From M&Ms to Mathematics, or, How I learned to answer questions and help my kids love math.

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Some Issues for the Future

- World rapidly changing – powerful computing cheap & available.
- What skills are we teaching? What should we?
- One of hardest skills: how to think / attack a new problem, how to see connections, what data to gather.

Goals of the Talk: Opportunities Everywhere!

- Ask Questions! Often simple questions lead to good math.
- Gather data: observe, program and simulate.
- Use games to get to mathematics.
- Discuss implementation: `Please interrupt!`

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Joint work with Cameron (was 4) and Kayla (was 2) Miller:
The M&M Game: From Morsels to Modern Mathematics (Ivan Badinski, Nathan McCue, Cameron Miller, Kayla Miller, Michael Stone), Math. Magazine **90** (2017), no. 3, 197–207:
https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/MMProblem10.pdf.

Pre-requisite: Logarithms

The logarithm of x base b is the power we raise b to get x; it is the inverse of exponentiation.

$$x = b^y \quad \text{if and only if} \quad \log_b(x) = y.$$

$$\log_{10}(10^4) = 4, \quad \log_{10}(1/100) = -2.$$

Properties:

- $\log_{10} 1 = 0.$
- $10^{\log_{10} x} = x.$
- $\log_{10}(A \cdot B) = \log_{10} A + \log_{10} B.$

Pre-requisites: Combinatorics Review

- $n!$: number of ways to order n people, order matters.
- $\frac{n!}{k!(n-k)!} = n\mathrm{C}k = \binom{n}{k}$: number of ways to choose k from n, order doesn't matter.
- Examples: $\binom{n}{1} = n$, $\binom{4}{2} = 6$, in general $\binom{n}{2} = \frac{n(n-1)}{2}$.

Pre-requisites: Pascal's Triangle: (4min26sec): https://www.youtube.com/watch?v=_vkGakVt1RA&t=216s

Pascal's Triangle: k^{th} entry in row n is $\binom{n}{k}$.

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Pascal's Triangle: k^{th} entry in row n is $\binom{n}{k}$.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1
1 11 55 165 330 462 462 330 165 55 11 1
1 12 66 220 495 792 924 792 495 220 66 12 1
1 13 78 286 715 1287 1716 1716 1287 715 286 78 13 1
1 14 91 364 1001 2002 3003 3432 3003 2002 1001 364 91 14 1
1 15 105 455 1365 3003 5005 6435 6435 5005 3003 1365 455 105 15 1

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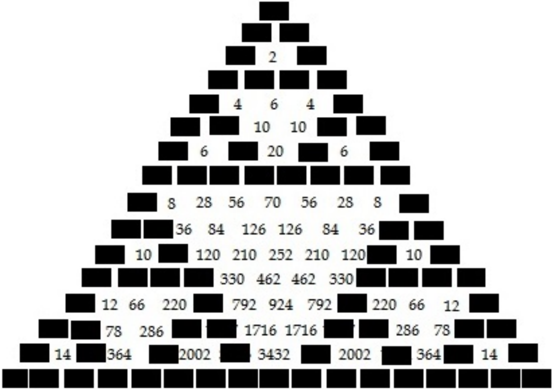
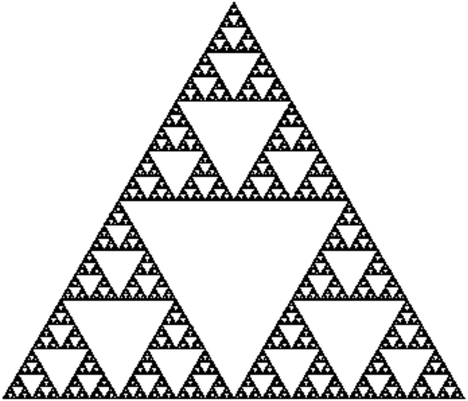


Blocking Out the Odd Entries
to create a Sierpinski Triangle

Pre-requisites: Pascal's Triangle: (4min26sec): https://www.youtube.com/watch?v=_vkGakVt1RA&t=216s

Sierpinski's triangle: Look at Pascal's triangle modulo 2: https://www.youtube.com/watch?v=tt4_4YajqRM.



Sierpinki's Gasket
http://www.math.sunysb.edu/~scott/Book331/Sierpinski_gasket.html

The M&M Game

Motivating Question

Cam (4 years): `If you're born on the same day, do you die on the same day?`

M&M Game Rules

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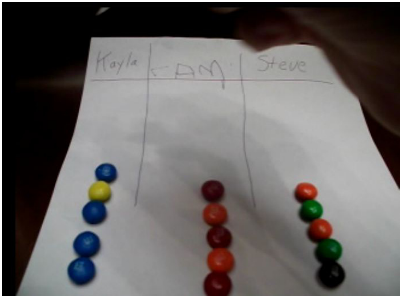

(1) Everyone starts off with *k* M&Ms (we did 5).
(2) All toss fair coins, eat an M&M if and only if head.

Be active – ask questions!

What are natural questions to ask?

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What are natural questions to ask?

Question 1: How likely is a tie (as a function of k)?

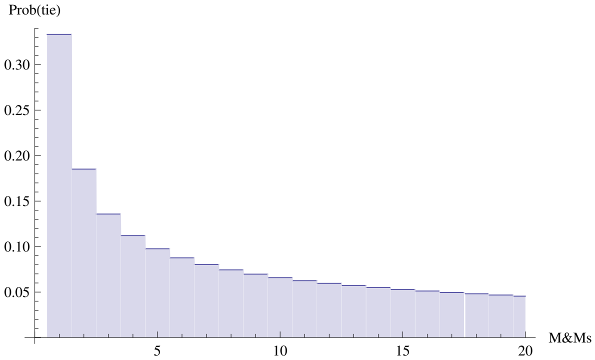
Question 2: How long until one dies?

Question 3: Generalize the game: More people? Biased coin?

Important to ask questions – curiousity is good and to be encouraged! Value to the journey and not knowing the answer.

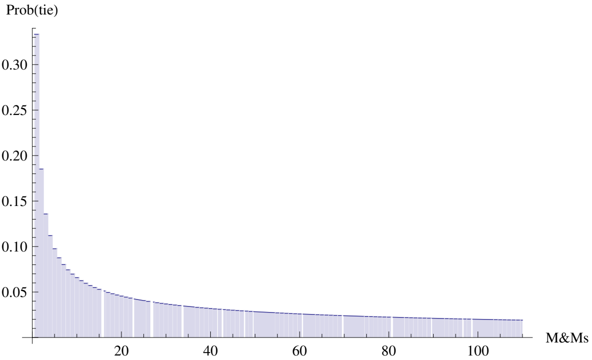
Let’s gather some data! Let’s play!

Probability of a tie in the M&M game (2 players)



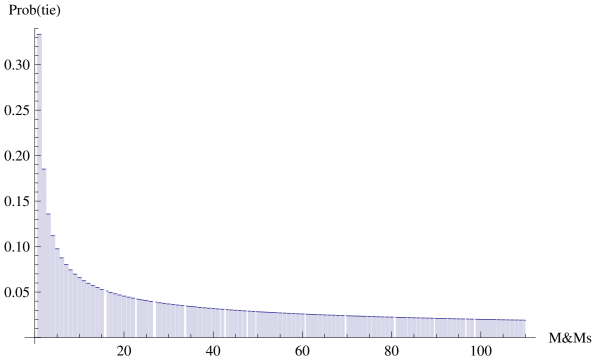
Prob(tie) $\approx$ 33% (1 M&M), 19% (2 M&Ms), 14% (3 M&Ms), 10% (4 M&Ms).

Probability of a tie in the M&M game (2 players)



Gave at a 110th anniversary talk....

Probability of a tie in the M&M game (2 players)

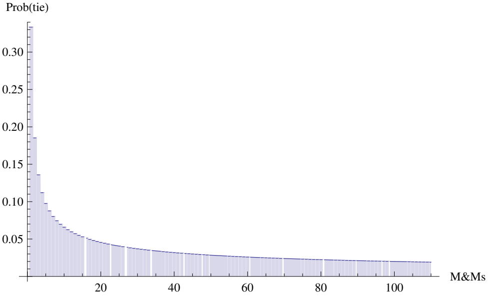


... asked them: what will the next 110 bring us?
Never too early to lay foundations for future classes.

Welcome to Statistics and Inference!

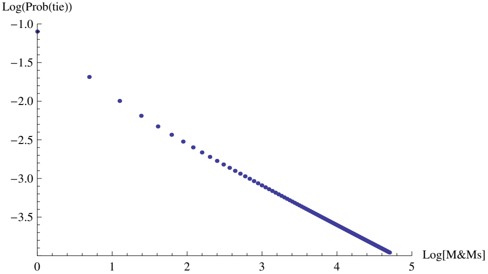
- Goal: Gather data, see pattern, extrapolate.
- Methods: Simulation, analysis of special cases.
- Presentation: It matters how we show data, and which data we show.

Viewing M&M Plots



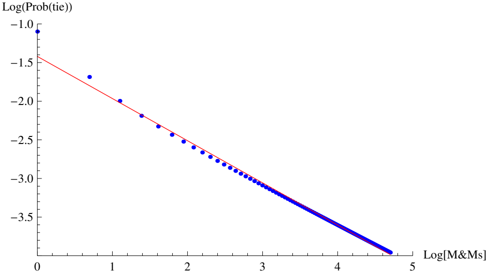
Hard to predict what comes next.

Viewing M&M Plots: Log-Log Plot



Not *just* sadistic teachers: logarithms useful!

Viewing M&M Plots: Log-Log Plot



Best fit line:

$\log\left(\text{Prob}(\text{tie})\right) \;=\; -1.42022 - 0.545568 \log\left(\#\text{M\&Ms}\right)$ or $\text{Prob}(k) \approx 0.2412/k^{.5456}$.

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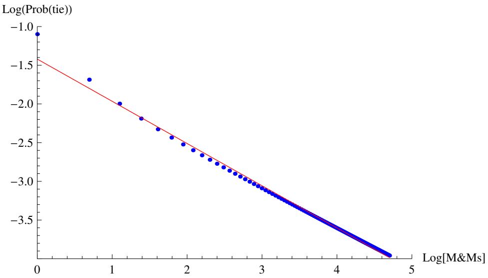

Best fit line:

$\log(\mathrm{Prob(tie)}) = -1.42022 - 0.545568 \log(\#\mathrm{M\&Ms})$ or $\mathrm{Prob}(k) \approx 0.2412/k^{.5456}$.

Predicts probability of a tie when $k = 220$ is 0.01274, but answer is 0.01347. **What gives?**

Statistical Inference: Too Much Data Is Bad!

Small values can mislead / distort. Let's go from $k = 50$ to 110.

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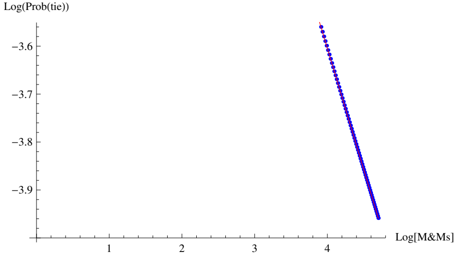

Best fit line:

$\log(\text{Prob}(\text{tie})) \;=\; -1.58261 - 0.50553 \log(\#\text{M\&Ms})$ or $\text{Prob}(k) \approx 0.205437/k^{.50553}$ (had $0.241662/k^{.5456}$).

Statistical Inference: Too Much Data Is Bad!

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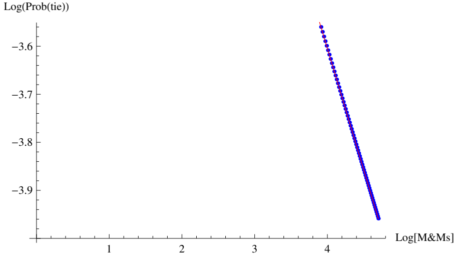

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$\text{Prob}(k) \approx 0.205437/k^{.50553}$ (had $0.241662/k^{.5456}$).

Get 0.01344 for $k = 220$ (answer 0.01347); **much better!**

From Shooting Hoops
to the Geometric Series Formula

Simpler Game: Hoops

Game of hoops: first basket wins, alternate shooting.

Simpler Game: Hoops: Mathematical Formulation

Bird and **Magic** (I'm old!) alternate shooting; first basket wins.

- **Bird** always gets basket with probability p.
- **Magic** always gets basket with probability q.

Let x be the probability **Bird** wins – what is x?

Solving the Hoop Game

Classic solution involves the geometric series.

Break into cases:

Solving the Hoop Game

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- **Bird** wins on 1^{st} shot: p.

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- **Bird** wins on 3rd shot: $(1-p)(1-q)\cdot(1-p)(1-q)\cdot p$.

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- **Bird** wins on 3rd shot: $(1-p)(1-q) \cdot (1-p)(1-q) \cdot p$.
- **Bird** wins on nth shot:
  $(1-p)(1-q) \cdot (1-p)(1-q) \cdots (1-p)(1-q) \cdot p$.

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- **Bird** wins on nth shot:
  $(1-p)(1-q)\cdot(1-p)(1-q)\cdots(1-p)(1-q)\cdot p$.

Let $r=(1-p)(1-q)$. Then

$$\begin{aligned} x &= \text{Prob}(\textbf{Bird}\text{ wins}) \\ &= p+rp+r^2p+r^3p+\cdots \\ &= p\left(1+r+r^2+r^3+\cdots\right), \end{aligned}$$

the geometric series.

Solving the Hoop Game: The Power of Perspective

Showed, for $r = (1-p)(1-q)$,

$$x \ = \ \text{Prob}(\textbf{Bird} \text{ wins}) \ = \ p(1 + r + r^2 + r^3 + \cdots);$$

will solve without the geometric series formula.

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$$x = \mathrm{Prob}(\textbf{Bird}\text{ wins}) = p + (1-p)(1-q)x$$

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Thus

$$(1-r)x \;=\; p \quad \text{or} \quad x \;=\; \frac{p}{1-r}.$$

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Thus

$$(1-r)x \;=\; p \quad \text{or} \quad x \;=\; \frac{p}{1-r}.$$

As $x = p(1+r+r^2+r^3+\cdots)$, find

$$1+r+r^2+r^3+\cdots \;=\; \frac{1}{1-r}.$$

Lessons from Hoop Problem

- Power of Perspective: Memoryless process.
- Can circumvent algebra with deeper understanding! (Hard)
- Depth of a problem not always what expect.
- Importance of knowing more than the minimum: connections.
- Math is fun!

Danger

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If $p = q = 0$ then $x = \frac{p}{1-r} = \frac{0}{0}$, how to interpret?

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If $p = q = 0$ then $x = \frac{p}{1-r} = \frac{0}{0}$, how to interpret?

Extend to $-1 < r < 1$ by grouping: let $r = -\rho$ with $\rho > 0$:

$$\begin{aligned} & 1 + \rho + \rho^2 + \cdots \\ &= \left(1 + \rho^2 + \rho^4 + \cdots\right) - \rho\left(1 + \rho^2 + \rho^4 + \cdots\right) \\ &= \frac{1}{1-\rho^2}(1-\rho) \;=\; \frac{1-\rho}{(1-\rho)(1+\rho)} \;=\; \frac{1}{1-r}. \end{aligned}$$

The M&M Game

Solving the M&M Game

Overpower with algebra: Assume k M&Ms, two people, fair coins:

$$\text{Prob(tie)} \ = \ \sum_{n=k}^{\infty} \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-1}\frac{1}{2} \cdot \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-1}\frac{1}{2},$$

where as always binomial coefficient

$$\binom{n}{r} \ = \ \frac{n!}{r!(n-r)!}.$$

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$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

"Simplifies" to $4^{-k}\ {}_2F_1(k, k, 1, 1/4)$, a special value of a hypergeometric function!
If $k = 100,000$ about .063% (predict .061%).

A look at your future classes, but is there a better way?

Solving the M&M Game (cont)

Where did formula come from? Each turn one of four equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.
- Neither eat.

Probability of each event is 1/4 or 25%.

Solving the M&M Game (cont)

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Each person has exactly $k-1$ heads in first $n-1$ tosses, then ends with a head.

$$\text{Prob(tie)} \ = \ \sum_{n=k}^{\infty} \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-1}\frac{1}{2} \cdot \binom{n-1}{k-1}\left(\frac{1}{2}\right)^{n-1}\frac{1}{2}.$$

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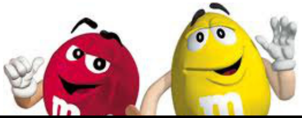
If neither eat, as if toss didn't happen. Now game is finite.

Much better perspective: each "turn" one of three equally likely events happens:

- Both eat an M&M.
- Cam eats and M&M but Kayla does not.
- Kayla eats an M&M but Cam does not.

Probability of each event is 1/3 or about 33%

$$\sum_{n=0}^{k-1} \binom{2k-n-2}{n} \left(\frac{1}{3}\right)^{n} \binom{2k-2n-2}{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \left(\frac{1}{3}\right)^{k-n-1} \binom{1}{1} \frac{1}{3}.$$



Solving the M&M Game (cont)

Interpretation: Let Cam have c M&Ms and Kayla have k; write as (c, k).

Then each of the following happens 1/3 of the time after a 'turn':

- $(c, k) \longrightarrow (c - 1, k - 1)$.
- $(c, k) \longrightarrow (c - 1, k)$.
- $(c, k) \longrightarrow (c, k - 1)$.

Solving the M&M Game (cont): Assume $k = 4$

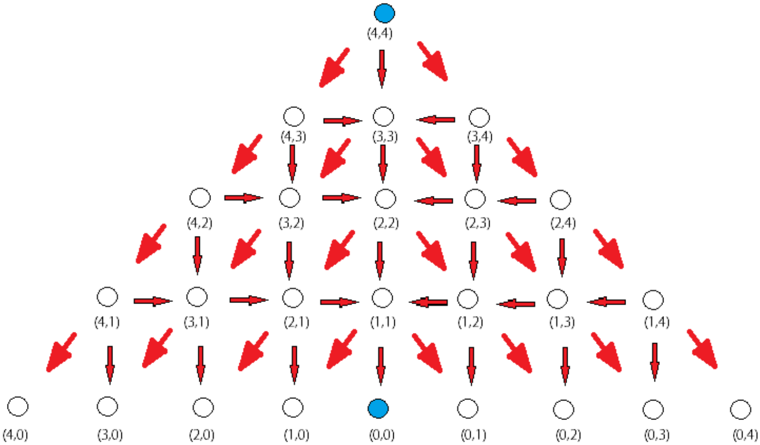


Figure: The M&M game when $k = 4$. Count the paths! Answer 1/3 of probability hit (1,1).

Solving the M&M Game (cont): Assume $k = 4$

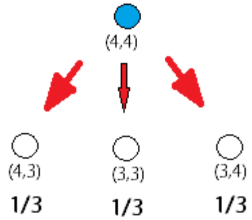


Figure: The M&M game when $k = 4$, going down one level.

Solving the M&M Game (cont): Assume $k = 4$

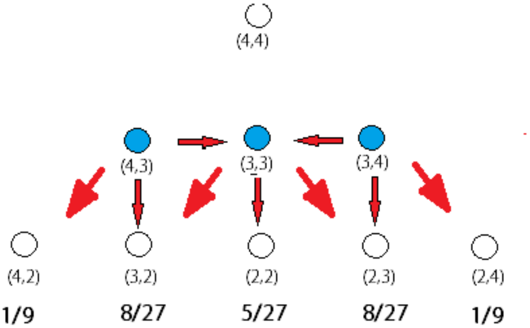


Figure: The M&M game when $k = 4$, removing probability from the second level.

Solving the M&M Game (cont): Assume $k = 4$

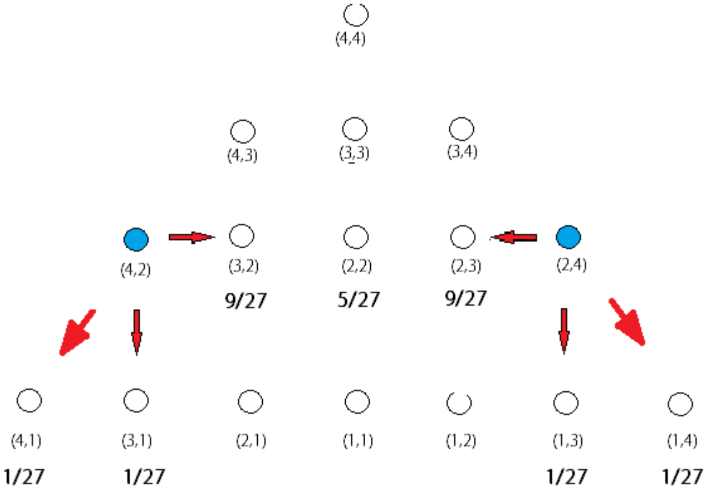


Figure: Removing probability from two outer on third level.

Solving the M&M Game (cont): Assume $k = 4$

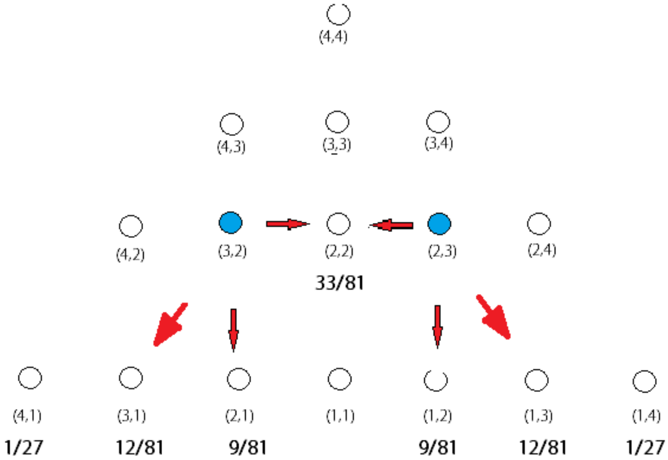


Figure: Removing probability from the (3,2) and (2,3) vertices.

Solving the M&M Game (cont): Assume $k = 4$

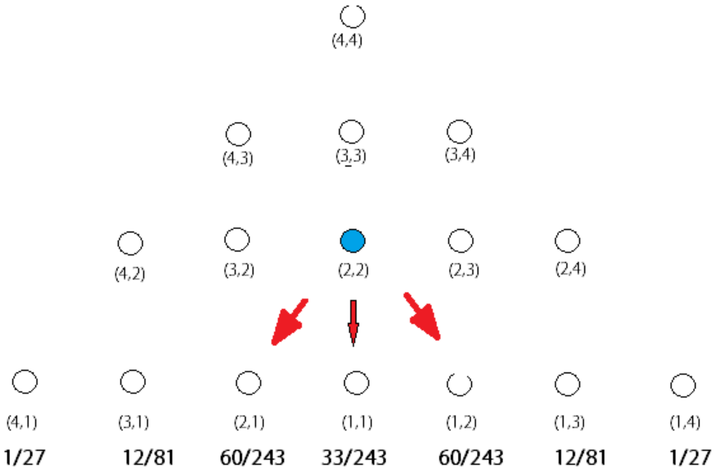


Figure: Removing probability from the (2,2) vertex.

Solving the M&M Game (cont): Assume $k = 4$

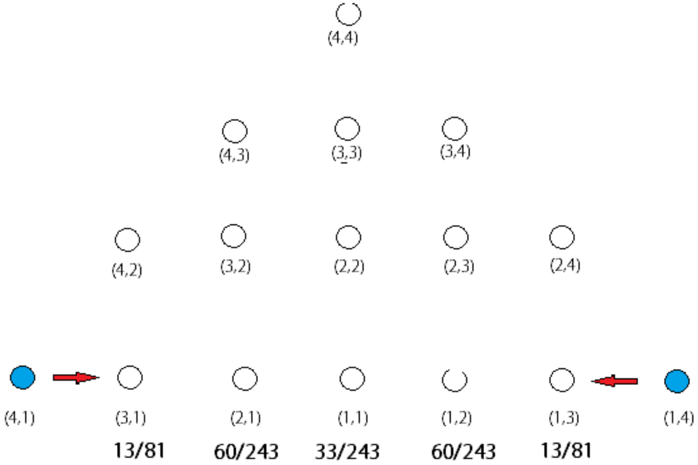


Figure: Removing probability from the (4,1) and (1,4) vertices.

Solving the M&M Game (cont): Assume $k = 4$

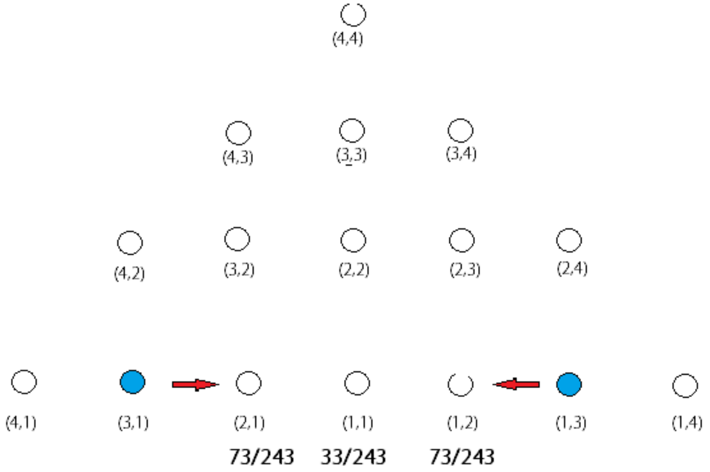


Figure: Removing probability from the (3,1) and (1,3) vertices.

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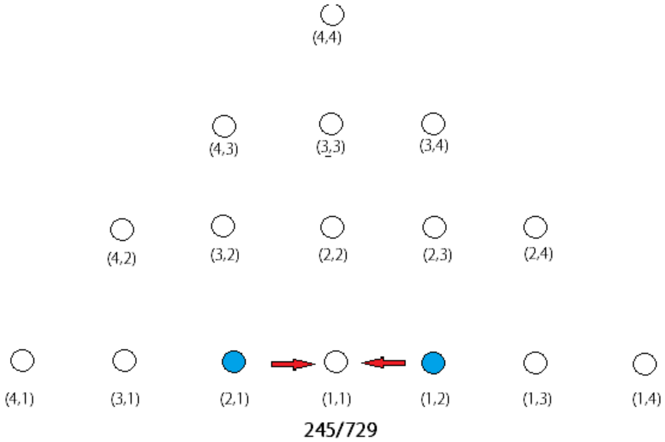

Figure: Removing probability from (2,1) and (1,2) vertices. Answer is 1/3 of (1,1) vertex, or 245/2187 (about 11%).

Interpreting Proof: Connections to the Fibonacci Numbers!

Fibonaccis: $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0, F_1 = 1$.

Starts 0, 1, 1, 2, 3, 5, 8, 13, 21, ...
http://www.youtube.com/watch?v=kkGeOWYOFoA.

Binet's Formula (can prove via 'generating functions'):

$$F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Interpreting Proof: Connections to the Fibonacci Numbers!

Fibonaccis: $F_{n+2} = F_{n+1} + F_n$ with $F_0 = 0, F_1 = 1$.

Starts 0, 1, 1, 2, 3, 5, 8, 13, 21, ...
http://www.youtube.com/watch?v=kkGeOWYOFoA.

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M&Ms: For $c, k \geq 1$: $x_{c,0} = x_{0,k} = 0$; $x_{0,0} = 1$, and if $c, k \geq 1$:

$$x_{c,k} \ = \ \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

Reproduces the tree but a lot 'cleaner'.

Interpreting Proof: Finding the Recurrence

What if we didn't see the 'simple' recurrence?

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

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The following recurrence is 'natural':

$$x_{c,k} \;=\; \frac{1}{4}x_{c,k} + \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1}.$$

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Obtain 'simple' recurrence by algebra: subtract $\frac{1}{4}x_{c,k}$:

$$\begin{aligned} \frac{3}{4}x_{c,k} \quad &= \quad \frac{1}{4}x_{c-1,k-1} + \frac{1}{4}x_{c-1,k} + \frac{1}{4}x_{c,k-1} \\ \text{therefore } x_{c,k} \quad &= \quad \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}. \end{aligned}$$

Solving the Recurrence

$$x_{c,k} \;=\; \frac{1}{3}x_{c-1,k-1} + \frac{1}{3}x_{c-1,k} + \frac{1}{3}x_{c,k-1}.$$

Solving the Recurrence

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- $x_{0,0} = 1$.

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- $x_{0,0} = 1$.
- $x_{1,0} = x_{0,1} = 0$.
- $x_{1,1} = \frac{1}{3}x_{0,0} + \frac{1}{3}x_{0,1} + \frac{1}{3}x_{1,0} = \frac{1}{3} \approx 33.3\%$.

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- $x_{2,0} = x_{0,2} = 0$.
- $x_{2,1} = \frac{1}{3}x_{1,0} + \frac{1}{3}x_{1,1} + \frac{1}{3}x_{2,0} = \frac{1}{9} \;=\; x_{1,2}$.
- $x_{2,2} = \frac{1}{3}x_{1,1} + \frac{1}{3}x_{1,2} + \frac{1}{3}x_{2,1} = \frac{1}{9} + \frac{1}{27} + \frac{1}{27} \;=\; \frac{5}{27} \approx 18.5\%$.

Try Simpler Cases!!!

Try and find an easier problem and build intuition.

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Walking from (0,0) to (k, k) with allowable steps (1,0), (0,1) and (1,1), hit (k, k) before hit top or right sides.

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Generalization of the Catalan problem. There don't have (1,1) and stay on or below the main diagonal.

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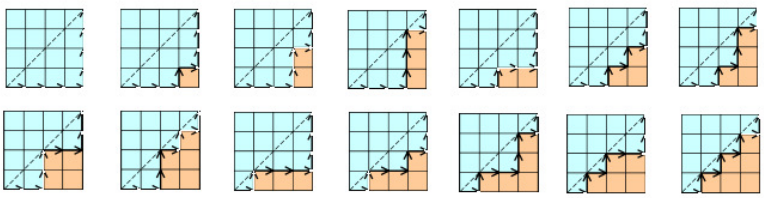

Interpretation: Catalan numbers are valid placings of (and).

Aside: Fun Riddle Related to Catalan Numbers

Young Saul, a budding mathematician and printer, is making himself a fake ID. He needs it to say he's 21. The problem is he's not using a computer, but rather he has some symbols he's bought from the store, and that's it. He has one 1, one 5, one 6, one 7, and an unlimited supply of + - $*$ / (the operations addition, subtraction, multiplication and division). Using each number exactly once (but you can use any number of +, any number of -, ...) how, oh how, can he get 21 from 1,5, 6,7? Note: you can't do things like 15+6 = 21. You have to use the four operations as 'binary' operations: ((1+5)$*$6) + 7. Problem submitted by ohadbp@infolink.net.il, phrasing by yours truly.

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Solution involves valid sentences: $((w + x) + y) + z$, $w + ((x + y) + z)$, $\dots$.

For more riddles see my riddles page: http://mathriddles.williams.edu/.

Examining Probabilities of a Tie

When $k = 1$, Prob(tie) = 1/3.

When $k = 2$, Prob(tie) = 5/27.

When $k = 3$, Prob(tie) = 11/81.

When $k = 4$, Prob(tie) = 245/2187.

When $k = 5$, Prob(tie) = 1921/19683.

When $k = 6$, Prob(tie) = 575/6561.

When $k = 7$, Prob(tie) = 42635/531441.

When $k = 8$, Prob(tie) = 355975/4782969.

Examining Ties: Multiply by 3^{2k-1} to clear denominators.

When $k = 1$, get 1.

When $k = 2$, get 5.

When $k = 3$, get 33.

When $k = 4$, get 245.

When $k = 5$, get 1921.

When $k = 6$, get 15525.

When $k = 7$, get 127905.

When $k = 8$, get 1067925.

OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ...

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OEIS: http://oeis.org/.

OEIS

Get sequence of integers: 1, 5, 33, 245, 1921, 15525, ...

OEIS: http://oeis.org/.

Our sequence: http://oeis.org/A084771.

The web exists! Use it to build conjectures, suggest proofs....

OEIS (continued)

A084771 Coefficients of 1/sqrt(1-10*x+9*x^2); also, a(n) is the central coefficient of (1+5*x+4*x^2)^n. 5

1, 5, 33, 245, 1921, 15525, 127905, 1067925, 9004545, 76499525, 653808673, 5614995765, 48416454529, 418895174885, 3634723102113, 31616937184725, 275621102802945, 2407331941640325, 21061836725455905, 184550106298084725 (list; graph; refs; listen; history; text; internal format)

OFFSET 0,2

COMMENTS Also number of paths from (0,0) to (n,0) using steps U=(1,1), H=(1,0) and D=(1,-1), the U steps come in four colors and the H steps come in five colors. - N-E. Fahssi, Mar 30 2008

Number of lattice paths from (0,0) to (n,n) using steps (1,0), (0,1), and three kinds of steps (1,1). [Joerg Arndt, Jul 01 2011]

Sums of squares of coefficients of (1+2*x)^n. [Joerg Arndt, Jul 06 2011]

The Hankel transform of this sequence gives A103488 . - Philippe DELEHAM, Dec 02 2007

REFERENCES Paul Barry and Aoife Hennessy, Generalized Narayana Polynomials, Riordan Arrays, and Lattice Paths, Journal of Integer Sequences, Vol. 15, 2012, #12.4.8.- From N. J. A. Sloane, Oct 08 2012

Michael Z. Spivey and Laura L. Steil, The k-Binomial Transforms and the Hankel Transform, Journal of Integer Sequences, Vol. 9 (2006), Article 06.1.1.

LINKS Table of n, a(n) for n=0..19.

Tony D. Noe, On the Divisibility of Generalized Central Trinomial Coefficients, Journal of Integer Sequences, Vol. 9 (2006), Article 06.2.7.

FORMULA G.f.: 1/sqrt(1-10*x+9*x^2).

Binomial transform of A059304. G.f.: Sum_{k>=0} binomial(2*k, k)*(2*x)^k/(1-x)^(k+1). E.g.f.: exp(5*x)*BesselI(0, 4*x). - Vladeta Jovovic (vladeta(AT)eunet.rs), Aug 20 2003

a(n) = sum(k=0..n, sum(j=0..n-k, C(n,j)*C(n-j,k)*C(2*n-2*j,n-j))). - Paul Barry, May 19 2006

a(n) = sum(k=0..n, 4^k*(C(n,k))^2) [From heruneedollar (heruneedollar(AT)gmail.com), Mar 20 2010]

Asymptotic: a(n) ~ 3^(2*n+1)/(2*sqrt(2*Pi*n)). [Vaclav Kotesovec, Sep 11 2012]

Conjecture: n*a(n) +5*(-2*n+1)*a(n-1) +9*(n-1)*a(n-2)=0. - R. J. Mathar,

Future Work
and Takeaways

Future Work

Possible Future Projects (inspired by conversations at Texas State with April Yang):

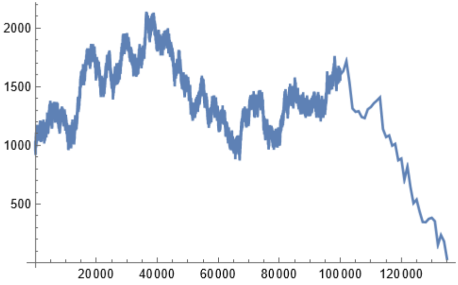
- What if each person tosses several coins simultaneously of different denominations?
- What if some of the coins are positive and some are negative, and have a starting value? Is a +5 and a -3 and a -2 different than a -5 and a +3 and a +2?

Numerics: Code

```
fibmandmgame[start_, val1_, val2_, val3_, numdo_] := Module[{},
   list = {{0, start}};
   current = start;
   Print["Welcome to the Fibonacci M&M Game. You have chosen coin values of ", coin1, ", ", coin2, " and ", coin3, "."];
   Print["We toss all three coins, independently, if coin k comes up heads you get its value."];
   Print["We are doing this for ", numdo, " tosses with starting value ", start, " and will record what happens."];
   Print[
    "Notice the coin values are chosen so that on average there is no change BUT is it more likely to go
      below zero if the big value is the negative versus instead having the two smaller ones be negative?"];
   For[n = 1, n ≤ numdo, n++,
    {
     coin1 = If[Random[] ≤ .5, val1, 0];
     coin2 = If[Random[] ≤ .5, val2, 0];
     coin3 = If[Random[] ≤ .5, val3, 0];
     current = current + (coin1 + coin2 + coin3);
     If[n < 100 000, list = AppendTo[list, {n, current}]];
     If[n ≥ 100 000 && Mod[n, 1000] == 0, list = AppendTo[list, {n, current}]];
     If[current < 0,
      {
       Print["YOU LOSE! SURVIVED TILL n = ", n, "."];
       n = numdo + 1000;
      }]; (* end of if statement *)
    }]; (* end of n loop *)
   Print[ListLinePlot[list]];
  ];
```

Numerics: Simulation

```
fibmandmgame[1000, -5, 3, 2, 200 000]

Welcome to the Fibonacci M&M Game. You have chosen coin values of 0, 3 and 0.

We toss all three coins, independently, if coin k comes up heads you get its value.

We are doing this for 200 000 tosses with starting value 1000 and will record what happens.

Notice the coin values are chosen so that on average there is no change BUT is it more likely
  to go below zero if the big value is the negative versus instead having the two smaller ones be negative

YOU LOSE! SURVIVED TILL n = 135 260.
```



Lessons

◇ Always ask questions.

◇ Many ways to solve a problem.

◇ Experience is useful and a great guide.

◇ Need to look at the data the right way.

◇ Often don't know where the math will take you.

◇ Value of continuing education: more math is better.

◇ Connections: My favorite quote: `If all you have is a hammer, pretty soon every problem looks like a nail.`

Generating Functions

Generating Function (Example: Binet's Formula)

Binet's Formula

$$F_1 = F_2 = 1;\ F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right].$$

- Recurrence relation: $F_{n+1} = F_n + F_{n-1}$
- Generating function: $g(x) = \sum_{n>0} F_n x^n$.

$$\begin{aligned}
(1) \quad &\Rightarrow \sum_{n\geq 2} F_{n+1}x^{n+1} = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 2} F_{n-1}x^{n+1} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = \sum_{n\geq 2} F_n x^{n+1} + \sum_{n\geq 1} F_n x^{n+2} \\
&\Rightarrow \sum_{n\geq 3} F_n x^n = x\sum_{n\geq 2} F_n x^n + x^2\sum_{n\geq 1} F_n x^n \\
&\Rightarrow g(x) - F_1 x - F_2 x^2 = x(g(x) - F_1 x) + x^2 g(x) \\
&\Rightarrow g(x) = x/(1 - x - x^2).
\end{aligned}$$

Partial Fraction Expansion (Example: Binet's Formula)

- Generating function: $g(x) = \sum_{n>0} F_n x^n = \frac{x}{1-x-x^2}$.
- Partial fraction expansion:

$$\Rightarrow \; g(x) \;=\; \frac{x}{1-x-x^2} \;=\; \frac{1}{\sqrt{5}}\left(\frac{\frac{1+\sqrt{5}}{2}x}{1-\frac{1+\sqrt{5}}{2}x} - \frac{\frac{-1+\sqrt{5}}{2}x}{1-\frac{-1+\sqrt{5}}{2}x}\right).$$

Coefficient of x^n (power series expansion):

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{-1+\sqrt{5}}{2}\right)^n\right]$$ - Binet's Formula!

(using geometric series: $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \cdots$).