From Zombies to Fibonacci: An Introduction to the Theory of Games

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http://www.williams.edu/Mathematics/sjmiller/public_html

Michigan Math Circle: April 23, 2020
General Advice: What are your tools and how can they be used?

Law of the Hammer:

- Abraham Kaplan: I call it the law of the instrument, and it may be formulated as follows: Give a small boy a hammer, and he will find that everything he encounters needs pounding.

- Abraham Maslow: I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.

- Bernard Baruch: If all you have is a hammer, everything looks like a nail.
Zombie Infection: Rules

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.
Zombie Infection: Rules

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*Initial Configuration*
Zombie Infection: Rules

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Zombie Infection: Rules

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![Initial Configuration](image)

![One moment later](image)

![Two moments later](image)
Zombie Infection: Rules

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.

Initial Configuration  One moment later

Two moments later  Three moments later
Zombie Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?
Zombie Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?
Zombie Infection: Conquering The World

Next simplest initial state ensuring all eventually infected...?
Zombie Infection: Conquering The World

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Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?

![Grid with red squares indicating infections](image-url)
Zombie Infection: Conquering The World

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![Matrix with red and pink squares]
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?
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Fewest number of initial infections needed to get all...?
Zombie Infection: Can $n - 1$ infect all?
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Perimeter of infection unchanged.
Zombie Infection: Can $n - 1$ infect all?

Perimeter of infection unchanged.
Zombie Infection: Can $n - 1$ infect all?

Perimeter of infection decreases by 2.
Zombie Infection: Can \( n - 1 \) infect all?

Perimeter of infection decreases by 4.
Zombie Infection: $n - 1$ cannot infect all

- If $n - 1$ infected, maximum perimeter is $4(n - 1) = 4n - 4$. 
Zombie Infection: $n - 1$ cannot infect all

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- Mono-variant: As time passes, perimeter of infection never increases.
Zombie Infection: $n - 1$ cannot infect all

- If $n - 1$ infected, maximum perimeter is $4(n - 1) = 4n - 4$.

- **Mono-variant**: As time passes, perimeter of infection never increases.

- Perimeter of $n \times n$ square is $4n$, so at least 1 square safe!
Triangle Game
Rules for Triangle Game

Take an equilateral triangle, label corners 0, 1 and 2.

Subdivide however you wish into triangles.

Add labels, if a sub-triangle labeled 0–1–2 then Player 1 wins, else Player 2.

Take turns adding labels, subject to:
- On 0–1 boundary must use 0 or 1
- On 1–2 boundary must use 1 or 2
- On 0–2 boundary must use 0 or 2

Who has the winning strategy? What is it?
Rules for Triangle Game
Rules for Triangle Game
The Line Game

Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.
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Player 1 wins, else Player 2 wins.
The Line Game

Consider one-dimensional analogue: if have a 0–1 segment Player 1 wins, else Player 2 wins.

Cannot prevent at least one 0–1 segment.
The Line Game (cont)

Mono-variant: as add labels, number of 0–1 segments stays the same or increases by 2.

![Diagram of line segments showing various cases from the 1-dimensional Sperner proof](image)

**Figure 3.** The various cases from the 1-dimensional Sperner proof. Notice in each of the six cases, the change in the number of 0–1 segments is even.

Can also view this as a parity argument.
Zeckendorf Games with Cordwell, Epstein, Flynt, Hlavacek, Huynh, Peterson, Vu
Zeckendorf’s Theorem

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .
Zeckendorf’s Theorem

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Zeckendorf’s Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.
Zeckendorf’s Theorem

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Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: 51 =?
Zeckendorf’s Theorem

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Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: 51 = 34 + 17 = F_8 + 17.
Zeckendorf’s Theorem

Fibonacci Numbers: $F_{n+1} = F_n + F_{n-1}$;
First few: $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$.

Zeckendorf’s Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: $51 = 34 + 13 + 4 = F_8 + F_6 + 4$. 
Zeckendorf’s Theorem

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First few: 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, . . .

Zeckendorf’s Theorem

Every positive integer can be written uniquely as a sum of non-consecutive Fibonacci numbers.

Example: 51 = 34 + 13 + 3 + 1 = \( F_8 + F_6 + F_3 + 1 \).
Zeckendorf’s Theorem

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Example: 51 = 34 + 13 + 3 + 1 = \( F_8 + F_6 + F_3 + F_1 \).
Example: 83 = 55 + 21 + 5 + 2 = \( F_9 + F_7 + F_4 + F_2 \).
Observe: 51 miles \( \approx 82.1 \) kilometers.
Introduction: Summand Minimality

Fibonacci Numbers: \( F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n. \)

Zeckendorf’s Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
\[ 2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2. \]
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Conversely, we can construct the Fibonacci sequence using this property:

1
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$1, 2$
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**Zeckendorf’s Theorem**

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

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\[ 2018 = 1597 + 377 + 34 + 8 + 2 = F_{16} + F_{13} + F_8 + F_5 + F_2. \]

Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3
Introduction: Summand Minimality

Fibonaccis: \( F_1 = 1, \ F_2 = 2, \ F_3 = 3, \ F_4 = 5, \ F_{n+2} = F_{n+1} + F_n. \)

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Conversely, we can construct the Fibonacci sequence using this property:
$1, 2, 3, 5, 8, 13, \ldots$
Summand Minimality

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

Theorem

The Zeckendorf decomposition is summand minimal.

Overall Question

What other recurrences are summand minimal?
Definition

A **positive linear recurrence sequence (PLRS)** is the sequence given by a recurrence \( \{a_n\} \) with

\[
a_n := c_1 a_{n-1} + \cdots + c_t a_{n-t}
\]

and each \( c_i \geq 0 \) and \( c_1, c_t > 0 \). We use **ideal initial conditions** \( a_{-(n-1)} = 0, \ldots, a_{-1} = 0, a_0 = 1 \) and call \( (c_1, \ldots, c_t) \) the **signature of the sequence**.

Theorem (Cordwell, Hlavacek, Huynh, M., Peterson, Vu)

For a PLRS with signature \( (c_1, c_2, \ldots, c_t) \), the Generalized Zeckendorf Decompositions are summand minimal if and only if

\[
c_1 \geq c_2 \geq \cdots \geq c_t.
\]
Proof for Fibonacci Case

Idea of proof:

- \( \mathcal{D} = b_1 F_1 + \cdots + b_n F_n \) decomposition of \( N \), set \( \text{Ind}(\mathcal{D}) = b_1 \cdot 1 + \cdots + b_n \cdot n \).

- Move to \( \mathcal{D}' \) by
  - \( 2F_k = F_{k+1} + F_{k-2} \) (and \( 2F_2 = F_3 + F_1 \)).
  - \( F_k + F_{k+1} = F_{k+2} \) (and \( F_1 + F_1 = F_2 \)).

- Monovariant: Note \( \text{Ind}(\mathcal{D}') \leq \text{Ind}(\mathcal{D}) \).
  - \( 2F_k = F_{k+1} + F_{k-2} \): \( 2k \) vs \( 2k - 1 \).
  - \( F_k + F_{k+1} = F_{k+2} \): \( 2k + 1 \) vs \( k + 2 \).

- If not at Zeckendorf decomposition can continue, if at Zeckendorf cannot. Better: \( \text{Ind}'(\mathcal{D}) = b_1 \sqrt{1} + \cdots + b_n \sqrt{n} \).
Rules

- Two player game, alternate turns, last to move wins.
Rules

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- Bins $F_1, F_2, F_3, \ldots$, start with $N$ pieces in $F_1$ and others empty.
Rules

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- Bins $F_1$, $F_2$, $F_3$, ..., start with $N$ pieces in $F_1$ and others empty.

- A turn is one of the following moves:
  - If have two pieces on $F_k$ can remove and put one piece at $F_{k+1}$ and one at $F_{k-2}$
    (if $k = 1$ then $2F_1$ becomes $1F_2$)
  - If pieces at $F_k$ and $F_{k+1}$ remove and add one at $F_{k+2}$. 
Rules

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- A turn is one of the following moves:
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    \hspace{1cm} (if $k = 1$ then $2F_1$ becomes $1F_2$)
  - If pieces at $F_k$ and $F_{k+1}$ remove and add one at $F_{k+2}$.

Questions:
- Does the game end? How long?
- For each $N$ who has the winning strategy?
- What is the winning strategy?
Sample Game

Start with 10 pieces at $F_1$, rest empty.

$\begin{array}{cccccc}
10 & 0 & 0 & 0 & 0 \\
\end{array}$

Next move: Player 1: $F_1 + F_1 = F_2$
Sample Game

Start with 10 pieces at $F_1$, rest empty.

<table>
<thead>
<tr>
<th></th>
<th>8</th>
<th>1</th>
<th>0</th>
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<tbody>
<tr>
<td>$F_1 = 1$</td>
<td>$F_2 = 2$</td>
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<td>$F_4 = 5$</td>
<td>$F_5 = 8$</td>
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</table>

Next move: Player 2: $F_1 + F_1 = F_2$
## Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{ccccc}
6 & 2 & 0 & 0 & 0 \\
\end{array}
\]

Next move: Player 1: $2F_2 = F_3 + F_1$
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{ccccc}
7 & 0 & 1 & 0 & 0 \\
\end{array}
\]

Next move: Player 2: $F_1 + F_1 = F_2$
Sample Game

Start with 10 pieces at \( F_1 \), rest empty.

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<tr>
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<th>5</th>
<th>1</th>
<th>1</th>
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<td>([F_1 = 1])</td>
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<td>([F_3 = 3])</td>
<td>([F_4 = 5])</td>
<td>([F_5 = 8])</td>
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</tbody>
</table>

Next move: Player 1: \( F_2 + F_3 = F_4 \).
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{cccccc}
5 & 0 & 0 & 1 & 0 \\
\end{array}
\]

Next move: Player 2: $F_1 + F_1 = F_2$. 
Sample Game

Start with 10 pieces at $F_1$, rest empty.

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</table>

Next move: Player 1: $F_1 + F_1 = F_2$. 
Sample Game

Start with 10 pieces at $F_1$, rest empty.

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<tr>
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<th>1</th>
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<td>$[F_3 = 3]$</td>
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Next move: Player 2: $F_1 + F_2 = F_3$. 
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{cccccc}
0 & 1 & 1 & 1 & 0 \\
\left[F_1 = 1\right] & \left[F_2 = 2\right] & \left[F_3 = 3\right] & \left[F_4 = 5\right] & \left[F_5 = 8\right] \\
\end{array}
\]

Next move: Player 1: $F_3 + F_4 = F_5$. 
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{ccccc}
0 & 1 & 0 & 0 & 1 \\
\end{array}
\]

No moves left, Player One wins.
Sample Game

Player One won in 9 moves.

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\[ F_1 = 1 \] \[ F_2 = 2 \] \[ F_3 = 3 \] \[ F_4 = 5 \] \[ F_5 = 8 \]
Sample Game

Player Two won in 10 moves.

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\[F_1 = 1\] \[F_2 = 2\] \[F_3 = 3\] \[F_4 = 5\] \[F_5 = 8\]
Games end

Theorem

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: \((\sqrt{k} + \sqrt{k}) - \sqrt{k} + 2 < 0\).
- Splitting: \(2\sqrt{k} - (\sqrt{k + 1} + \sqrt{k + 1}) < 0\).
- Adding 1’s: \(2\sqrt{1} - \sqrt{2} < 0\).
- Splitting 2’s: \(2\sqrt{2} - (\sqrt{3} + \sqrt{1}) < 0\).
Upper bound: At most $n \log_\phi (n \sqrt{5} + 1/2)$ moves.

Fastest game: $n - Z(n)$ moves ($Z(n)$ is the number of summands in $n$’s Zeckendorf decomposition).
   From always moving on the largest summand possible (deterministic).
Games Lengths: II

Figure: Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when \( n = 60 \) vs a Gaussian. Natural conjecture....
Winning Strategy

Theorem

Payer Two Has a Winning Strategy

Idea is to show if not, Player Two could steal Player One’s strategy.

Non-constructive!

Will highlight idea with a simpler game.
Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at \((i, j)\) and coloring every dot \((m, n)\) with \(i \leq m\) and \(j \leq n\).

Once all dots colored game ends; whomever goes last loses.

Prove Player 1 has a winning strategy!
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Sketch of Proof for Player Two’s Winning Strategy

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1^n

1^{(n-2)} \land 2

1^{(n-3)} \land 3  
1^{(n-4)} \land 2^2

1^{(n-5)} \land 2 \land 3  
1^{(n-3)} \land 3  
1^{(n-6)} \land 2^3

1^{(n-7)} \land 5  
1^{(n-7)} \land 2^2 \land 3  
1^{(n-6)} \land 3^2  
1^{(n-5)} \land 2 \land 3  
1^{(n-8)} \land 2^4

1^{(n-7)} \land 2 \land 5  
1^{(n-9)} \land 2^3 \land 3  
1^{(n-6)} \land 3^2  
1^{(n-8)} \land 2 \land 3^2  
1^{(n-5)} \land 5  
1^{(n-7)} \land 2^2 \land 3  
1^{(n-10)} \land 2^5
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81
Sketch of Proof for Player Two’s Winning Strategy
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Tree diagram showing a strategy for winning the Triangle Game.
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Future Work

- What if $p \geq 3$ people play the Fibonacci game?

- Does the number of moves in random games converge to a Gaussian?

- Define $k$-nacci numbers by $S_{i+1} = S_i + S_{i-1} + \cdots + S_{i-k}$; game terminates but who has the winning strategy?
Games
Games: Coins on a line

You have $2N$ coins of varying denominations (each is a non-negative real number) in a line. Players A and B take turns choosing one coin from either end. Does Player A or B have a winning strategy (i.e., a way to ensure they get at least as much as the other?) If yes, who has it and find it if possible!
Games: Devilish Coins

You die and the devil comes out to meet you. In the middle of the room is a giant circular table and next to the walls are many sacks of coins. The devil speaks. *We’ll take turns putting coins down flat on the table. I’ll put down a coin and then you’ll put down a coin, and so on. The coins cannot overlap and they cannot hang over the edge of the table. The last person to put down a coin wins, or equivalently, the last person who can no longer put a coin down on the table loses. You decide if you want to go first.*

Do you have a winning strategy for the game? If yes, what?
Alice and Bob play a game in which they take turns removing stones from a heap that initially has $n$ stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many such $n$ such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)
\( \sqrt{2} \) is Irrational

Standard Proof: Assume \( \sqrt{2} = \frac{a}{b} \).

WLOG, assume \( b \) is the smallest denominator among all fractions that equal \( \sqrt{2} \).

\[ 2b^2 = a^2 \text{ thus } a = 2m \text{ is even.} \]

Then \( 2b^2 = 4m^2 \) so \( b^2 = 2m^2 \) so \( b = 2n \) is even.

Thus \( \sqrt{2} = \frac{a}{b} = \frac{2m}{2n} = \frac{m}{n} \), contradicts minimality of \( n \).

(Could also do by contradiction from \( a, b \) relatively prime.)
Tennenbaum’s Proof

Assume $\sqrt{2} = a/b$ with $b$ minimal.

Figure: $2b^2 = a^2$ so $(2b - a)^2 = 2(a - b)^2$ and $\sqrt{2} = \frac{2b-a}{a-b}$.

As $0 < a - b < b$ (if not, $a - b \geq b$ so $a \geq 2b$ and $\sqrt{2} = a/b \geq 2$), contradicts minimality of $b$. 
WHAT OTHER NUMBERS HAVE GEOMETRIC IRRATIONALITY PROOFS?
More Irrationals
Assume $\sqrt{3} = a/b$ with $b$ minimal.

\[ 3(2b - a)^2 = (2a - 3b)^2 \]

Figure: Geometric proof of the irrationality of $\sqrt{3}$. The white equilateral triangle in the middle has sides of length $2a - 3b$.

Have $3(2b - a)^2 = (2a - 3b)^2$ so $\sqrt{3} = (2a - 3b)/(2b - a)$, note $2b - a < b$ (else $b \geq a$), violates minimality.
\[ \sqrt{5} \]

**Figure:** Geometric proof of the irrationality of \( \sqrt{5} \).
Figure: Geometric proof of the irrationality of \( \sqrt{5} \): the kites, triangles and the small pentagons. This leaves five doubly covered pentagons, and one larger pentagon uncovered.
A straightforward analysis shows that the five doubly covered pentagons are all regular, with side length $a - 2b$, and the middle pentagon is also regular, with side length $b - 2(a - 2b) = 5b - 2a$.

We now have a smaller solution, with the five doubly counted regular pentagons having the same area as the omitted pentagon in the middle. Specifically, we have $5(a - 2b)^2 = (5b - 2a)^2$; as $a = b\sqrt{5}$ and $2 < \sqrt{5} < 3$, note that $a - 2b < b$ and thus we have our contradiction.
Figure: Geometric proof of the irrationality of $\sqrt{6}$. 
Closing Thoughts

Could try to do $\sqrt{10}$ but eventually must break down. Note 3, 6, 10 are triangular numbers ($T_n = n(n + 1)/2$).

$T_8 = 36$ and thus $\sqrt{T_8}$ is an integer!

Can you get a cube-root?

What other numbers?