Gaps Between Zeros of GL(2) L-functions

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Gaps between Critical Zeros

Paper: *Gaps between zeros of GL(2) L-functions* (with Owen Barrett, Brian McDonald, Ryan Patrick, Caroline Turnage-Butterbaugh and Karl Winsor), Journal of Mathematical Analysis and Applications **429** (2015), 204–232. http://arxiv.org/pdf/1410.7765.pdf.

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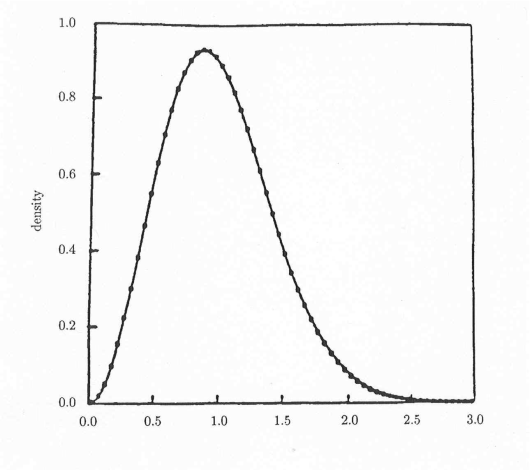
The Random Matrix Theory Connection

Philosophy: Critical-zero statistics of L-functions agree with eigenvalue statistics of large random matrices.

- Montgomery - pair-correlations of zeros of $\zeta(s)$ and eigenvalues of the Gaussian Unitary Ensemble.
- Hejhal, Rudnick and Sarnak - Higher correlations and automorphic L-functions.
- Odlyzko - further evidence through extensive numerical computations.

Consecutive Zero Spacings



Consecutive zero spacings of $\zeta(s)$ vs. GUE predictions (Odlyzko).

Large Gaps between Zeros

Let $0 \leq \gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_i \leq \cdots$ be the ordinates of the critical zeros of an L-function.

Conjecture

Gaps between consecutive zeros that are arbitrarily large, relative to the average gap size, appear infinitely often.

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$$\text{Letting} \quad \Lambda = \limsup_{n\to\infty} \frac{\gamma_{n+1} - \gamma_n}{\text{average spacing}},$$

this conjecture is equivalent to $\Lambda = \infty$.

- Best unconditional result for the Riemann zeta function is $\Lambda > 2.69$.

Degree 2 Case

Higher degree L-functions are mostly unexplored.

Theorem (Turnage-Butterbaugh '14)

Let $T \geq 2$, $\varepsilon > 0$, $\zeta_K(s)$ the Dedekind zeta function attached to a quadratic number field K with discriminant d with $|d| \leq T^{\varepsilon}$, and $\mathcal{S}_T := \{\gamma_1, \gamma_2, \ldots, \gamma_N\}$ be the distinct zeros of $\zeta_K\left(\frac{1}{2} + it, f\right)$ in the interval $[T, 2T]$. Let κ_T denote the maximum gap between consecutive zeros in $\mathcal{S}_T$. Then

$$\kappa_T \geq \sqrt{6}\frac{\pi}{\log\sqrt{|d|}T}\left(1 + O(d^{\varepsilon}\log T)^{-1}\right).$$

- Assuming GRH, this means $\Lambda \geq \sqrt{6} \approx 2.449$.

A Lower Bound on Large Gaps

We proved the following unconditional theorem for an L-function associated to a holomorphic cusp form f on $\mathrm{GL}(2)$.

Theorem (BMMRTW '14)

Let $\mathcal{S}_T := \{\gamma_1, \gamma_2, ..., \gamma_N\}$ be the set of distinct zeros of $L\left(\frac{1}{2} + it, f\right)$ in the interval $[T, 2T]$. Let κ_T denote the maximum gap between consecutive zeros in $\mathcal{S}_T$. Then

$$\kappa_T \geq \frac{\sqrt{3}\pi}{\log T}\left(1 + O\left(\frac{1}{c_f}(\log T)^{-\delta}\right)\right),$$

where c_f is the residue of the Rankin-Selberg convolution $L(s, f \times \bar{f})$ at $s = 1$.

Assuming GRH, there are infinitely many normalized gaps between consecutive zeros at least $\sqrt{3}$ times the mean spacing, i.e.,

$$\Lambda \geq \sqrt{3} \approx 1.732.$$

An Upper Bound on Small Gaps

Theorem (BMMRTW '14)

L in Selberg class primitive of degree m_L. Assume GRH for $\log L(s) = \sum_{n=1}^{\infty} b_L(n)/n^s$, $\sum_{n\le x} |b_L(n)\log n|^2 = (1+o(1))x\log x$. Have a computable nontrivial upper bound on μ_L (liminf of smallest average gap) depending on m_L.

m_L	upper bound for μ_L
1	0.606894
2	0.822897
3	0.905604
4	0.942914
5	0.962190
⋮	⋮

($m_L = 1$ due to Carneiro, Chandee, Littmann and Milinovich).

Key idea: use pair correlation analysis.

Results on Gaps and Shifted Second Moments

Shifted Moment Result

To prove our theorem, use a method due to R.R. Hall and the following shifted moment result.

Theorem (BMMRTW '14)

$$\int_T^{2T} L\left(\frac{1}{2}+it+\alpha, f\right) L\left(\frac{1}{2}-it+\beta, f\right) dt$$
$$= c_f T \sum_{n\geq 0} \frac{(-1)^n 2^{n+1}(\alpha+\beta)^n (\log T)^{n+1}}{(n+1)!} + O\left(T(\log T)^{1-\delta}\right),$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|, |\beta| \ll 1/\log T$.

Key idea: differentiate wrt parameters, yields formulas for integrals of products of derivatives.

Shifted Moments Proof Technique

- Approximate functional equation:

$$L(s+\alpha, f) \;=\; \sum_{n\geq 1} \frac{\lambda_f(n)}{n^{s+\alpha}} e^{-\frac{n}{X}} + F(s) \sum_{n\leq X} \frac{\lambda_f(n)}{n^{1-s-\alpha}} + E(s),$$

  where $\lambda_f(n)$ are the Fourier coefficients of $L(s, f)$, $F(s)$ is a functional equation term, and $E(s)$ is an error term.

- We have an analogous expression for $L(1-s+\beta, f)$.

Shifted Moments Proof Technique

- Analyze product

$$L(s+\alpha, f)L(1-s+\beta, f),$$

  where each factor gives rise to four products (so sixteen products to estimate).

- Use a generalization of Montgomery and Vaughan's mean value theorem and contour integration to estimate product and compute the resulting moments.

Shifted Moment Result for Derivatives

- Shifted moment result yields lower order terms and moments of derivatives of L-functions by differentiation and Cauchy's integral formula.

- Derive an expression for

$$\int_T^{2T} L^{(\mu)}\left(\frac{1}{2}+it, f\right) L^{(\nu)}\left(\frac{1}{2}-it, f\right) dt,$$

  where $T \geq 2$ and $\mu, \nu \in \mathbb{Z}^+$. Use this in Hall's method to obtain the lower bound stated in our theorem.

- Need $(\mu, \nu) \in \{(0,0),(1,0),(1,1)\}$; other cases previously done (Good did $(0,0)$ and Yashiro did $\mu = \nu$).

Modified Wirtinger Inequality

Using Hall's method, we bound the gaps between zeroes. This requires the following result, due to Wirtinger and modified by Bredberg.

Lemma (Bredberg)

Let $y : [a, b] \to \mathbb{C}$ be a continuously differentiable function and suppose that $y(a) = y(b) = 0$. Then

$$\int_a^b |y(x)|^2 dx \leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b |y'(x)|^2 dx.$$

Proving our Result

- For ρ a real parameter to be determined later, define
$$g(t) \ := \ e^{i\rho t \log T} L\left(\tfrac{1}{2} + it, f\right),$$
Fix f and let $\tilde{\gamma}_f(k)$ denote an ordinate zero of $L(s, f)$ on the critical line $\Re(s) = \frac{1}{2}$.

- $g(t)$ has same zeros as $L(s, f)$ (at $t = \tilde{\gamma}_f(k)$). Use in the modified Wirtinger's inequality.

- For adjacent zeros have
$$\sum_{n=1}^{N-1} \int_{\tilde{\gamma}_f(n)}^{\tilde{\gamma}_f(n+1)} |g(t)|^2 dt \ \leq \ \sum_{n=1}^{N-1} \frac{\kappa_T^2}{\pi^2} \int_{\tilde{\gamma}_f(n)}^{\tilde{\gamma}_f(n+1)} |g'(t)|^2 dt.$$

- Summing over zeros with $n \in \{1, \dots, N\}$ and trivial estimation yields integrals from T to $2T$.

Proving our Result

- $|g(t)|^2 = |L(1/2+it,f)|^2$ and

$$|g'(t)|^2 = |L'(1/2+it,f)|^2 + \rho^2 \log^2 T \cdot |L(1/2+it,f)|^2 + 2\rho \log T \cdot \mathrm{Re}\left(L'(1/2+it,f)\overline{L(1/2+it,f)}\right).$$

- Apply sub-convexity bounds to $L(1/2+it,f)$:

$$\int_T^{2T} |g(t)|^2 dt \leq \frac{\kappa_T^2}{\pi^2} \int_T^{2T} |g'(t)|^2 dt + O\left(T^{\frac{2}{3}} (\log T)^{\frac{5}{6}}\right).$$

- As $g(t)$ and $g'(t)$ may be expressed in terms of $L\left(\frac{1}{2}+it,f\right)$ and its derivatives, can write our inequality explicitly in terms of formula given by our mixed moment theorem.

Finishing the Proof

- After substituting our formula, we have

$$\frac{\kappa_T^2}{\pi^2} \;\geq\; \frac{3}{3\rho^2 - 6\rho + 4} (\log T)^{-2} \left(1 + O(\log T)^{-\delta}\right).$$

- The polynomial in ρ is minimized at $\rho = 1$, yielding

$$\kappa_T \;\geq\; \frac{\sqrt{3}\pi}{\log T} \left(1 + O\left(\frac{1}{c_f} (\log T)^{-\delta}\right)\right).$$

Essential $\mathrm{GL}(2)$ properties

Properties

For primitive f on $\mathrm{GL}(2)$ over $\mathbb{Q}$ (Hecke or Maass) with

$$L(s,f) \ = \ \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s}, \quad \Re(s) > 1,$$

we isolate needed crucial properties (all are known).

1. $L(s,f)$ has an analytic continuation to an entire function of order 1.

2. $L(s,f)$ satisfies a functional equation of the form

$$\begin{aligned} \Lambda(s,f) \ &:= \ L(s,f_\infty)L(s,f) \ = \ \epsilon_f \Lambda(1-s,\bar{f}) \\ \text{with } L(s,f_\infty) \ &= \ Q^s \Gamma\left(\frac{s}{2}+\mu_1\right)\Gamma\left(\frac{s}{2}+\mu_2\right). \end{aligned}$$

Properties (continued)

3. Convolution L-function $L(s, f \times \bar{f})$,

$$\sum_{n=1}^{\infty} \frac{|a_f(n)|^2}{n^s}, \quad \Re(s) > 1,$$

is entire except for a simple pole at $s = 1$.

4. The Dirichlet coefficients (normalized so that the critical strip is $0 \le \Re(s) \le 1$) satisfy

$$\sum_{n \le x} |a_f(n)|^2 \ll x.$$

5. For some small $\delta > 0$, we have a subconvexity bound

$$\left| L\left(\frac{1}{2} + it, f\right) \right| \ll |t|^{\frac{1}{2} - \delta}.$$

Properties (status)

- Mœglin and Waldspurger prove the needed properties of $L(s, f \times \overline{f})$ (in greater generality).
- Dirichlet coefficient asymptotics follow for Hecke forms essentially from the work of Rankin and Selberg, and for Maass by spectral theory.
- Michel and Venkatesh proved a subconvexity bound for primitive $GL(2)$ L-functions over $\mathbb{Q}$.
- Other properties are standard and are valid for $\mathrm{GL}(2)$.