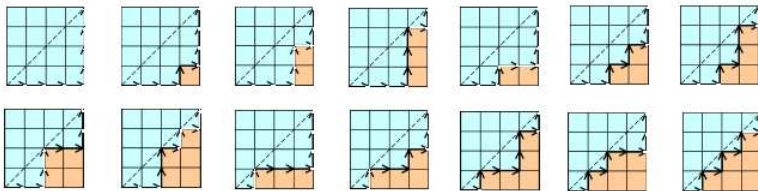
Generalization of the Catalan problem. There don't have $(1,1)$ and stay on or below the main diagonal.

Try Simpler Cases!!!

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Generalization of the Catalan problem. There don't have $(1,1)$ and stay on or below the main diagonal.



Interpretation: Catalan numbers are valid placings of (and).

Aside: Fun Riddle Related to Catalan Numbers

Young Saul, a budding mathematician and printer, is making himself a fake ID. He needs it to say he's 21. The problem is he's not using a computer, but rather he has some symbols he's bought from the store, and that's it. He has one 1, one 5, one 6, one 7, and an unlimited supply of $+$ $-$ $*$ $/$ (the operations addition, subtraction, multiplication and division). Using each number exactly once (but you can use any number of $+$, any number of $-$, ...) how, oh how, can he get 21 from 1,5, 6,7? Note: you can't do things like $15+6 = 21$. You have to use the four operations as 'binary' operations: $((1+5)*6) + 7$. Problem submitted by ohadbp@infolink.net.il, phrasing by yours truly.

Solution involves valid sentences: $((w + x) + y) + z, w + ((x + y) + z), \dots$

For more riddles see my riddles page:
<http://mathriddles.williams.edu/>.

Examining Probabilities of a Tie

When $k = 1$, $\text{Prob}(\text{tie}) = 1/3$.

When $k = 2$, $\text{Prob}(\text{tie}) = 5/27$.

When $k = 3$, $\text{Prob}(\text{tie}) = 11/81$.

When $k = 4$, $\text{Prob}(\text{tie}) = 245/2187$.

When $k = 5$, $\text{Prob}(\text{tie}) = 1921/19683$.

When $k = 6$, $\text{Prob}(\text{tie}) = 575/6561$.

When $k = 7$, $\text{Prob}(\text{tie}) = 42635/531441$.

When $k = 8$, $\text{Prob}(\text{tie}) = 355975/4782969$.

Examining Ties: Multiply by 3^{2k-1} to clear denominators.

When $k = 1$, get 1.

When $k = 2$, get 5.

When $k = 3$, get 33.

When $k = 4$, get 245.

When $k = 5$, get 1921.

When $k = 6$, get 15525.

When $k = 7$, get 127905.

When $k = 8$, get 1067925.

OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525,

OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525,

OEIS: <http://oeis.org/>.

OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525,

OEIS: <http://oeis.org/>.

Our sequence: <http://oeis.org/A084771>.

The web exists! Use it to build conjectures, suggest proofs....

85

A084771

Coefficients of $1/\sqrt{1-10*x+9*x^2}$; also, $a(n)$ is the central coefficient of $(1+5*x+4*x^2)^n$.

1, 5, 33, 245, 1921, 15525, 127905, 1067925, 9004545, 76499525, 653808673, 5614995765,
48416454529, 418895174885, 3634723102113, 31616937184725, 275621102802945,
2407331941640325, 21061836725455905, 184550106298084725 ([list](#); [graph](#); [refs](#); [listen](#); [history](#); [text](#); [internal format](#))

OFFSET 0,2

COMMENTS Also number of paths from (0,0) to (n,0) using steps U=(1,1), H=(1,0) and D=(1,-1), the U steps come in four colors and the H steps come in five colors. - [N.-E. Fahssi](#), Mar 30 2008
Number of lattice paths from (0,0) to (n,n) using steps (1,0), (0,1), and three kinds of steps (1,1). [[Joerg Arndt](#), Jul 01 2011]
Sums of squares of coefficients of $(1+2*x)^n$. [[Joerg Arndt](#), Jul 06 2011]
The Hankel transform of this sequence gives [A103488](#). - [Philippe DELEHAM](#), Dec 02 2007

REFERENCES [Paul Barry and Aoife Hennessy, Generalized Narayana Polynomials, Riordan Arrays, and Lattice Paths, Journal of Integer Sequences, Vol. 15, 2012, #12.4.8.-](#) From [N. J. A. Sloane](#), Oct 08 2012
[Michael Z. Spivey and Laura L. Steil, The k-Binomial Transforms and the Hankel Transform, Journal of Integer Sequences, Vol. 9 \(2006\), Article 06.1.1.](#)

LINKS [Table of n, a\(n\) for n=0..19.](#)
[Tony D. Noe, On the Divisibility of Generalized Central Trinomial Coefficients](#), Journal of Integer Sequences, Vol. 9 (2006), Article 06.2.7.

FORMULA G.f.: $1/\sqrt{1-10*x+9*x^2}$.
Binomial transform of [A059304](#). G.f.: $\sum_{k=0}^{\infty} \text{binomial}(2*k, k) * (2*x)^k * (1-x)^{(k+1)}$. E.g.f.: $\exp(5*x) * \text{BesselI}(0, 4*x)$. - [Vladeta Jovovic](#) ([vladeta\(AT\)eunet.rs](#)), Aug 20 2003
 $a(n) = \sum(k=0..n, \sum(j=0..n-k, C(n,j) * C(n-j, k) * C(2*n-2*j, n-j)))$. - [Paul Barry](#), May 19 2006
 $a(n) = \text{sum}(k=0..n, 4^k * (C(n, k))^2)$ [From [herunedollar](#) ([herunedollar\(AT\)gmail.com](#)), Mar 20 2010]
Asymptotic: $a(n) \sim 3^{n+1} / (2 * \sqrt{2 * \pi * n})$. [[Vaclav Kotesovec](#), Sep 11 2012]
Conjecture: $n*a(n) + 5*(-2*n+1)*a(n-1) + 9*(n-1)*a(n-2) = 0$. - [R. J. Mathar](#).

Takeaways

Lessons

- ◇ Always ask questions.
- ◇ Many ways to solve a problem.
- ◇ Experience is useful and a great guide.
- ◇ Need to look at the data the right way.
- ◇ Often don't know where the math will take you.
- ◇ Value of continuing education: more math is better.
- ◇ Connections: My favorite quote: If all you have is a hammer, pretty soon every problem looks like a nail.

Generating Functions

Generating Function (Example: Binet's Formula)

Binet's Formula

$$F_1 = F_2 = 1; F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{-1+\sqrt{5}}{2} \right)^n \right].$$

- **Recurrence relation:** $F_{n+1} = F_n + F_{n-1}$ (1)
- **Generating function:** $g(x) = \sum_{n \geq 0} F_n x^n$.

$$(1) \Rightarrow \sum_{n \geq 2} F_{n+1} x^{n+1} = \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 2} F_{n-1} x^{n+1}$$

$$\Rightarrow \sum_{n \geq 3} F_n x^n = \sum_{n \geq 2} F_n x^{n+1} + \sum_{n \geq 1} F_n x^{n+2}$$

$$\Rightarrow \sum_{n \geq 3} F_n x^n = x \sum_{n \geq 2} F_n x^n + x^2 \sum_{n \geq 1} F_n x^n$$

$$\Rightarrow g(x) - F_1 x - F_2 x^2 = x(g(x) - F_1 x) + x^2 g(x)$$

$$\Rightarrow g(x) = x/(1 - x - x^2).$$

Partial Fraction Expansion (Example: Binet's Formula)

- **Generating function:** $g(x) = \sum_{n \geq 0} F_n x^n = \frac{x}{1-x-x^2}$.
- **Partial fraction expansion:**

$$\Rightarrow g(x) = \frac{x}{1-x-x^2} = \frac{1}{\sqrt{5}} \left(\frac{\frac{1+\sqrt{5}}{2}x}{1 - \frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1 - \frac{-1+\sqrt{5}}{2}x} \right).$$

Coefficient of x^n (power series expansion):

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{-1+\sqrt{5}}{2} \right)^n \right] \text{ - Binet's Formula!}$$

(using geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \dots$).