From Zombies to Fibonaccis: An Introduction to the Theory of Games

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http://www.williams.edu/Mathematics/sjmiller/public_html

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Zombies

General Advice: What are your tools and how can they be used?

Law of the Hammer:

- Abraham Kaplan: I call it the law of the instrument, and it may be formulated as follows: Give a small boy a hammer, and he will find that everything he encounters needs pounding.
- Abraham Maslow: I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.
- Bernard Baruch: If all you have is a hammer, everything looks like a nail.

Zombie Infection: Rules

- If share walls with 2 or more infected, become infected.
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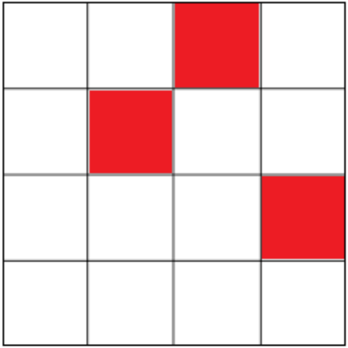

Initial Configuration

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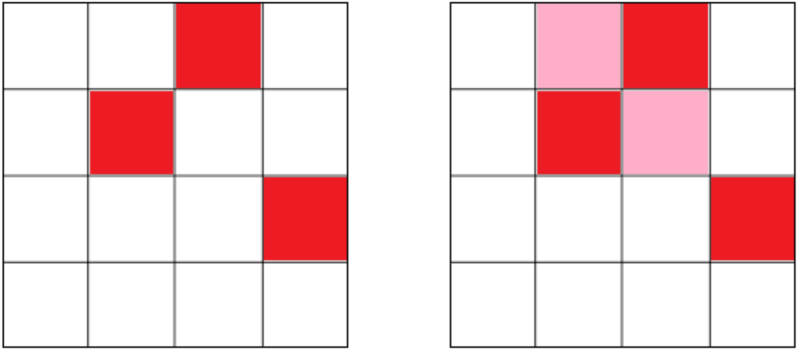

Initial Configuration One moment later

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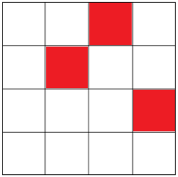

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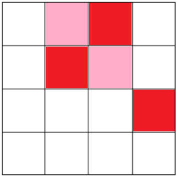

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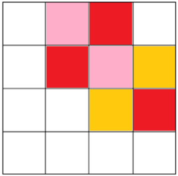

Two moments later

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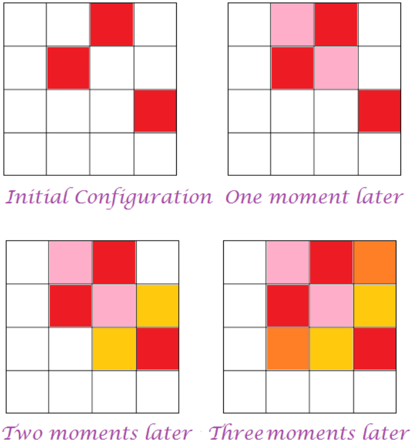
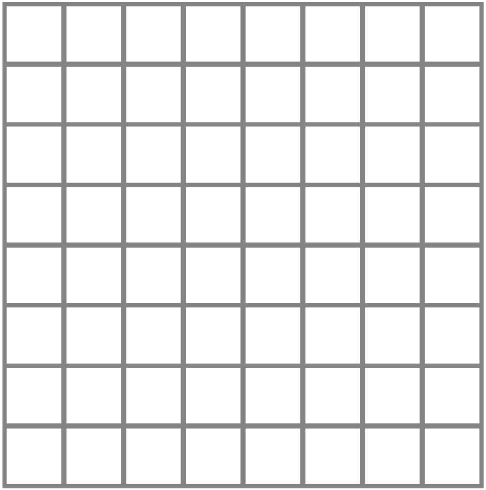


Zombie Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?



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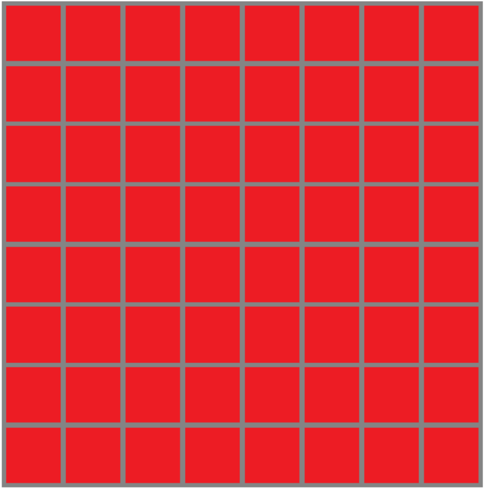

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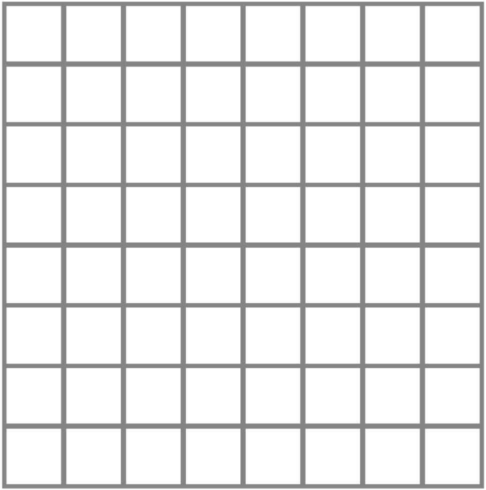

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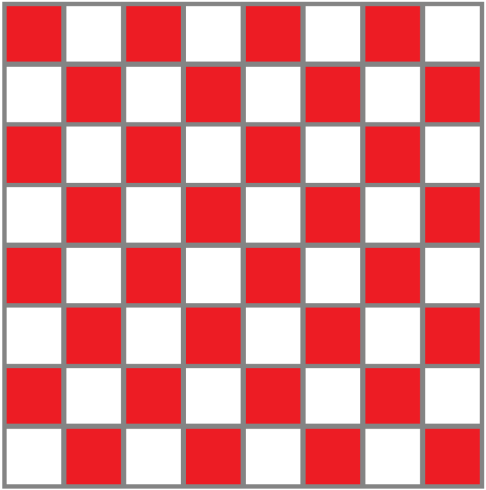

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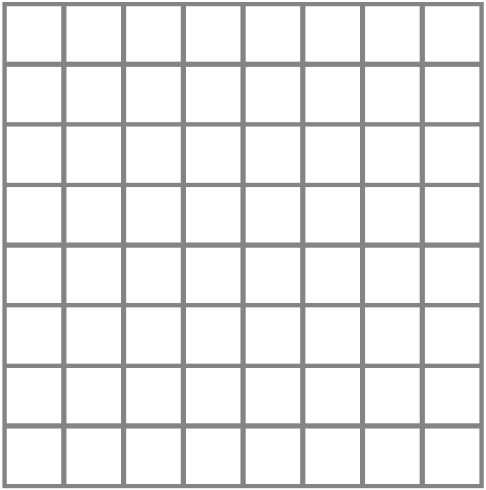
Fewest number of initial infections needed to get all...?



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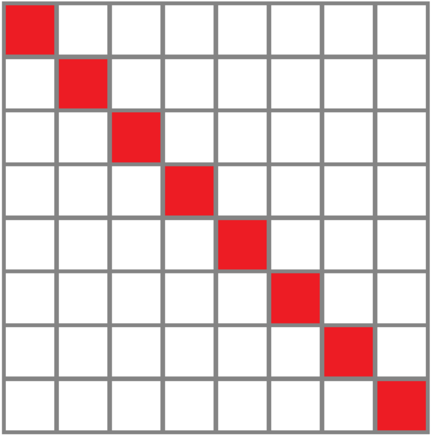

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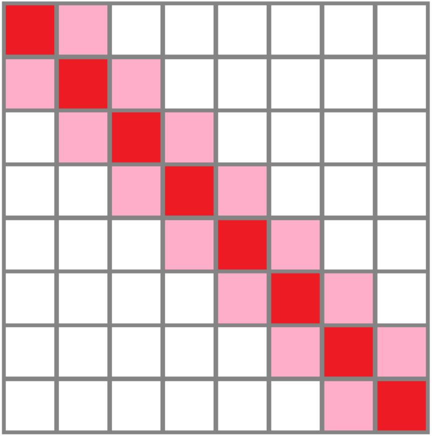

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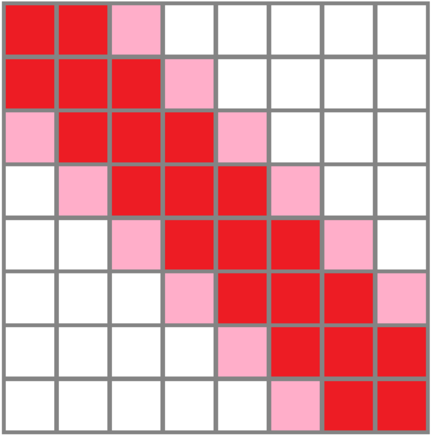

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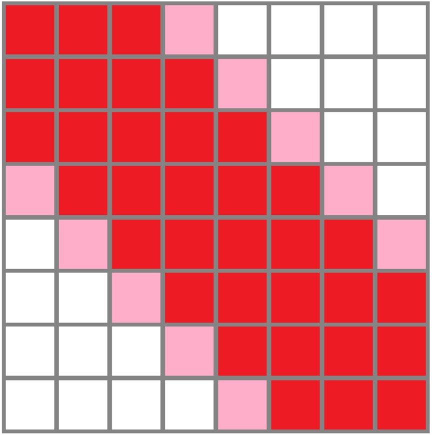

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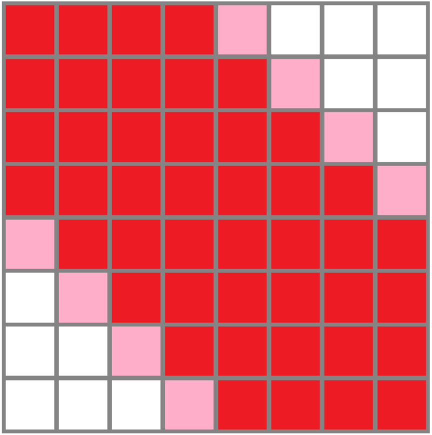

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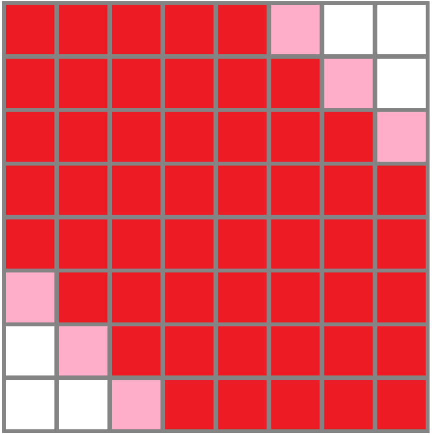

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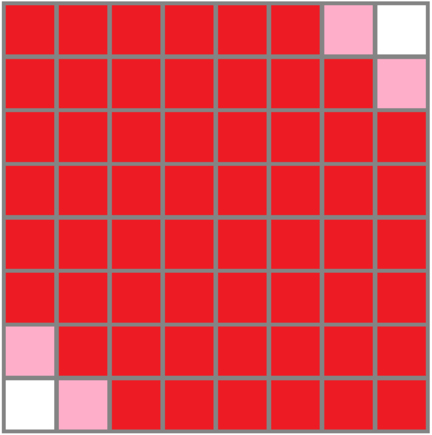

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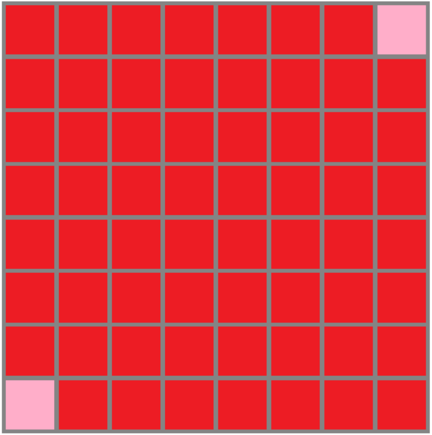

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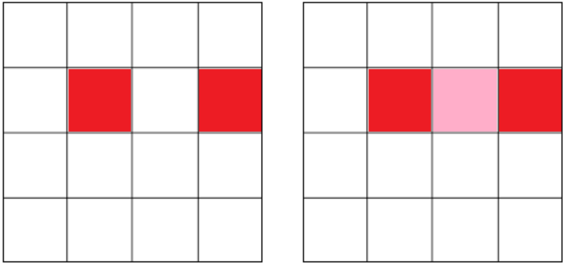

Perimeter of infection unchanged.

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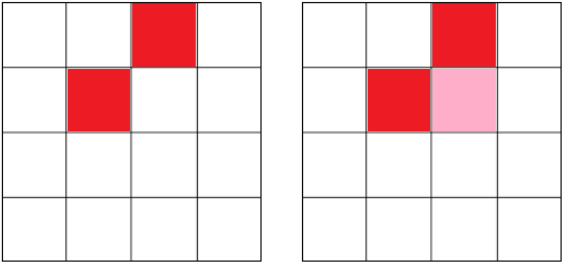

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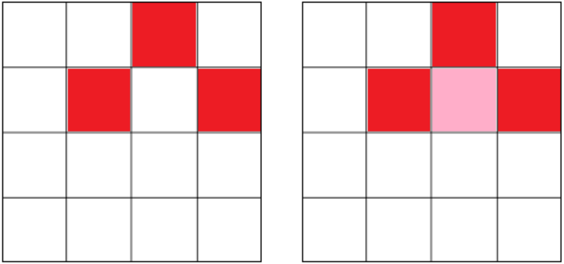
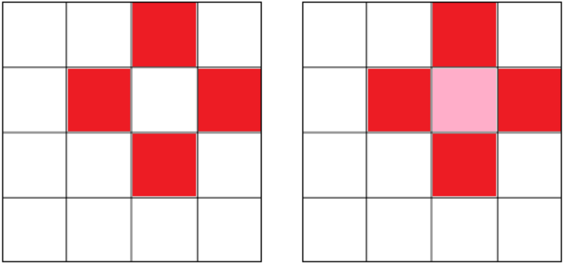


Perimeter of infection decreases by 2.

Zombie Infection: Can $n-1$ infect all?



Perimeter of infection decreases by 4.

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- Mono-variant: As time passes, perimeter of infection never increases.
- Perimeter of $n \times n$ square is $4n$, so at least 1 square safe!

Triangle Game

Rules for Triangle Game

Take an equilateral triangle, label corners 0, 1 and 2.

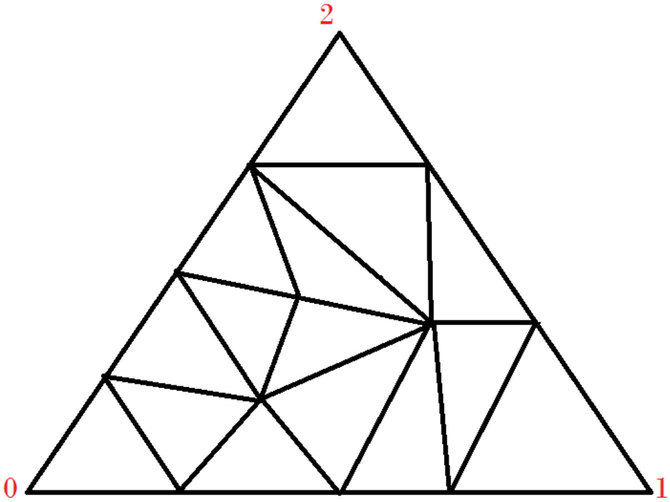
Subdivide however you wish into triangles.

Add labels, if a sub-triangle labeled 0–1–2 then Player 1 wins, else Player 2.

Take turns adding labels, subject to:
On 0–1 boundary must use 0 or 1
On 1–2 boundary must use 1 or 2
On 0–2 boundary must use 0 or 2

Who has the winning strategy? What is it?

Rules for Triangle Game



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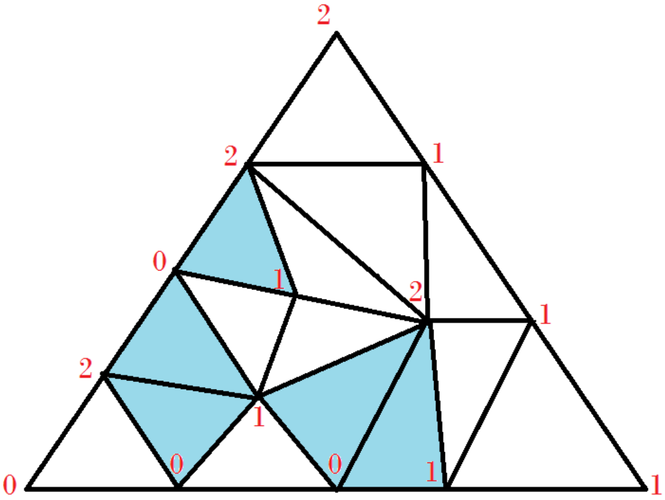
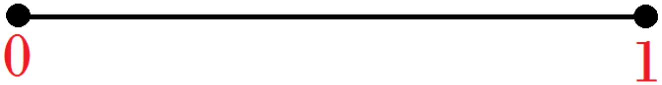


The Line Game

Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.



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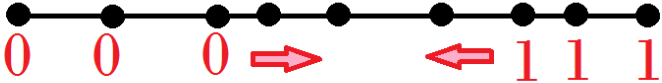

Cannot prevent at least one 0–1 segment.

The Line Game (cont)

Mono-variant: as add labels, number of 0–1 segments stays the same or increases by 2.

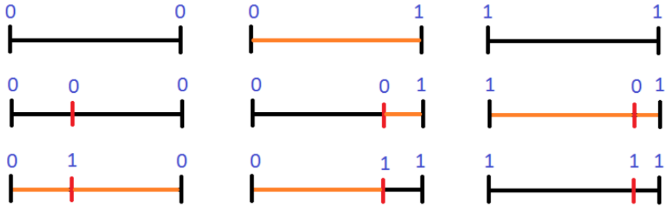

Figure 3. The various cases from the 1-dimensional Sperner proof. Notice in each of the six cases, the change in the number of 0–1 segments is even.

Can also view this as a parity argument.

Zeckendorf Games
with Cordwell, Epstein, Flynt, Hlavacek, Huynh, Peterson, Vu

Zeckendorf's Theorem

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$.

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Example: $51 = 34 + 13 + 4 = F_8 + F_6 + 4$.

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Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + 1$.

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Example: $51 = 34 + 13 + 3 + 1 = F_8 + F_6 + F_3 + F_1$.
Example: $83 = 55 + 21 + 5 + 2 = F_9 + F_7 + F_4 + F_2$.
Observe: 51 miles $\approx$ 82.1 kilometers.

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Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

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Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8, 13. . .

Summand Minimality

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

Theorem

The Zeckendorf decomposition is ***summand minimal****.*

Overall Question

What other recurrences are summand minimal?

Positive Linear Recurrence Sequences

Definition

A **positive linear recurrence sequence (PLRS)** is the sequence given by a recurrence $\{a_n\}$ with

$$a_n \;:=\; c_1 a_{n-1} + \cdots + c_t a_{n-t}$$

and each $c_i \geq 0$ and $c_1, c_t > 0$. We use **ideal initial conditions** $a_{-(n-1)} = 0, \ldots, a_{-1} = 0, a_0 = 1$ and call $(c_1, \ldots, c_t)$ the **signature of the sequence.**

Theorem (Cordwell, Hlavacek, Huynh, M., Peterson, Vu)

For a PLRS with signature $(c_1, c_2, \ldots, c_t)$, the Generalized Zeckendorf Decompositions are summand minimal if and only if

$$c_1 \geq c_2 \geq \cdots \geq c_t.$$

Proof for Fibonacci Case

Idea of proof:

- $\mathcal{D} = b_1 F_1 + \cdots + b_n F_n$ decomposition of N, set $\mathrm{Ind}(\mathcal{D}) = b_1 \cdot 1 + \cdots + b_n \cdot n$.

- Move to $\mathcal{D}'$ by
  - ◇ $2F_k = F_{k+1} + F_{k-2}$ (and $2F_2 = F_3 + F_1$).
  - ◇ $F_k + F_{k+1} = F_{k+2}$ (and $F_1 + F_1 = F_2$).

- Monovariant: Note $\mathrm{Ind}(\mathcal{D}') \leq \mathrm{Ind}(\mathcal{D})$.
  - ◇ $2F_k = F_{k+1} + F_{k-2}$: $2k$ vs $2k - 1$.
  - ◇ $F_k + F_{k+1} = F_{k+2}$: $2k + 1$ vs $k + 2$.

- If not at Zeckendorf decomposition can continue, if at Zeckendorf cannot. Better: $\mathrm{Ind}'(\mathcal{D}) = b_1\sqrt{1} + \cdots + b_n\sqrt{n}$.

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Questions:

- Does the game end? How long?
- For each N who has the winning strategy?
- What is the winning strategy?

Sample Game

Start with 10 pieces at F_1, rest empty.

10	0	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

8	1	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

6	2	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $2F_2 = F_3 + F_1$

Sample Game

Start with 10 pieces at F_1, rest empty.

7	0	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

5	1	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_2 + F_3 = F_4$.

Sample Game

Start with 10 pieces at F_1, rest empty.

5	0	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

3	1	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

1	2	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_2 = F_3$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	1	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_3 + F_4 = F_5$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

No moves left, Player One wins.

Sample Game

Player One won in 9 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
0	1	1	1	0
0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Sample Game

Player Two won in 10 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
2	0	1	1	0
0	1	1	1	0
0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Games end

Theorem

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: $\left(\sqrt{k}+\sqrt{k}\right)-\sqrt{k+2}<0$.
- Splitting: $2\sqrt{k}-\left(\sqrt{k+1}+\sqrt{k+1}\right)<0$.
- Adding 1's: $2\sqrt{1}-\sqrt{2}<0$.
- Splitting 2's: $2\sqrt{2}-\left(\sqrt{3}+\sqrt{1}\right)<0$.

Games Lengths: I

Upper bound: At most $n \log_\phi \left(n\sqrt{5} + 1/2\right)$ moves.

Fastest game: $n - Z(n)$ moves ($Z(n)$ is the number of summands in n's Zeckendorf decomposition).

From always moving on the largest summand possible (deterministic).

Games Lengths: II

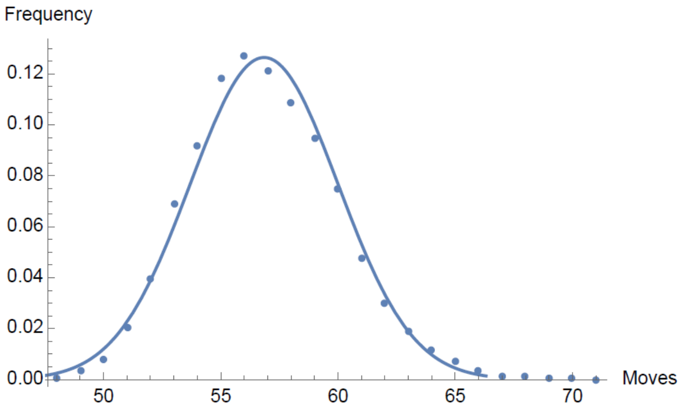


Figure: Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when $n = 60$ vs a Gaussian. Natural conjecture....

Winning Strategy

Theorem

Payer Two Has a Winning Strategy

Idea is to show if not, Player Two could steal Player One's strategy.

Non-constructive!

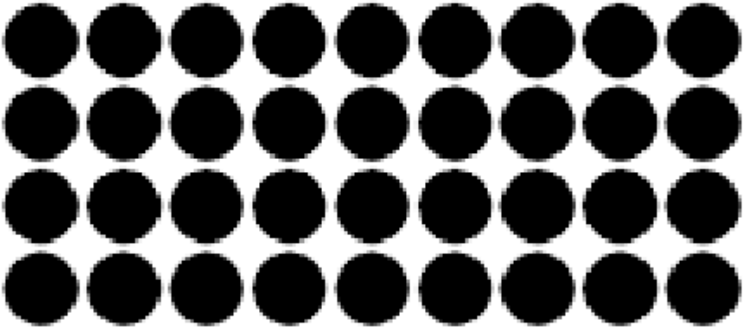
Will highlight idea with a simpler game.

Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at (i, j) and coloring every dot (m, n) with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Prove Player 1 has a winning strategy!



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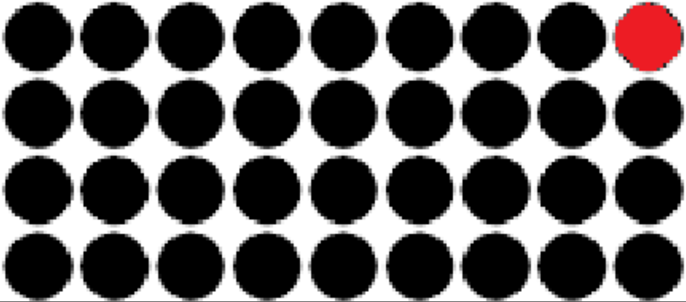
Proof Player 1 has a winning strategy. If have, play; if not, steal.



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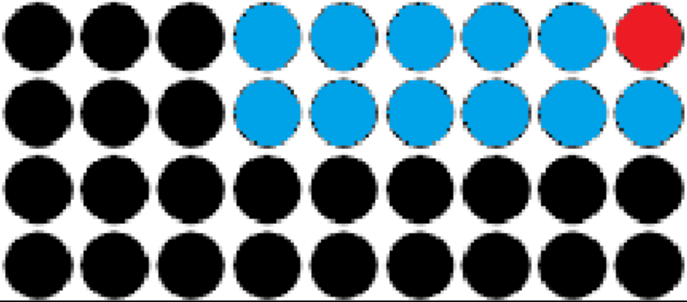

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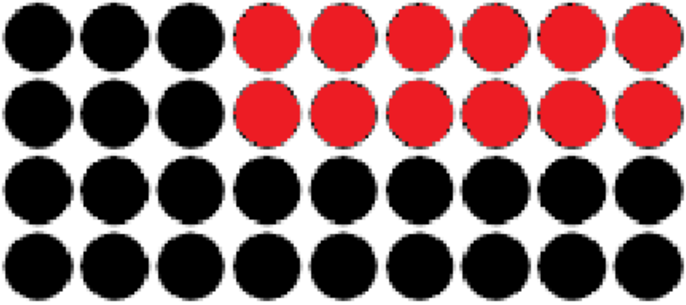
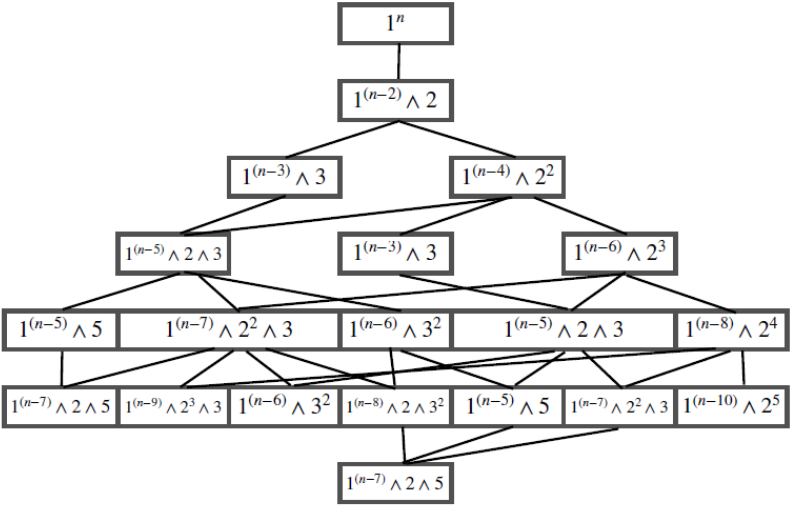


Sketch of Proof for Player Two's Winning Strategy



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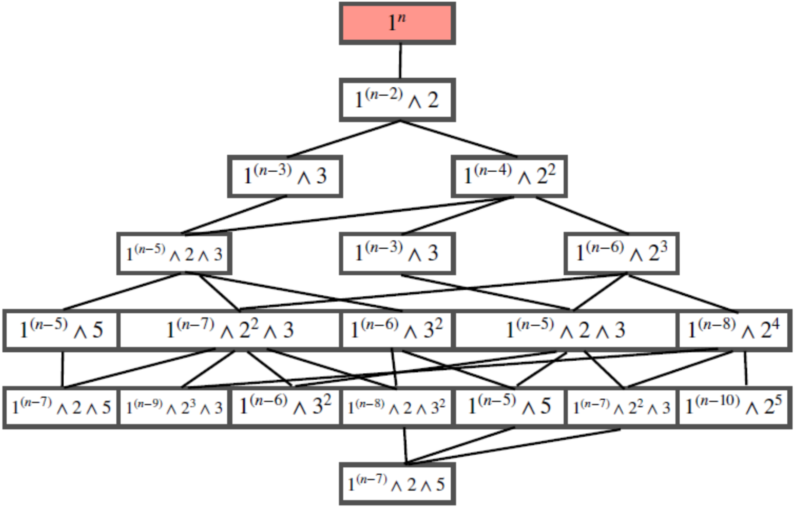

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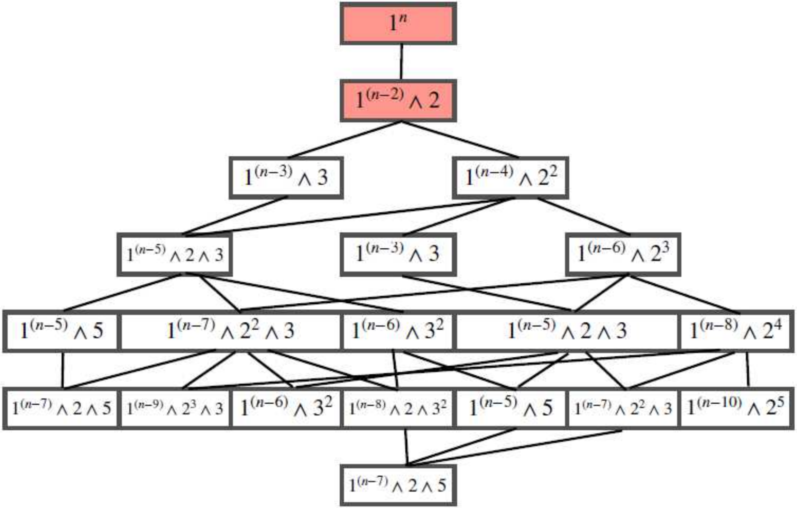

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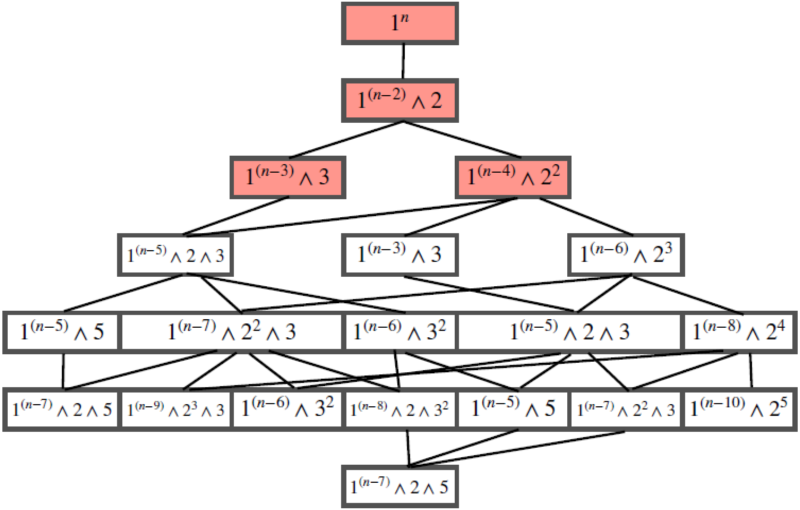

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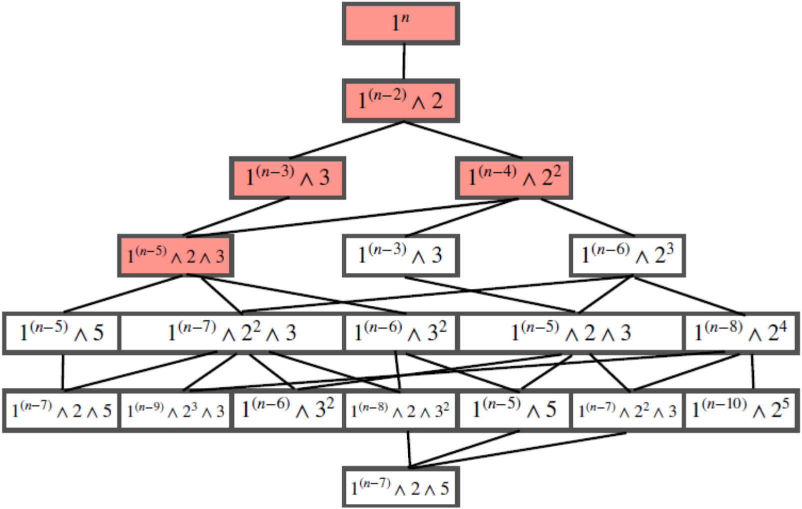

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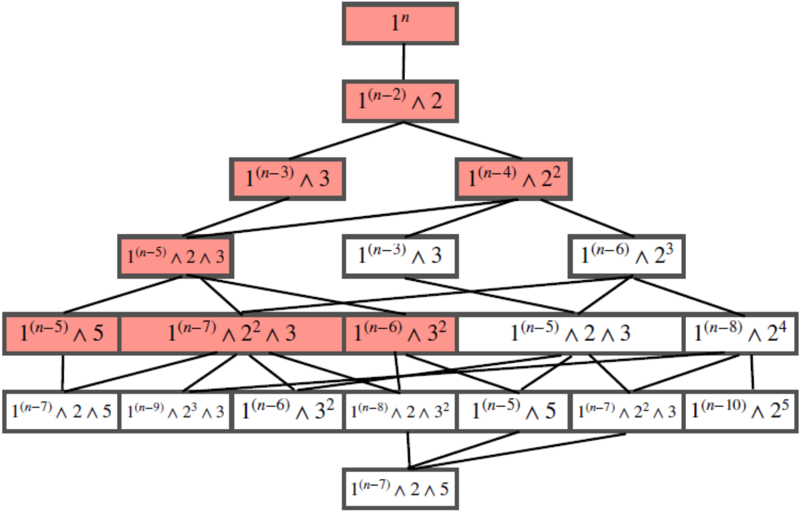

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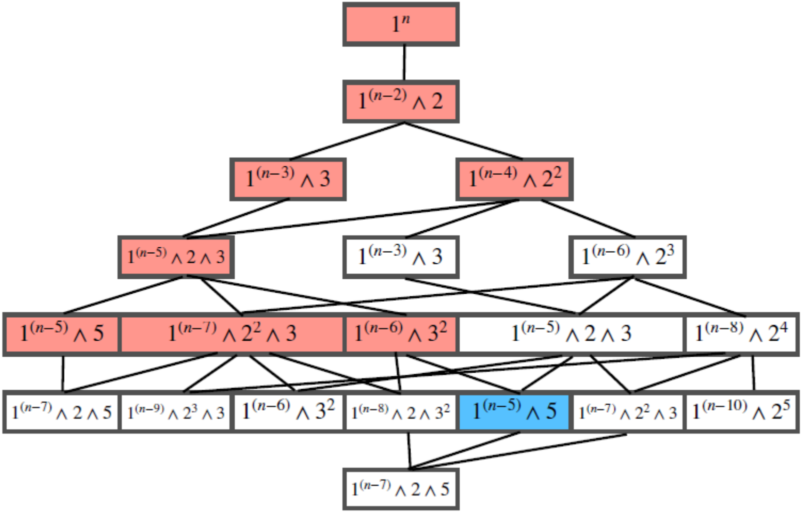

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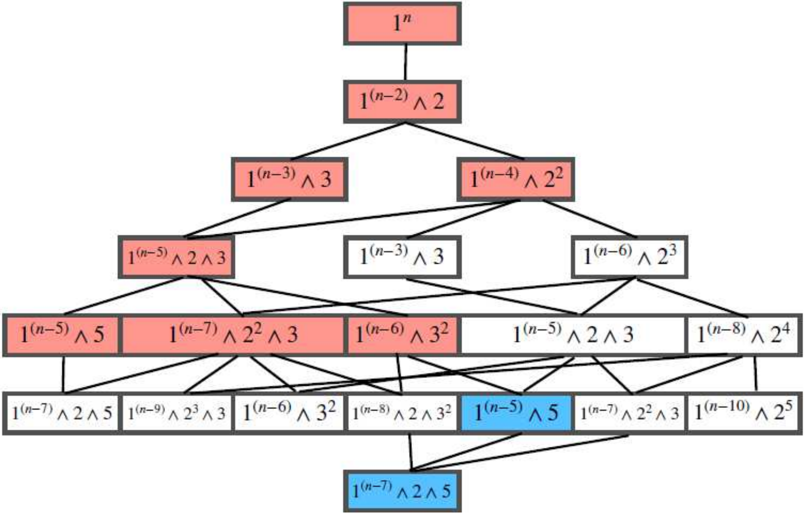

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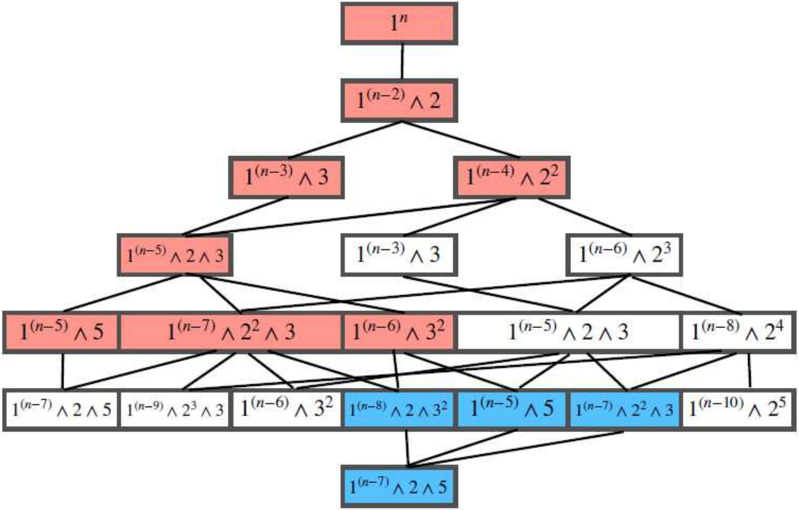

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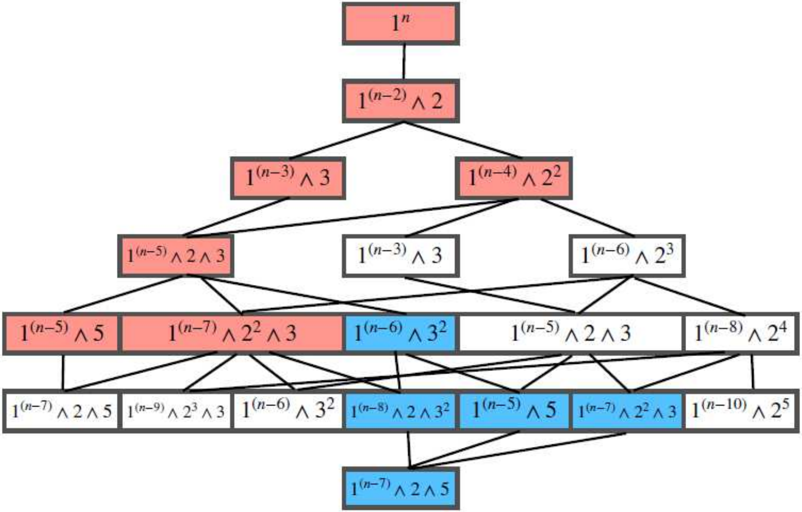


Future Work

- What if $p \geq 3$ people play the Fibonacci game?
- Does the number of moves in random games converge to a Gaussian?
- Define k-nacci numbers by $S_{i+1} = S_i + S_{i-1} + \cdots + S_{i-k}$; game terminates but who has the winning strategy?

Games

Games: Coins on a line

You have $2N$ coins of varying denominations (each is a non-negative real number) in a line. Players A and B take turns choosing one coin from either end. Does Player A or B have a winning strategy (i.e., a way to ensure they get at least as much as the other?) If yes, who has it and find it if possible!

Games: Devilish Coins

You die and the devil comes out to meet you. In the middle of the room is a giant circular table and next to the walls are many sacks of coins. The devil speaks. *We'll take turns putting coins down flat on the table. I'll put down a coin and then you'll put down a coin, and so on. The coins cannot overlap and they cannot hang over the edge of the table. The last person to put down a coin wins, or equivalently, the last person who can no longer put a coin down on the table loses. You decide if you want to go first.*

Do you have a winning strategy for the game? If yes, what?

Games: Prime Heaps

Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many such n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

$\sqrt{2}$

$\sqrt{2}$ Is Irrational

Standard Proof: Assume $\sqrt{2} = a/b$.

WLOG, assume b is the smallest denominator among all fractions that equal $\sqrt{2}$.

$2b^2 = a^2$ thus $a = 2m$ is even.

Then $2b^2 = 4m^2$ so $b^2 = 2m^2$ so $b = 2n$ is even.

Thus $\sqrt{2} = a/b = 2m/2n = m/n$, contradicts minimality of n.

(Could also do by contradiction from a, b relatively prime.)

Tennenbaum's Proof

Assume $\sqrt{2} = a/b$ with b minimal.

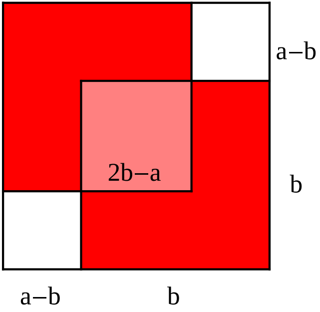


Figure: $2b^2 \;=\; a^2 \quad \text{so} \quad (2b-a)^2 = 2(a-b)^2 \quad \text{and} \quad \sqrt{2} = \frac{2b-a}{a-b}$.

As $0 < a - b < b$ (if not, $a - b \geq b$ so $a \geq 2b$ and $\sqrt{2} = a/b \geq 2$), contradicts minimality of b.

Challenge

WHAT OTHER NUMBERS HAVE GEOMETRIC IRRATIONALITY PROOFS?

More Irrationals

$\sqrt{3}$

Assume $\sqrt{3} = a/b$ with b minimal.

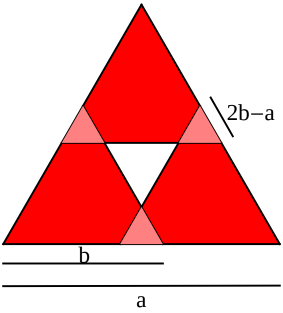


Figure: Geometric proof of the irrationality of $\sqrt{3}$. The white equilateral triangle in the middle has sides of length $2a - 3b$.

Have $3(2b-a)^2 = (2a-3b)^2$ so $\sqrt{3} = (2a-3b)/(2b-a)$, note $2b - a < b$ (else $b \geq a$), violates minimality.

$\sqrt{5}$

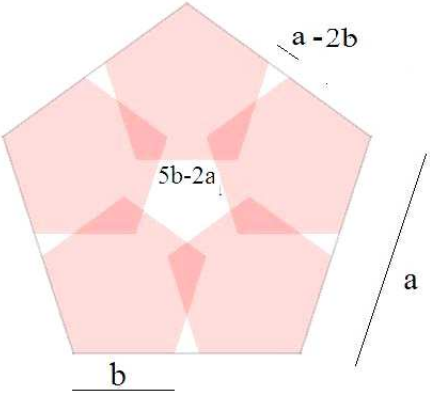
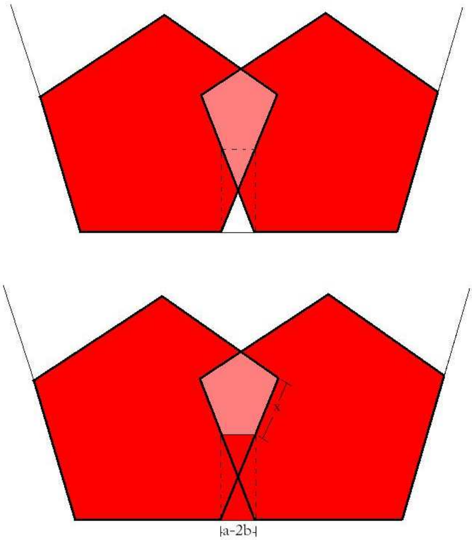


Figure: Geometric proof of the irrationality of $\sqrt{5}$.

$\sqrt{5}$



$\sqrt{5}$

A straightforward analysis shows that the five doubly covered pentagons are all regular, with side length $a - 2b$, and the middle pentagon is also regular, with side length $b - 2(a - 2b) = 5b - 2a$.

We now have a smaller solution, with the five doubly counted regular pentagons having the same area as the omitted pentagon in the middle. Specifically, we have $5(a - 2b)^2 = (5b - 2a)^2$; as $a = b\sqrt{5}$ and $2 < \sqrt{5} < 3$, note that $a - 2b < b$ and thus we have our contradiction.

$\sqrt{6}$

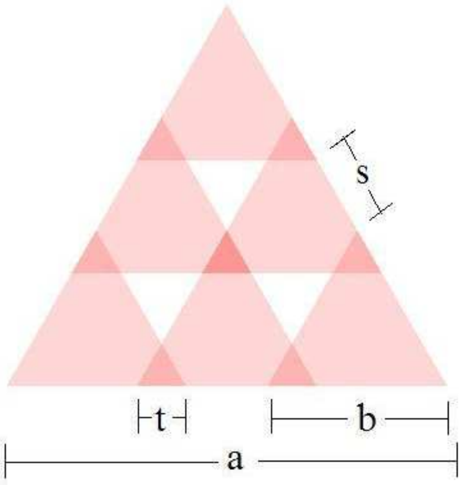


Figure: Geometric proof of the irrationality of $\sqrt{6}$.

Closing Thoughts

Could try to do $\sqrt{10}$ but eventually must break down. Note $3, 6, 10$ are triangular numbers ($T_n = n(n+1)/2$).

$T_8 = 36$ and thus $\sqrt{T_8}$ is an integer!

Can you get a cube-root?

What other numbers?