From Zombies to Fibonaccis: An Introduction to the Theory of Games

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Michigan Math Circle: April 23, 2020

Tic-Tac-Toe

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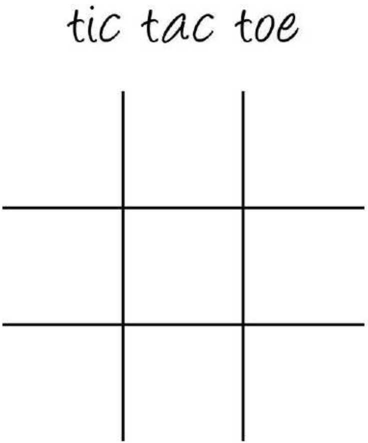


Figure: How many opening moves? How many first two moves?

Tic-Tac-Toe: First Move

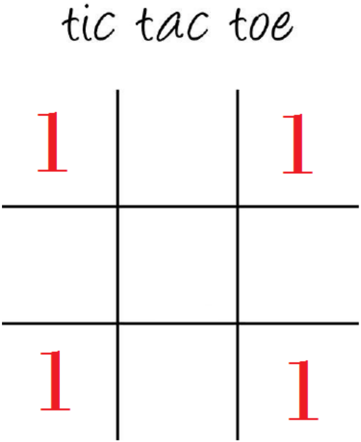


Figure: Analyzing Opening Moves: Corners all equivalent.

Tic-Tac-Toe: First Move

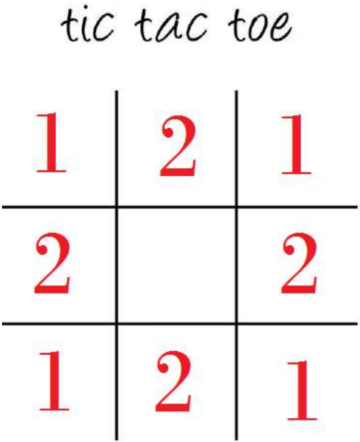


Figure: Analyzing Opening Moves: Middles all equivalent.

Tic-Tac-Toe: First Move

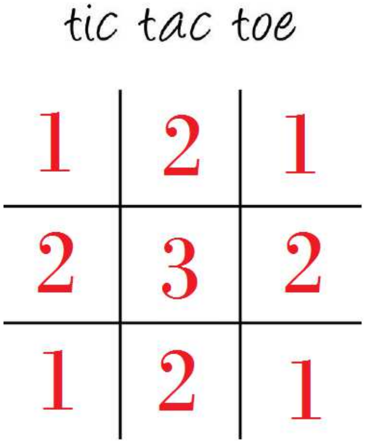


Figure: Analyzing Opening Moves: Only one center: 3 classes of moves.

Tic-Tac-Toe: Second Move

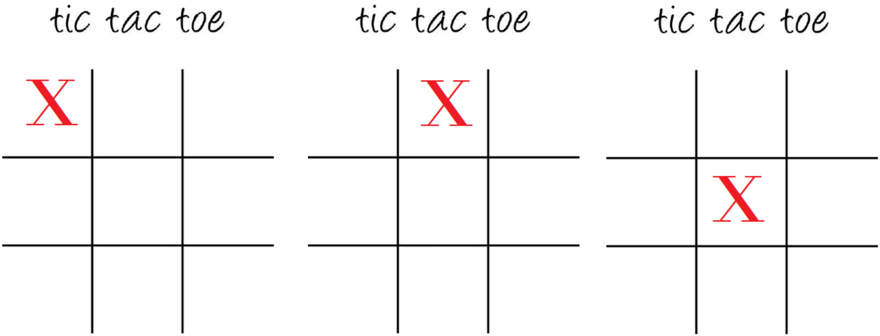
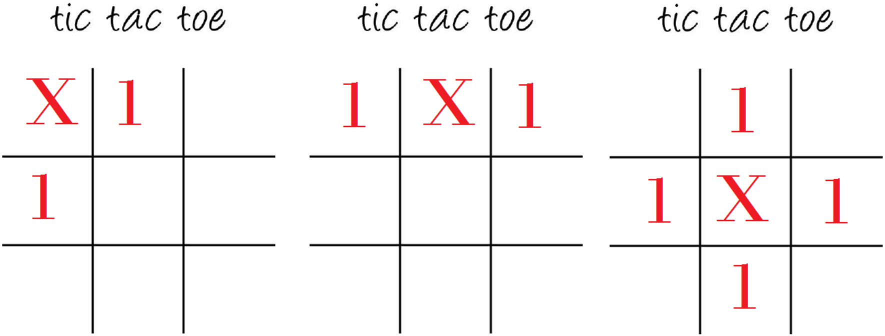


Figure: Analyzing Second Player Response.

Tic-Tac-Toe: Second Move



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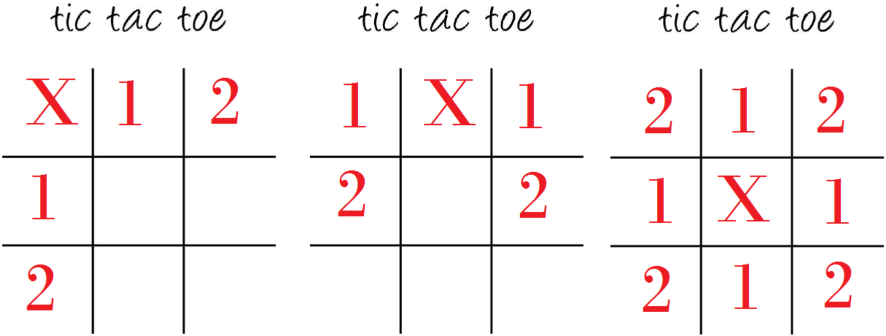

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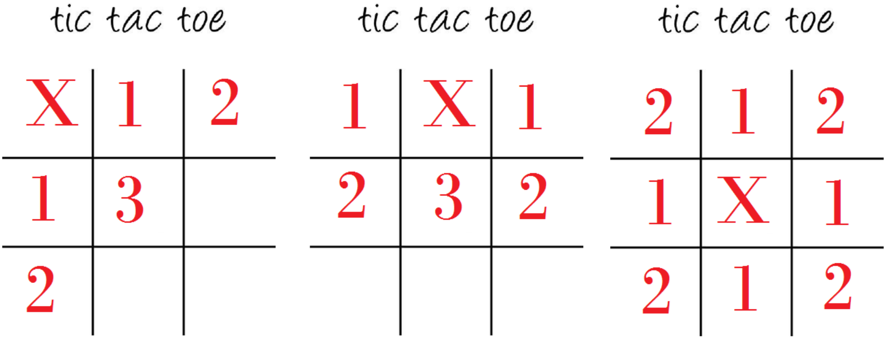

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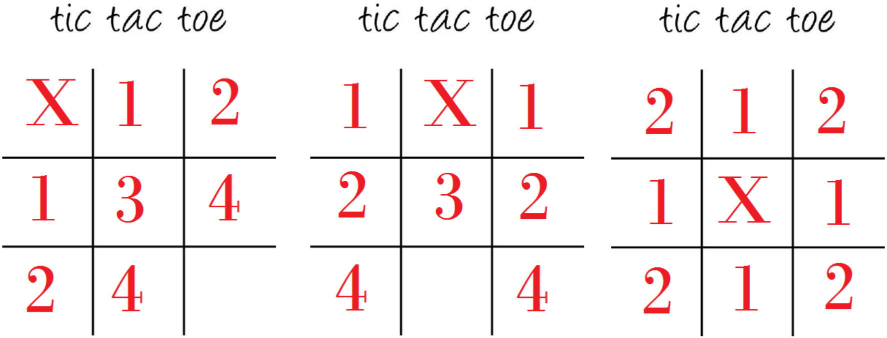


Figure: Analyzing Second Player Response.

Tic-Tac-Toe: Second Move

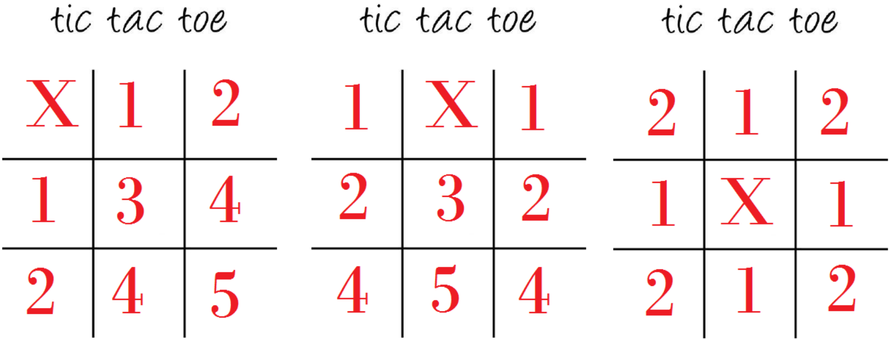


Figure: Analyzing Second Player Response: Thus there are 12 possible pairs of first two moves.

Tic-Tac-Toe: Questions

- Does either player have a winning strategy?
- What happens if we play randomly? Chance of Player 1 winning?
- How can we make it interesting?

Tic-Tac-Toe: Interesting Variants

- Gobble Tic-Tac-Toe
- Larger board (and handicaps)
- Tic-Tac-Toe in Tic-Tac-Toe
- Bidding Tic-Tac-Toe

Tic-Tac-Toe in Tic-Tac-Toe

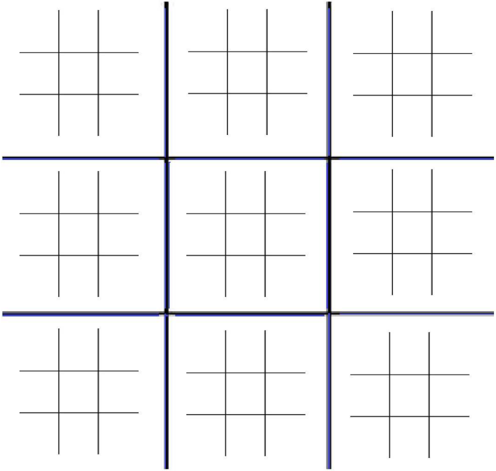
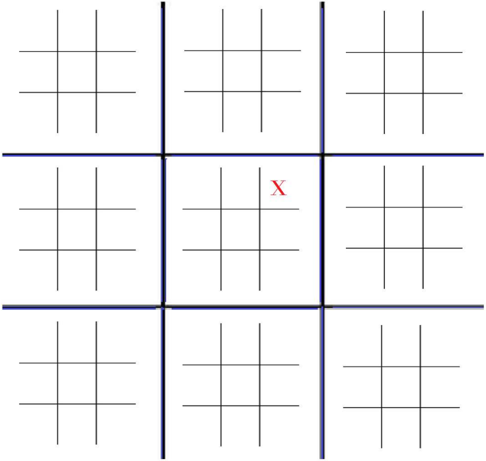


Figure: Rule: next move from position of previous.

Tic-Tac-Toe in Tic-Tac-Toe



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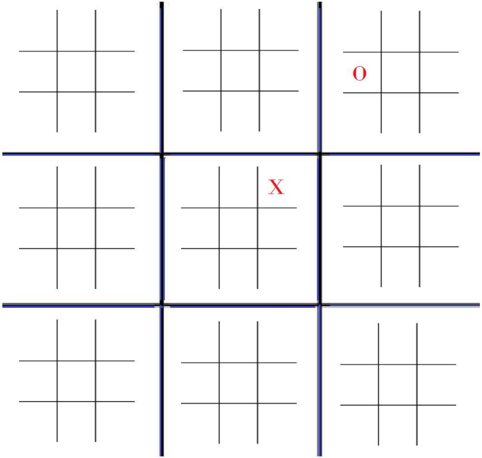

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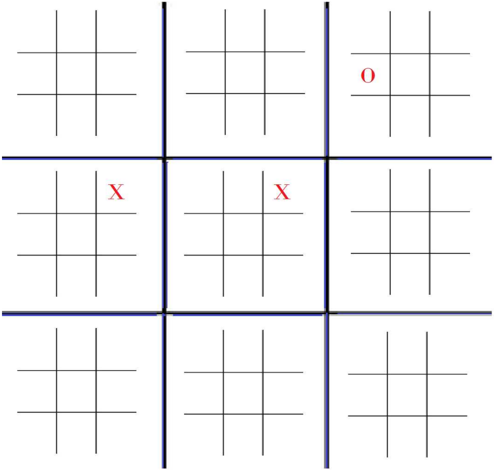

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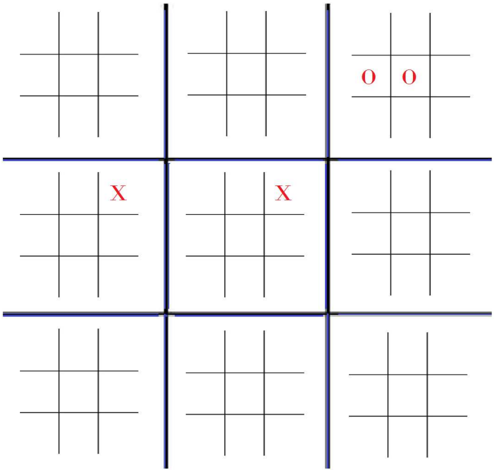

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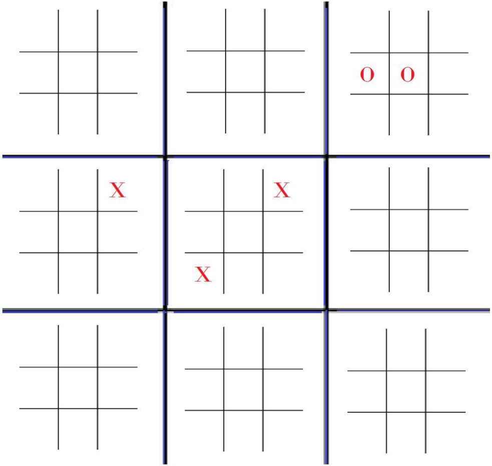

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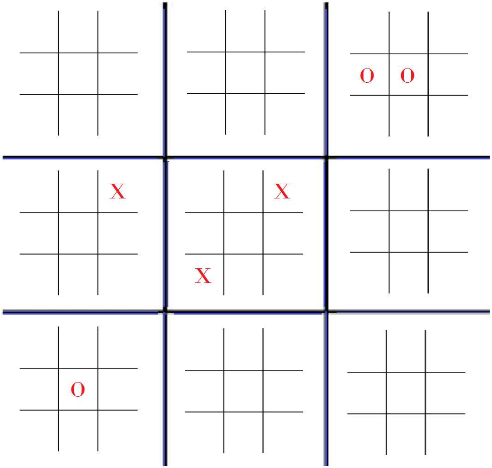
Tic-Tac-Toe in Tic-Tac-Toe



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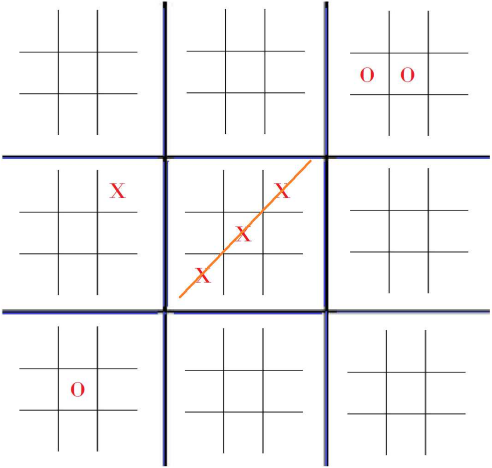


Figure: Rule: next move from position of previous.

Tic-Tac-Toe in Tic-Tac-Toe

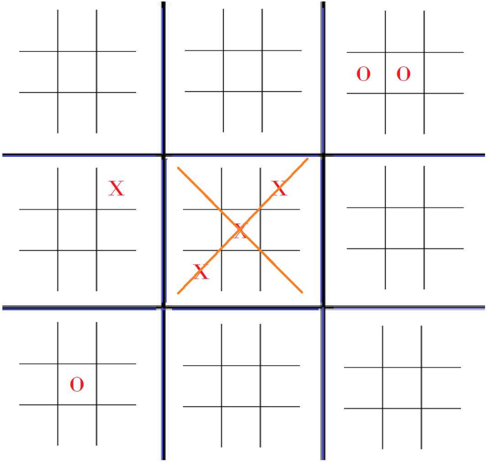
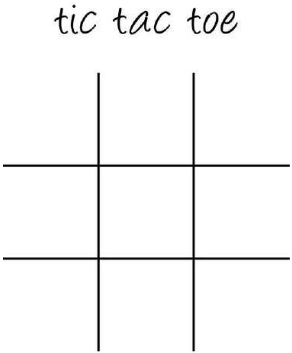


Figure: Rule: next move from position of previous.

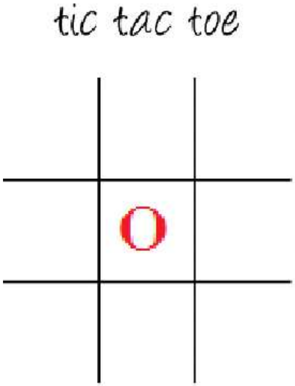
Bidding Tic-Tac-Toe



Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

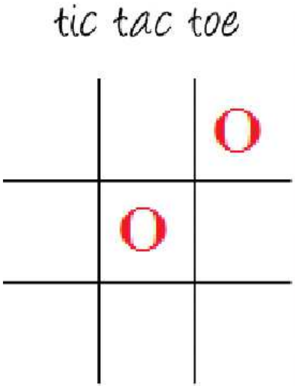
Bidding Tic-Tac-Toe



Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1	1000	1000	300	700
2				
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

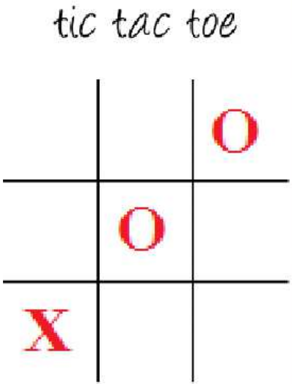
Bidding Tic-Tac-Toe



Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1	1000	1000	300	700
2	1700	300	99	100
3				
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

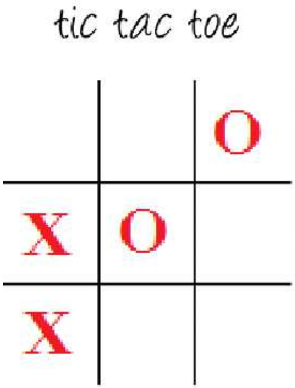
Bidding Tic-Tac-Toe



Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1	1000	1000	300	700
2	1700	300	99	100
3	1800	200	201	200
4				
5				
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

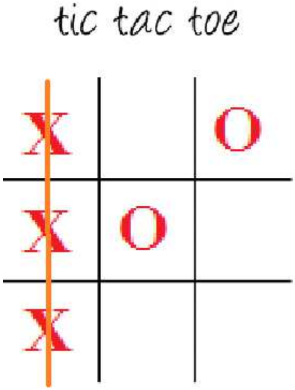
Bidding Tic-Tac-Toe



Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1	1000	1000	300	700
2	1700	300	99	100
3	1800	200	201	200
4	1599	401	402	400
5				
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

Bidding Tic-Tac-Toe



Move	#1's Money	#2's Money	#1's Bid	#2's Bid
1	1000	1000	300	700
2	1700	300	99	100
3	1800	200	201	200
4	1599	401	402	400
5	1197	803	804	803
6				
7				
8				
9				
10				
11				
12				
13				

Figure: Rules: Start with \$1000 and \$1000, whomever bids more gets move and gives money to other. Opening bid?

Bidding Tic-Tac-Toe: Overbidding for First Move

Question: If each have $1000, how much is too much to bid for the first move?

Easy answer:

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Question: If each have \$1000, how much is too much to bid for the first move?

Easy answer: \$1000.

Can we do better? Assume if tie Player 1 gets the move.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids x and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids x and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Player 1 bids $1000 - x$ and just wins:
$(1000 + x, 1000 - x) \longrightarrow (2x, 2000 - 2x)$.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids x and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Player 1 bids $1000 - x$ and just wins:
$(1000 + x, 1000 - x) \longrightarrow (2x, 2000 - 2x)$.

Player 1 bids $2000 - 2x$ and just wins:
$(2x, 2000 - 2x) \longrightarrow (4x - 2000, 4000 - 4x)$.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids x and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Player 1 bids $1000 - x$ and just wins:
$(1000 + x, 1000 - x) \longrightarrow (2x, 2000 - 2x)$.

Player 1 bids $2000 - 2x$ and just wins:
$(2x, 2000 - 2x) \longrightarrow (4x - 2000, 4000 - 4x)$.

Player 1 bids $4000 - 4x$ and just wins:
$(4x - 2000, 4000 - 4x) \longrightarrow (8x - 6000, 8000 - 8x)$.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids x and wins:
$(1000, 1000) \longrightarrow (1000 + x, 1000 - x)$.

Player 1 bids $1000 - x$ and just wins:
$(1000 + x, 1000 - x) \longrightarrow (2x, 2000 - 2x)$.

Player 1 bids $2000 - 2x$ and just wins:
$(2x, 2000 - 2x) \longrightarrow (4x - 2000, 4000 - 4x)$.

Player 1 bids $4000 - 4x$ and just wins:
$(4x - 2000, 4000 - 4x) \longrightarrow (8x - 6000, 8000 - 8x)$.

Need $4x - 2000 \geq 4000 - 4x$ or $8x \geq 6000$ or $x \geq 750$.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids 750 and wins: $(1000, 1000) \longrightarrow (1750, 250)$.

Player 1 bids 250 and just wins: $(1750, 250) \longrightarrow (1500, 500)$.

Player 1 bids 500 and just wins: $(1500, 500) \longrightarrow (1000, 1000)$.

Player 1 bids 1000 and just wins: $(1000, 1000) \longrightarrow (0, 2000)$.

Bidding Tic-Tac-Toe: Overbidding for First Move (cont)

Player 2 bids 750 and wins: $(1000, 1000) \longrightarrow (1750, 250)$.

Player 1 bids 250 and just wins: $(1750, 250) \longrightarrow (1500, 500)$.

Player 1 bids 500 and just wins: $(1500, 500) \longrightarrow (1000, 1000)$.

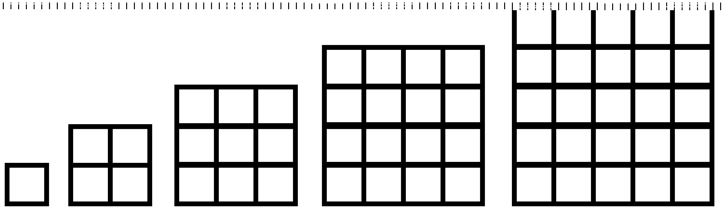
Player 1 bids 1000 and just wins: $(1000, 1000) \longrightarrow (0, 2000)$.

Note: If Player 2 spends $750 to win the first two moves then Player 1 can win!

I Love Rectangles

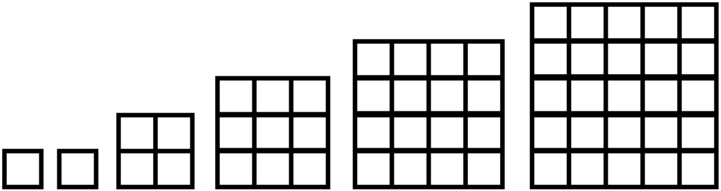
Tiling the Plane with Squares

Have $n \times n$ square for each n, place one at a time so that shape formed is always connected and a rectangle.



Tiling the Plane with Squares

Have $n \times n$ square for each n, extra 1×1 square, place one at a time so that shape formed is always connected and a rectangle.



Tiling the Plane with Squares: 1×1, 1×1, 2×2, 3×3, ...

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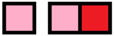
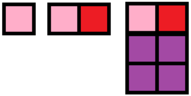
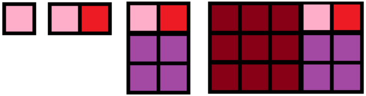


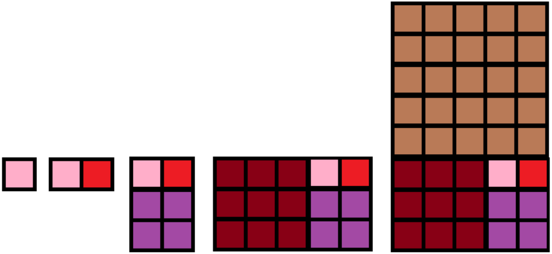
Tiling the Plane with Squares: 1×1, 1×1, 2×2, 3×3, ...



Tiling the Plane with Squares: 1×1, 1×1, 2×2, 3×3, ...



Tiling the Plane with Squares: 1×1, 1×1, 2×2, 3×3, ...



1, 1, 2, 3, 5, ...

Fibonacci Spiral:

https://www.youtube.com/watch?v=kkGeOWYOFoA

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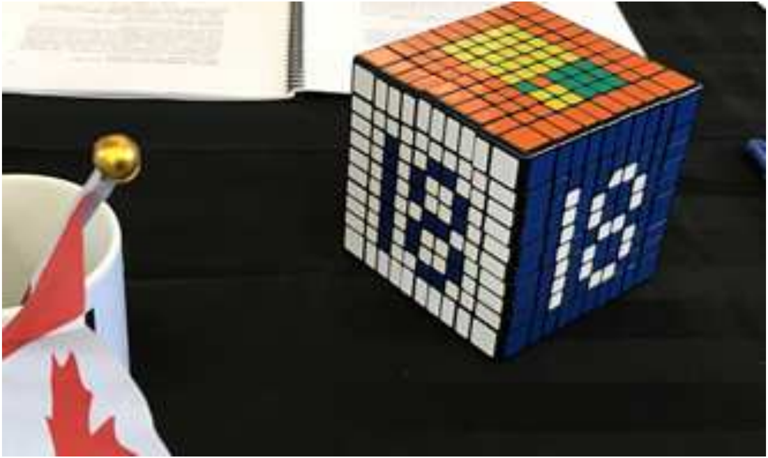

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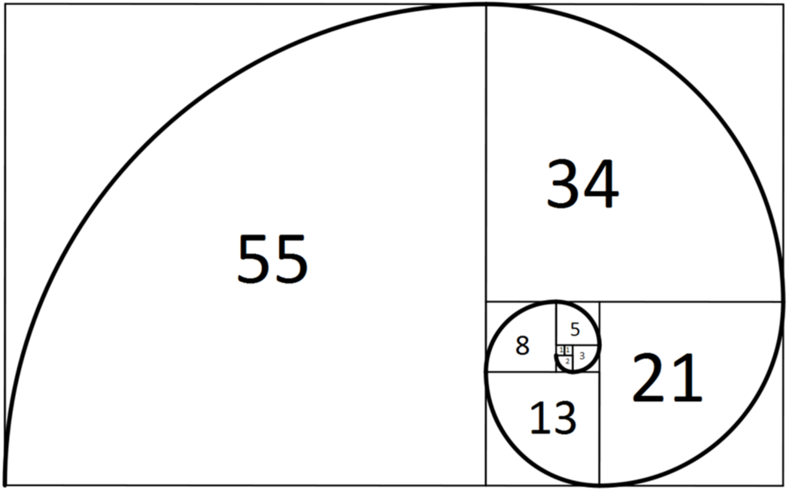

Rectangle Game

Rectangle Game

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

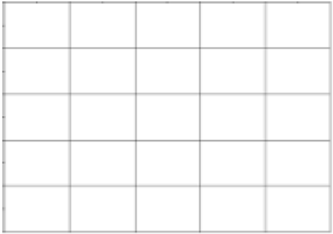

Figure: Winning strategy? Function of board dimension?

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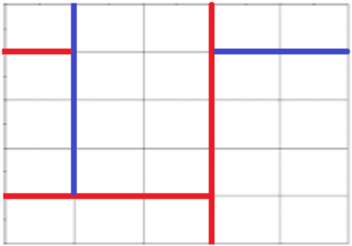

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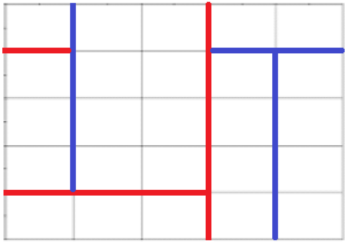
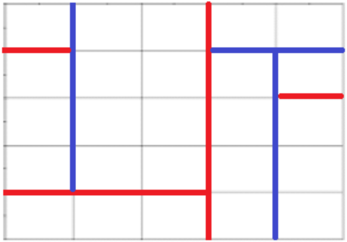


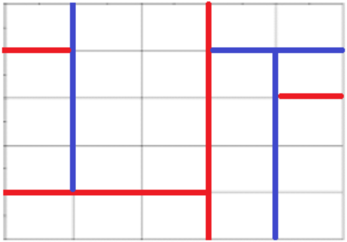
Rectangle Game

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



Rectangle Game

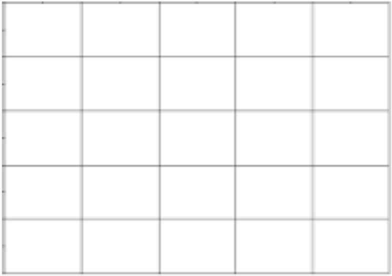
RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



Gather data! Try various sized boards, strategies.

Rectangle Game: Data

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



Length	Width	Winner
2	2	--1--
2	3	--1--
3	3	--2--
2	4	--1--
3	4	--1--
4	4	--1--
3	5	--2--

Figure: Do you see a pattern?

Mono-variant

A mono-variant is a quantity that moves on only one direction (either non-decreasing or non-increasing).

Idea: Associate a mono-variant to the rectangle game....

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

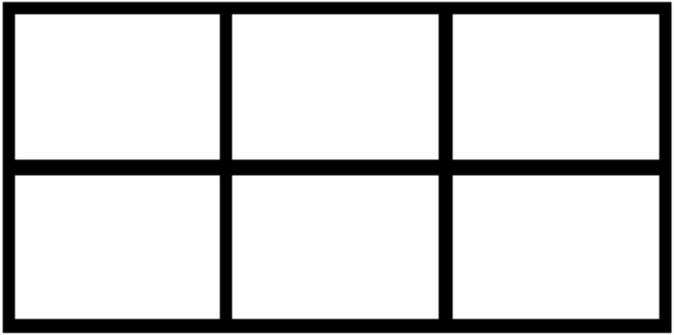


Figure: Move: 0; Pieces: 1.

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

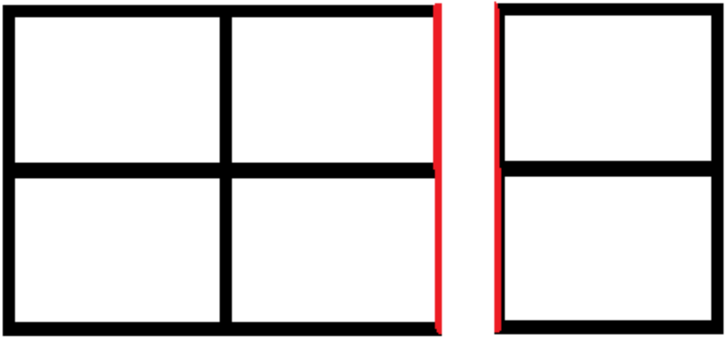


Figure: Move: 1; Pieces: 2.

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

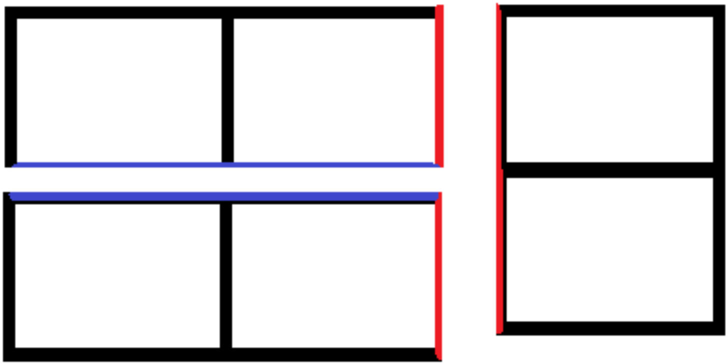


Figure: Move: 2; Pieces: 3.

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

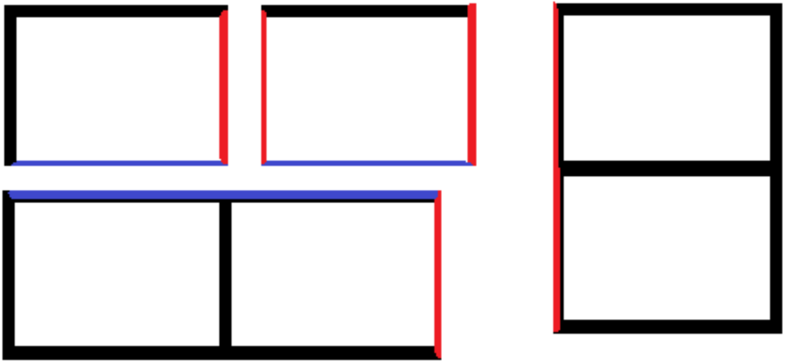


Figure: Move: 3; Pieces: 4.

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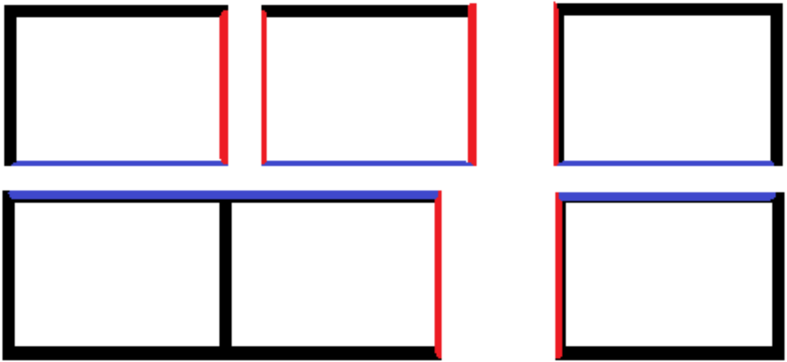

Figure: Move: 4; Pieces: 5.

Rectangle Game: Solution

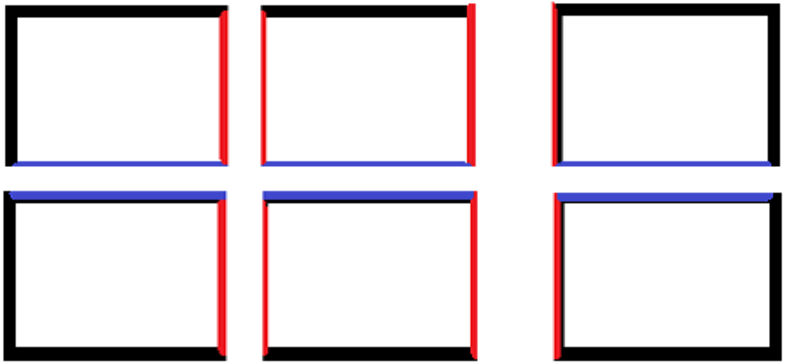

Figure: Move: 5; Pieces: 6. Player 1 Wins.

Rectangle Game: Solution (Continued)

Mono-variant is the number of pieces.

If board is $m \times n$, game ends with mn pieces.

Thus takes $mn - 1$ moves.

If mn even then Player 1 wins else Player 2 wins.

Zombies

General Advice: What are your tools and how can they be used?

Law of the Hammer:

- Abraham Kaplan: I call it the law of the instrument, and it may be formulated as follows: Give a small boy a hammer, and he will find that everything he encounters needs pounding.
- Abraham Maslow: I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.
- Bernard Baruch: If all you have is a hammer, everything looks like a nail.

Zombie Infection: Rules

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.

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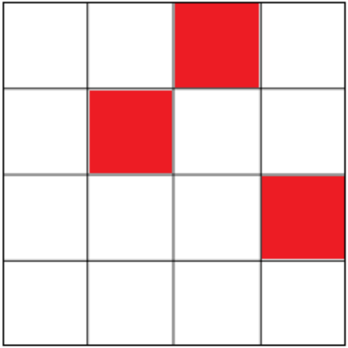

Initial Configuration

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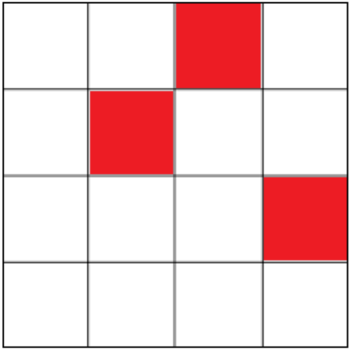

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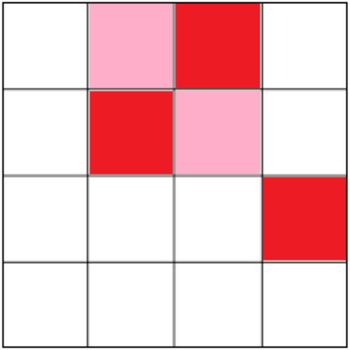

One moment later

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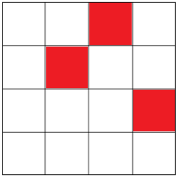

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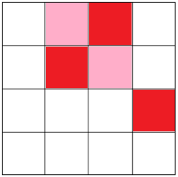

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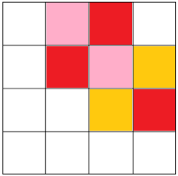

Two moments later

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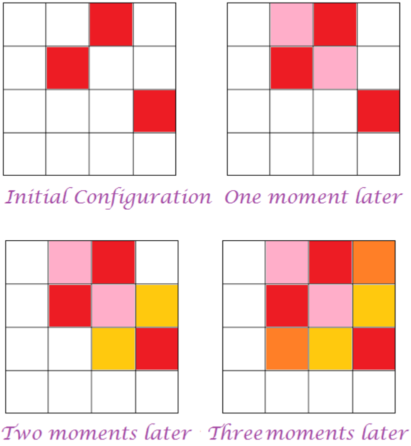
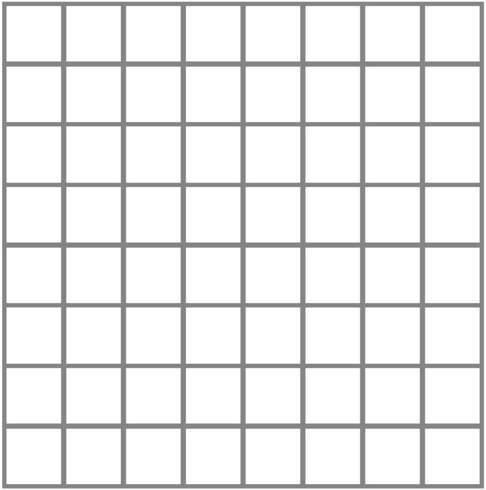
- If share walls with 2 or more infected, become infected.
- Once infected, always infected.



Zombie Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?



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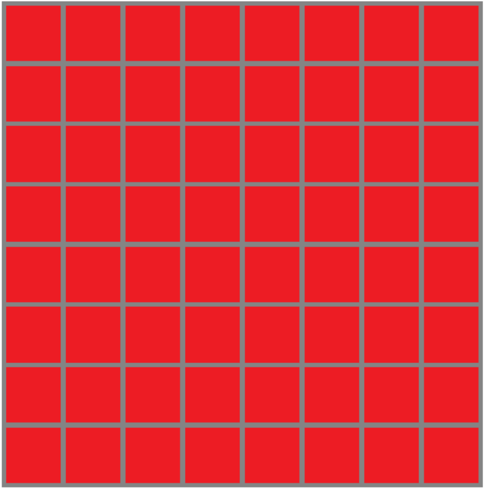
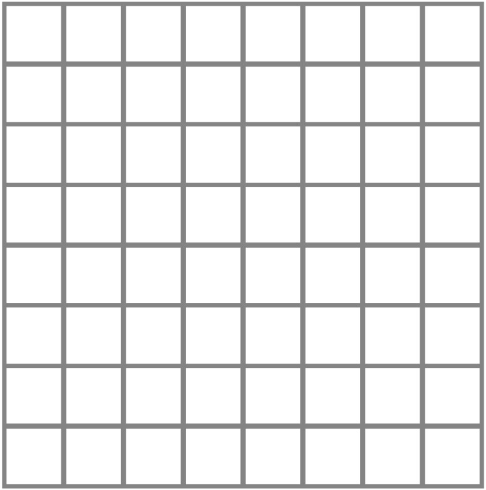


Zombie Infection: Conquering The World

Next simplest initial state ensuring all eventually infected...?



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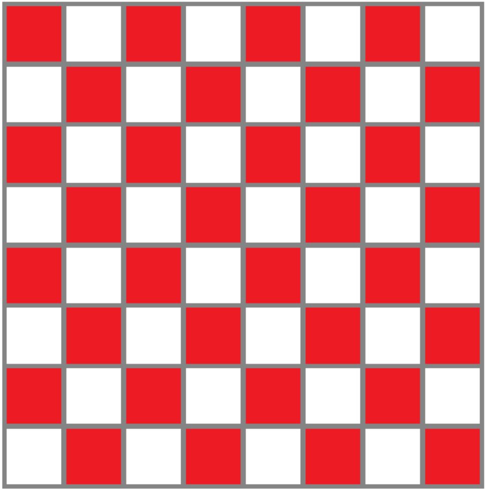

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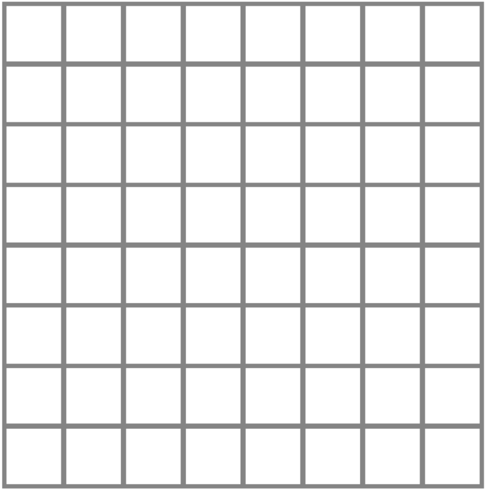
Fewest number of initial infections needed to get all...?



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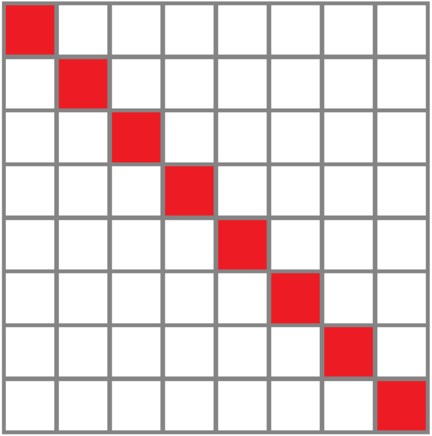

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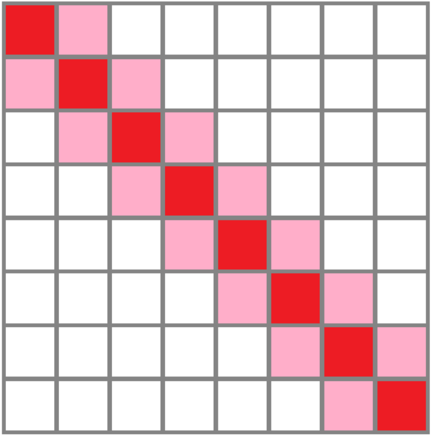

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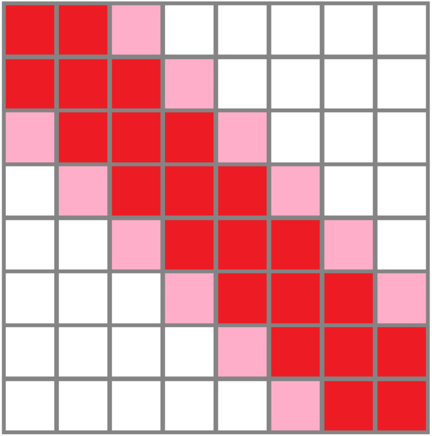

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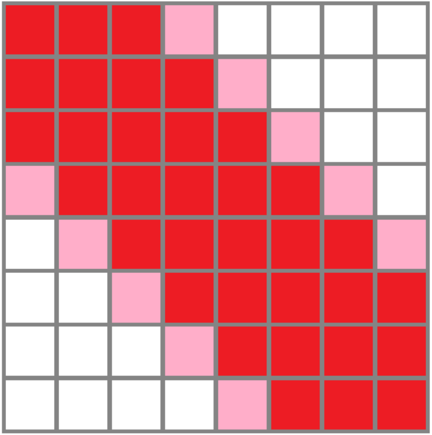

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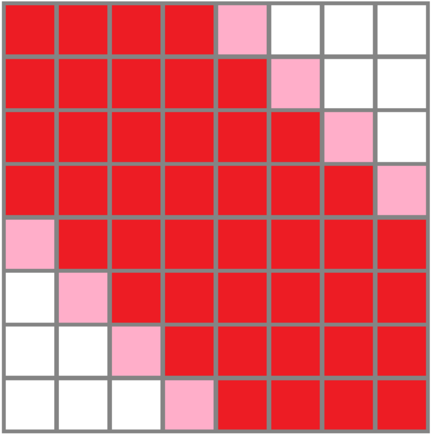

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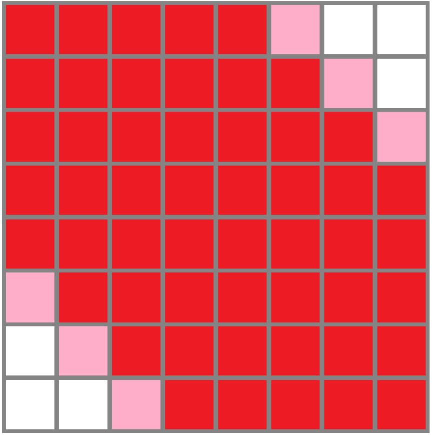

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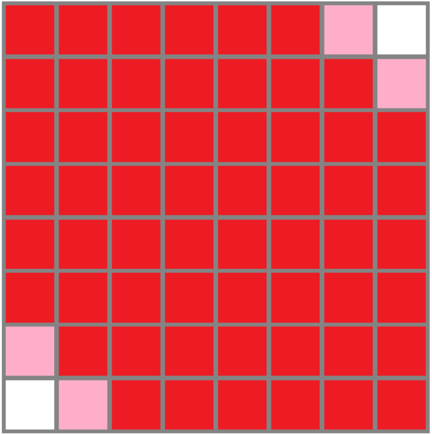

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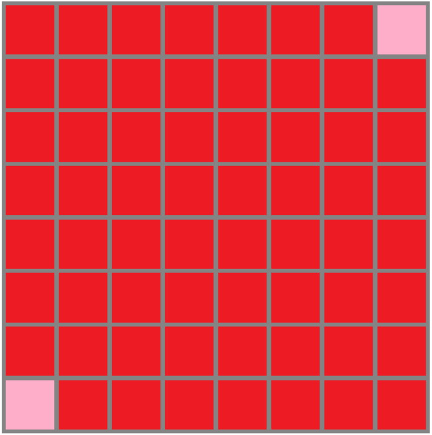

Zombie Infection: Can $n-1$ infect all?

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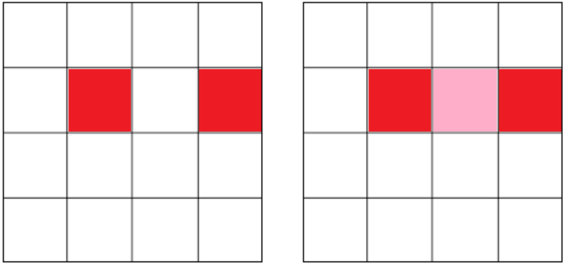

Perimeter of infection unchanged.

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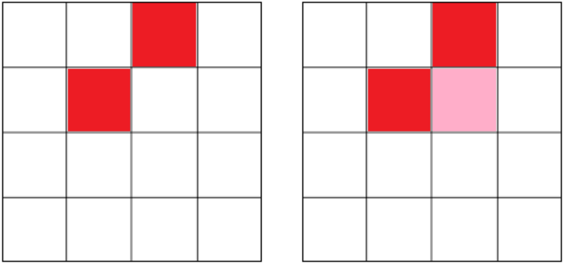

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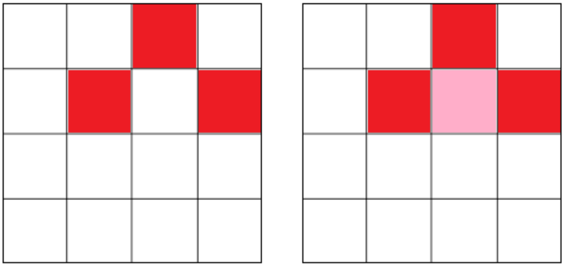
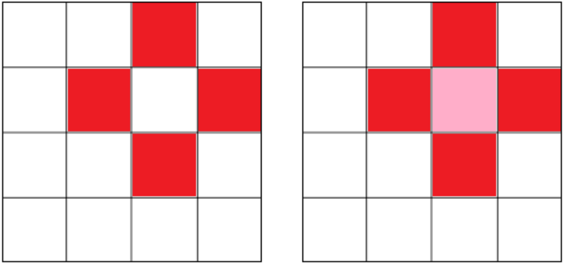


Perimeter of infection decreases by 2.

Zombie Infection: Can $n-1$ infect all?



Perimeter of infection decreases by 4.

Zombie Infection: $n - 1$ cannot infect all

- If $n - 1$ infected, maximum perimeter is $4(n - 1) = 4n - 4$.

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- Mono-variant: As time passes, perimeter of infection never increases.

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- If $n-1$ infected, maximum perimeter is $4(n-1) = 4n-4$.
- Mono-variant: As time passes, perimeter of infection never increases.
- Perimeter of $n \times n$ square is $4n$, so at least 1 square safe!

Triangle Game

Rules for Triangle Game

Take an equilateral triangle, label corners 0, 1 and 2.

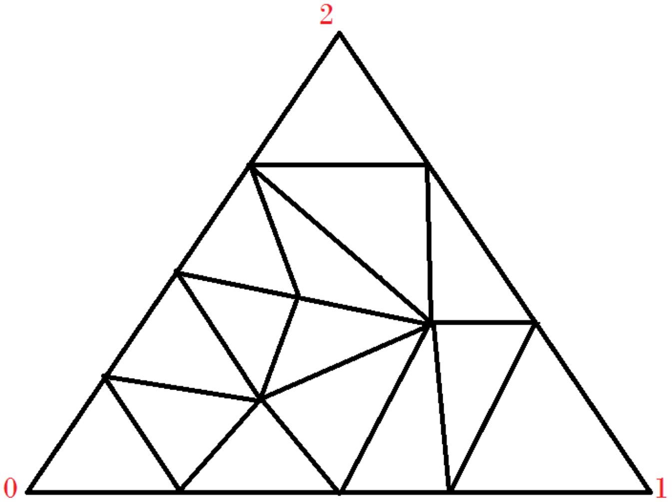
Subdivide however you wish into triangles.

Add labels, if a sub-triangle labeled 0–1–2 then Player 1 wins, else Player 2.

Take turns adding labels, subject to:
On 0–1 boundary must use 0 or 1
On 1–2 boundary must use 1 or 2
On 0–2 boundary must use 0 or 2

Who has the winning strategy? What is it?

Rules for Triangle Game



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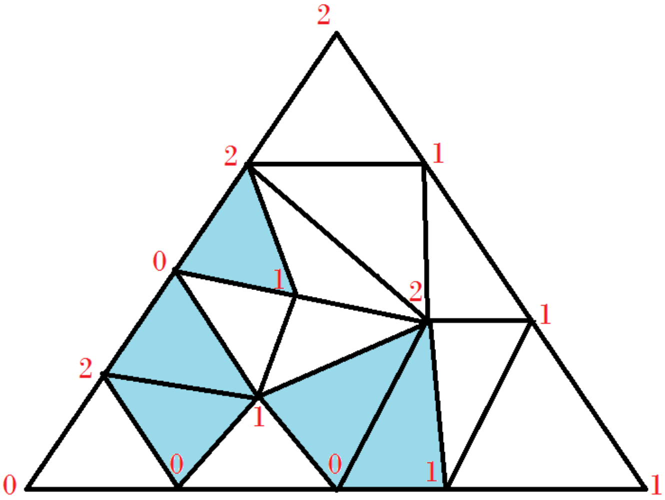
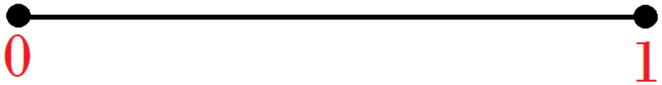


The Line Game

Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.



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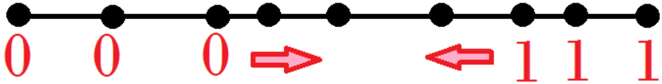

Cannot prevent at least one 0–1 segment.

The Line Game (cont)

Mono-variant: as add labels, number of 0–1 segments stays the same or increases by 2.

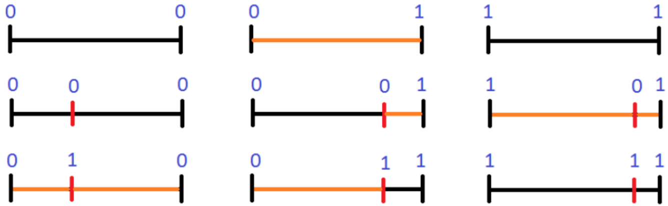

Figure 3. The various cases from the 1-dimensional Sperner proof. Notice in each of the six cases, the change in the number of 0–1 segments is even.

Can also view this as a parity argument.

Zeckendorf Games

with Cordwell, Epstein, Flynt, Hlavacek, Huynh, Peterson, Vu

Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

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$2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2$.

Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8, 13...

Summand Minimality

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

Theorem

*The Zeckendorf decomposition is **summand minimal**.*

Overall Question

What other recurrences are summand minimal?

Positive Linear Recurrence Sequences

Definition

A **positive linear recurrence sequence (PLRS)** is the sequence given by a recurrence $\{a_n\}$ with

$$a_n \;:=\; c_1 a_{n-1} + \cdots + c_t a_{n-t}$$

and each $c_i \geq 0$ and $c_1, c_t > 0$. We use **ideal initial conditions** $a_{-(n-1)} = 0, \ldots, a_{-1} = 0, a_0 = 1$ and call $(c_1, \ldots, c_t)$ the **signature of the sequence.**

Theorem (Cordwell, Hlavacek, Huynh, M., Peterson, Vu)

For a PLRS with signature $(c_1, c_2, \ldots, c_t)$, the Generalized Zeckendorf Decompositions are summand minimal if and only if

$$c_1 \geq c_2 \geq \cdots \geq c_t.$$

Proof for Fibonacci Case

Idea of proof:

- $\mathcal{D} = b_1 F_1 + \cdots + b_n F_n$ decomposition of N, set $\mathrm{Ind}(\mathcal{D}) = b_1 \cdot 1 + \cdots + b_n \cdot n$.

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- Move to $\mathcal{D}'$ by
  - ◇ $2F_k = F_{k+1} + F_{k-2}$ (and $2F_2 = F_3 + F_1$).
  - ◇ $F_k + F_{k+1} = F_{k+2}$ (and $F_1 + F_1 = F_2$).

- Monovariant: Note $\mathrm{Ind}(\mathcal{D}') \leq \mathrm{Ind}(\mathcal{D})$.
  - ◇ $2F_k = F_{k+1} + F_{k-2}$: $2k$ vs $2k - 1$.
  - ◇ $F_k + F_{k+1} = F_{k+2}$: $2k + 1$ vs $k + 2$.

- If not at Zeckendorf decomposition can continue, if at Zeckendorf cannot. Better: $\mathrm{Ind}'(\mathcal{D}) = b_1\sqrt{1} + \cdots + b_n\sqrt{n}$.

Rules

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  - ◇ If pieces at F_k and F_{k+1} remove and add one at F_{k+2}.

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  - ◇ If pieces at F_k and F_{k+1} remove and add one at F_{k+2}.

Questions:

- Does the game end? How long?
- For each N who has the winning strategy?
- What is the winning strategy?

Sample Game

Start with 10 pieces at F_1, rest empty.

10	0	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

8	1	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

6	2	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $2F_2 = F_3 + F_1$

Sample Game

Start with 10 pieces at F_1, rest empty.

7	0	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

5	1	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_2 + F_3 = F_4$.

Sample Game

Start with 10 pieces at F_1, rest empty.

5	0	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

3	1	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

1	2	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_2 = F_3$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	1	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_3 + F_4 = F_5$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

No moves left, Player One wins.

Sample Game

Player One won in 9 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
0	1	1	1	0
0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Sample Game

Player Two won in 10 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
2	0	1	1	0
0	1	1	1	0
0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Games end

Theorem

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: $\left(\sqrt{k}+\sqrt{k}\right)-\sqrt{k+2}<0.$
- Splitting: $2\sqrt{k}-(\sqrt{k+1}+\sqrt{k+1})<0.$
- Adding 1's: $2\sqrt{1}-\sqrt{2}<0.$
- Splitting 2's: $2\sqrt{2}-\left(\sqrt{3}+\sqrt{1}\right)<0.$

Games Lengths: I

Upper bound: At most $n \log_\phi \left(n\sqrt{5} + 1/2\right)$ moves.

Fastest game: $n - Z(n)$ moves ($Z(n)$ is the number of summands in n's Zeckendorf decomposition).

From always moving on the largest summand possible (deterministic).

Games Lengths: II

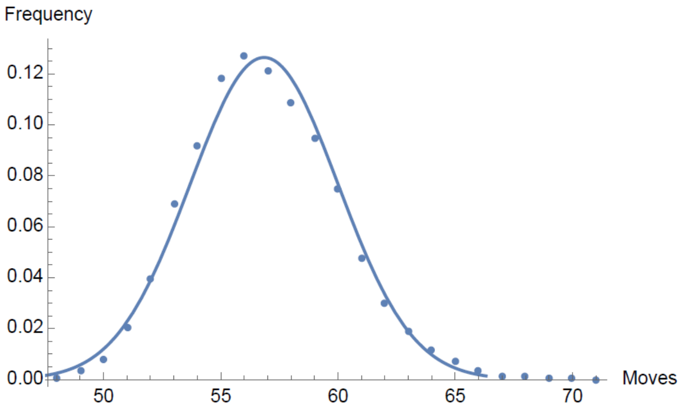


Figure: Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when $n = 60$ vs a Gaussian. Natural conjecture....

Winning Strategy

Theorem

Payer Two Has a Winning Strategy

Idea is to show if not, Player Two could steal Player One's strategy.

Non-constructive!

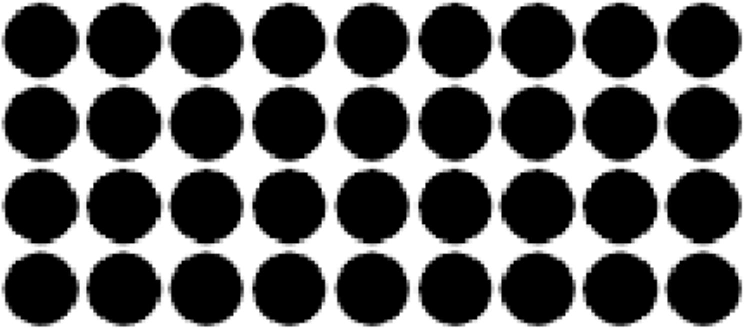
Will highlight idea with a simpler game.

Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at (i, j) and coloring every dot (m, n) with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Prove Player 1 has a winning strategy!



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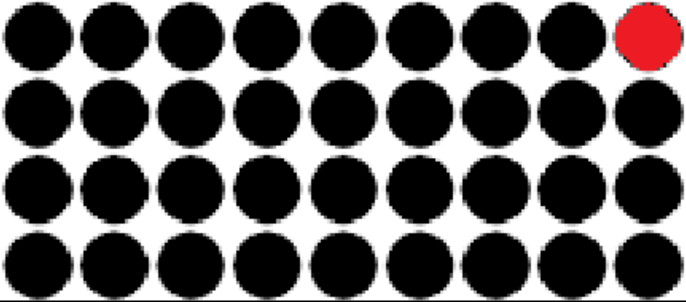
Proof Player 1 has a winning strategy. If have, play; if not, steal.



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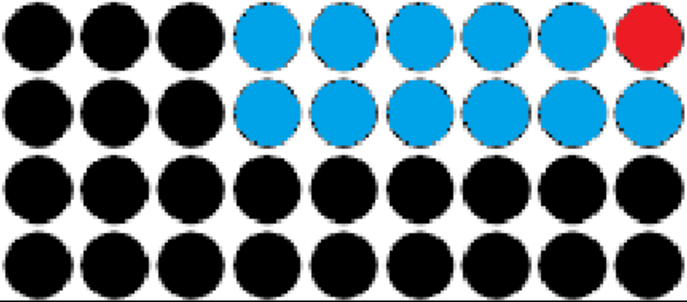

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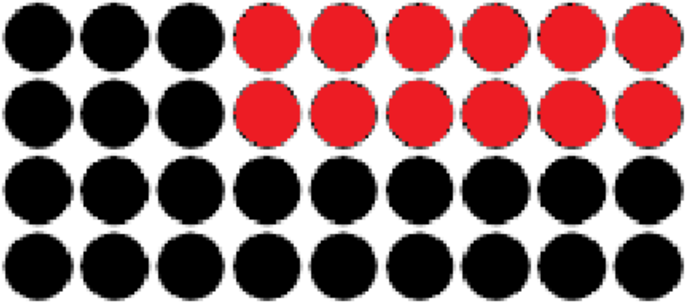
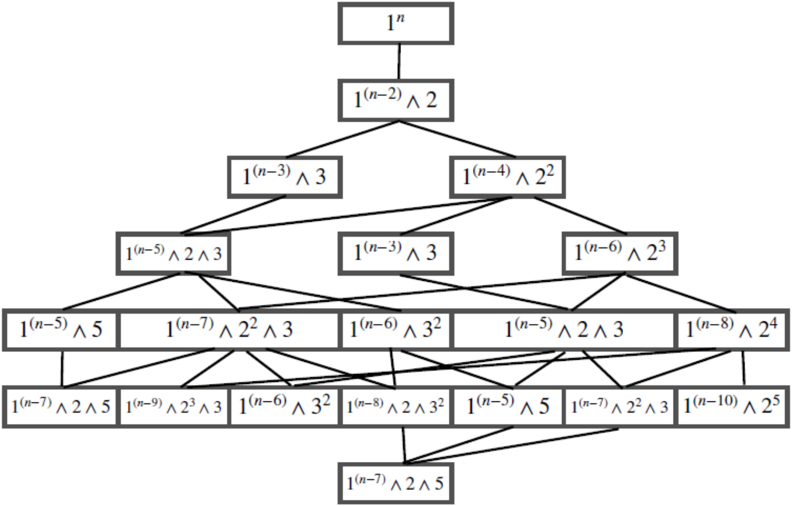


Sketch of Proof for Player Two's Winning Strategy



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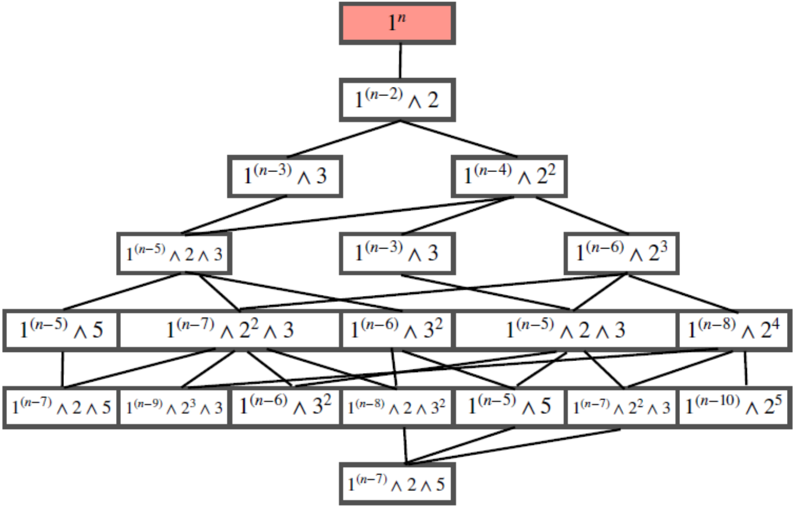

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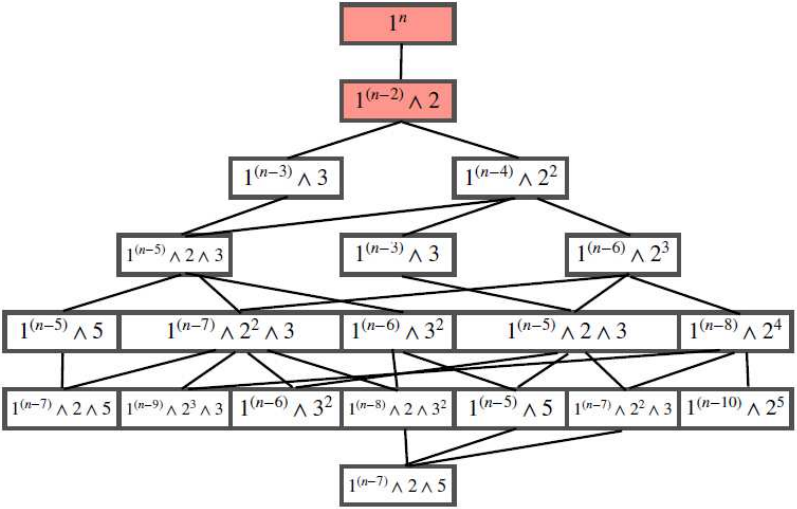

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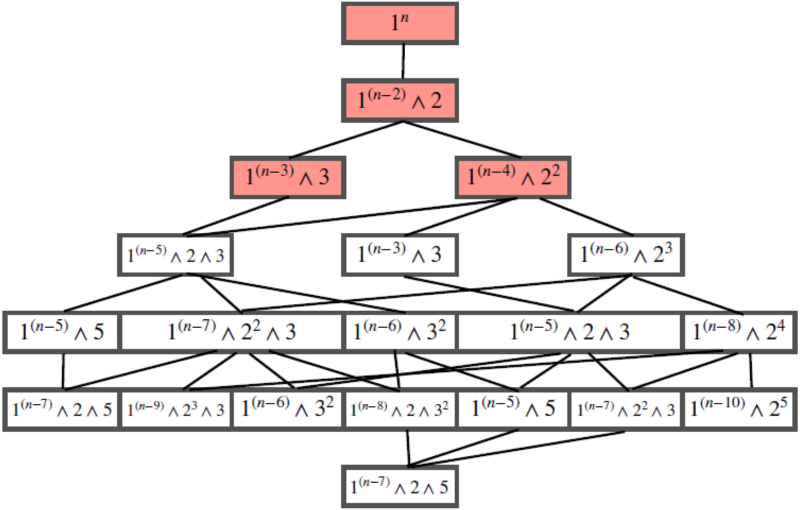

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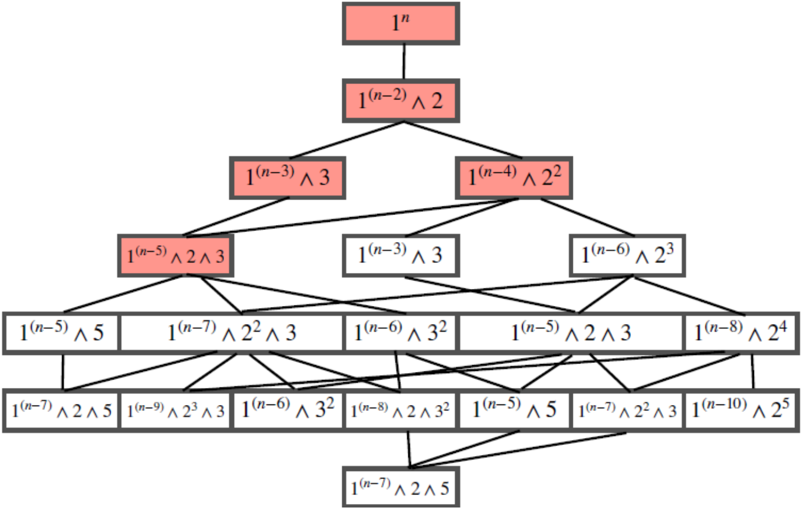

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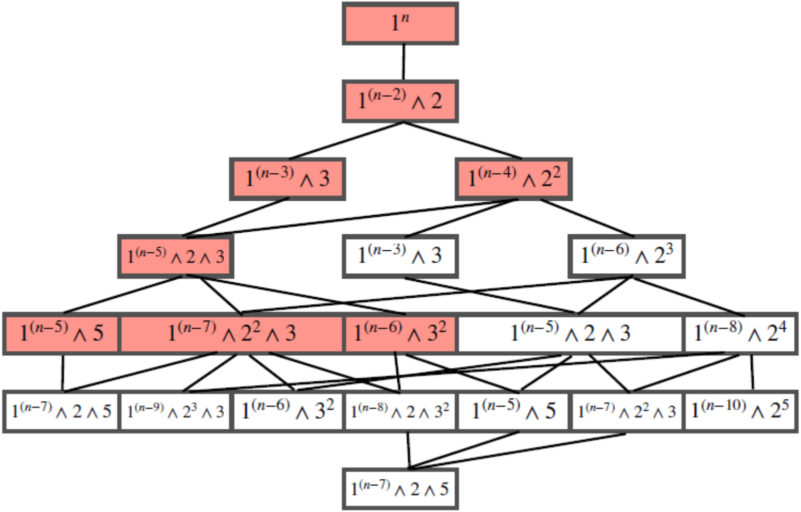

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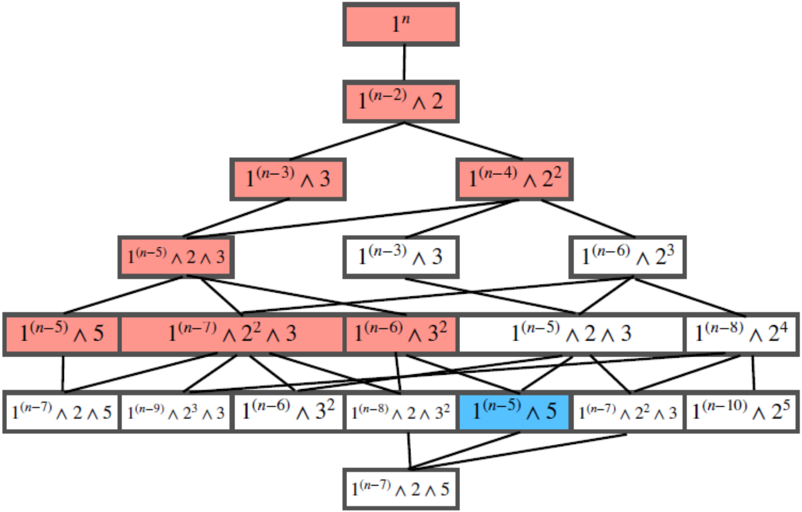

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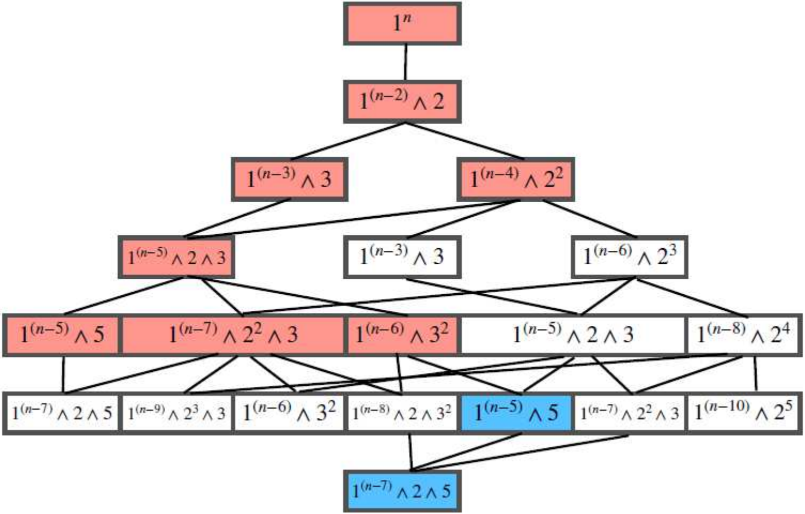

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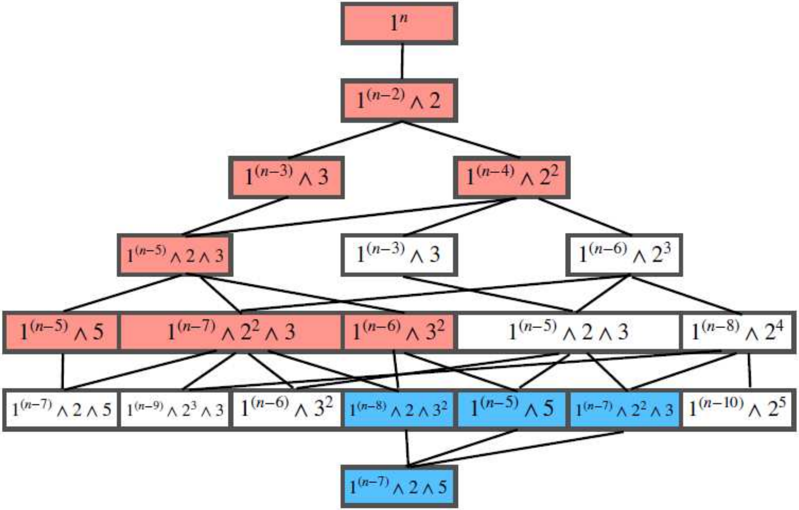

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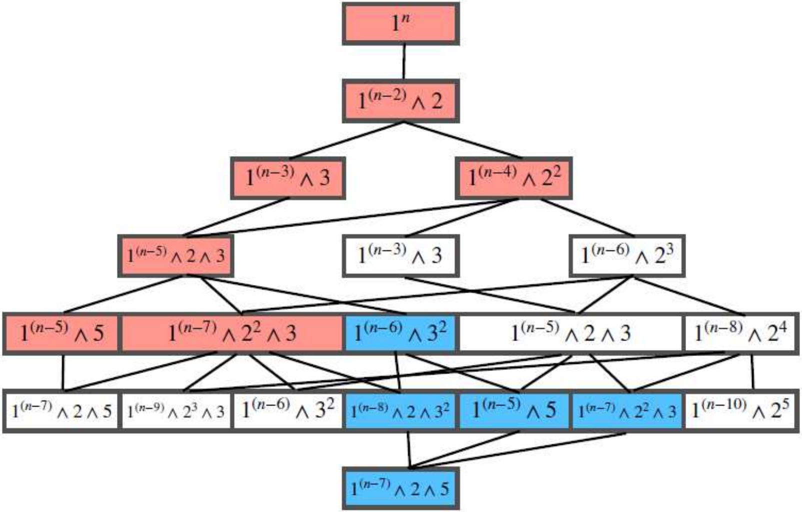


Future Work

- What if $p \geq 3$ people play the Fibonacci game?
- Does the number of moves in random games converge to a Gaussian?
- Define k-nacci numbers by $S_{i+1} = S_i + S_{i-1} + \cdots + S_{i-k}$; game terminates but who has the winning strategy?

Games

Games: Coins on a line

You have $2N$ coins of varying denominations (each is a non-negative real number) in a line. Players A and B take turns choosing one coin from either end. Does Player A or B have a winning strategy (i.e., a way to ensure they get at least as much as the other?) If yes, who has it and find it if possible!

Games: Devilish Coins

You die and the devil comes out to meet you. In the middle of the room is a giant circular table and next to the walls are many sacks of coins. The devil speaks. *We'll take turns putting coins down flat on the table. I'll put down a coin and then you'll put down a coin, and so on. The coins cannot overlap and they cannot hang over the edge of the table. The last person to put down a coin wins, or equivalently, the last person who can no longer put a coin down on the table loses. You decide if you want to go first.*

Do you have a winning strategy for the game? If yes, what?

Games: Prime Heaps

Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many such n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

$$\sqrt{2}$$

$\sqrt{2}$ Is Irrational

Standard Proof: Assume $\sqrt{2} = a/b$.

WLOG, assume b is the smallest denominator among all fractions that equal $\sqrt{2}$.

$2b^2 = a^2$ thus $a = 2m$ is even.

Then $2b^2 = 4m^2$ so $b^2 = 2m^2$ so $b = 2n$ is even.

Thus $\sqrt{2} = a/b = 2m/2n = m/n$, contradicts minimality of n.

(Could also do by contradiction from a, b relatively prime.)

Tennenbaum's Proof

Assume $\sqrt{2} = a/b$ with b minimal.

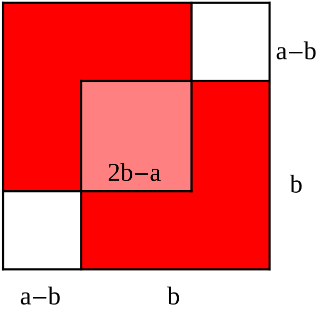


Figure: $2b^2 = a^2$ so $(2b-a)^2 = 2(a-b)^2$ and $\sqrt{2} = \frac{2b-a}{a-b}$.

As $0 < a - b < b$ (if not, $a - b \geq b$ so $a \geq 2b$ and $\sqrt{2} = a/b \geq 2$), contradicts minimality of b.

Challenge

WHAT OTHER NUMBERS HAVE GEOMETRIC IRRATIONALITY PROOFS?

More Irrationals

$\sqrt{3}$

Assume $\sqrt{3} = a/b$ with b minimal.

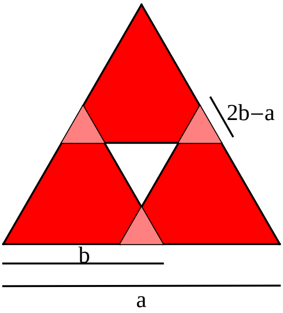


Figure: Geometric proof of the irrationality of $\sqrt{3}$. The white equilateral triangle in the middle has sides of length $2a - 3b$.

Have $3(2b-a)^2 = (2a-3b)^2$ so $\sqrt{3} = (2a-3b)/(2b-a)$, note $2b - a < b$ (else $b \geq a$), violates minimality.

$\sqrt{5}$

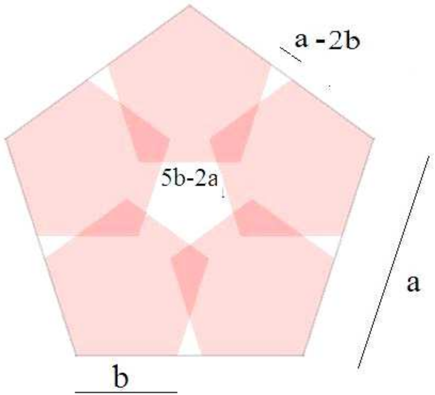
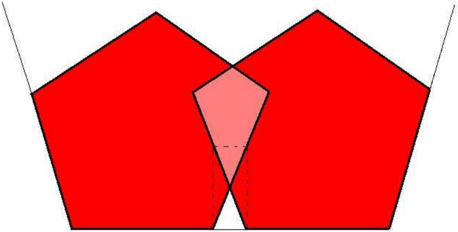
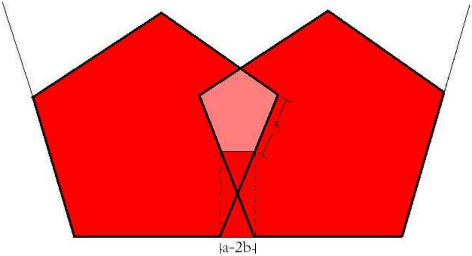


Figure: Geometric proof of the irrationality of $\sqrt{5}$.

$\sqrt{5}$





$\sqrt{5}$

A straightforward analysis shows that the five doubly covered pentagons are all regular, with side length $a - 2b$, and the middle pentagon is also regular, with side length $b - 2(a - 2b) = 5b - 2a$.

We now have a smaller solution, with the five doubly counted regular pentagons having the same area as the omitted pentagon in the middle. Specifically, we have $5(a - 2b)^2 = (5b - 2a)^2$; as $a = b\sqrt{5}$ and $2 < \sqrt{5} < 3$, note that $a - 2b < b$ and thus we have our contradiction.

$\sqrt{6}$

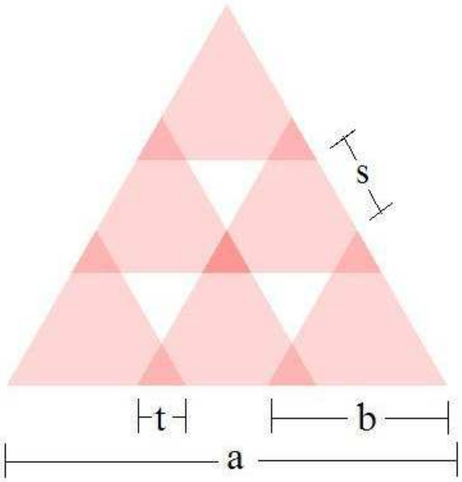


Figure: Geometric proof of the irrationality of $\sqrt{6}$.

Closing Thoughts

Could try to do $\sqrt{10}$ but eventually must break down. Note $3, 6, 10$ are triangular numbers ($T_n = n(n+1)/2$).

$T_8 = 36$ and thus $\sqrt{T_8}$ is an integer!

Can you get a cube-root?

What other numbers?