Why I love Monovariants: From Zombies to Conway's Soldiers to Fibonacci Games

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Invariants / Monovariants

Invariant: a quantity that is unchanged throughout the process / operations. (Big application: Noether's theorem).

Monovariant: a quantity that only changes in one direction throughout the process / operations. See

https://howardhalim.com/math/Invariants%20and%20Monovariants.pdf

for a nice collection of problems.

Often a challenge to find a useful monovariant.

Zombies

Zombie Infection: Rules

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.

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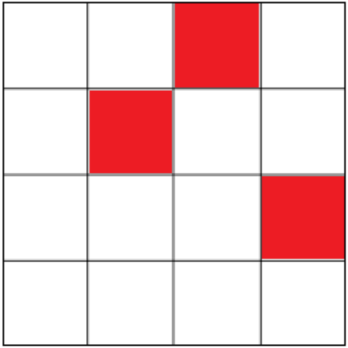

Initial Configuration

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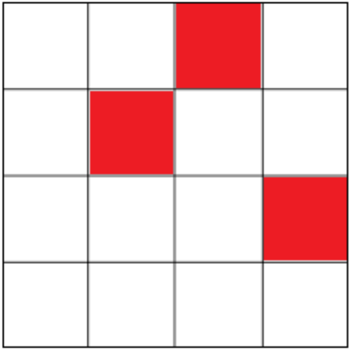
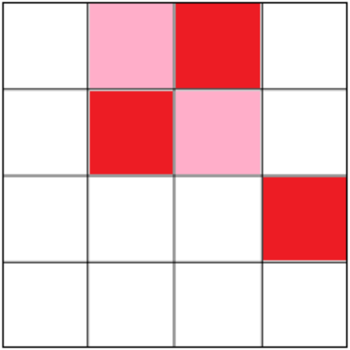


Initial Configuration



One moment later

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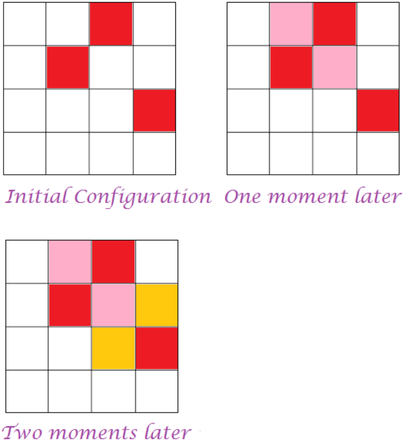

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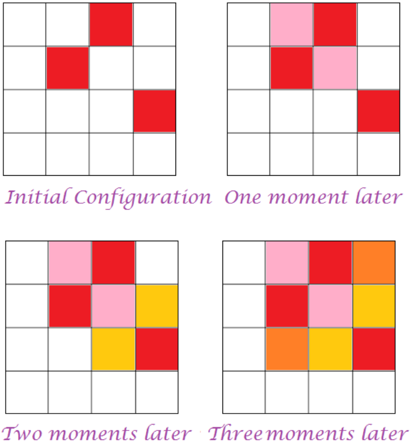
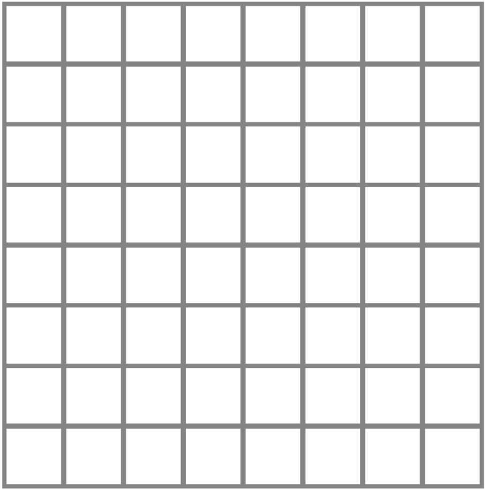


Zombie Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?



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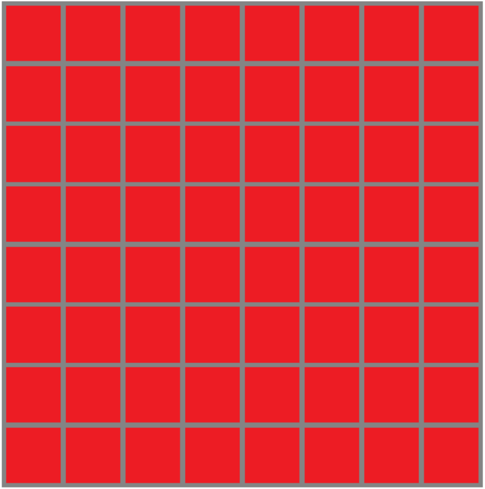

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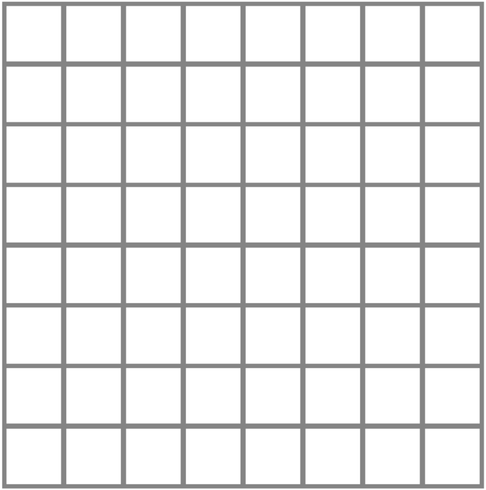
Next simplest initial state ensuring all eventually infected...?



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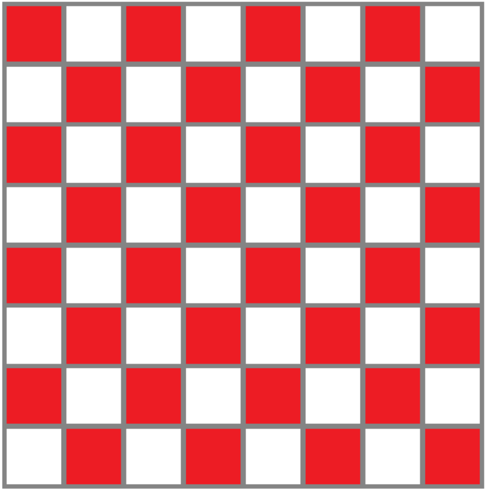

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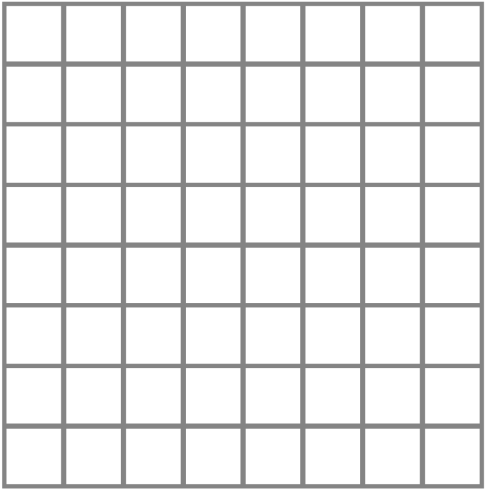
Fewest number of initial infections needed to get all...?



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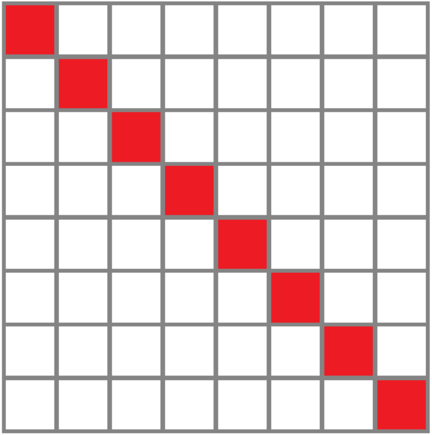

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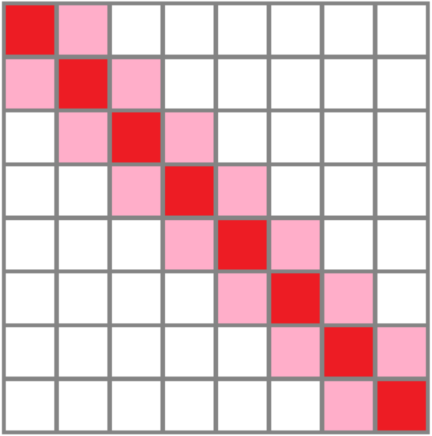

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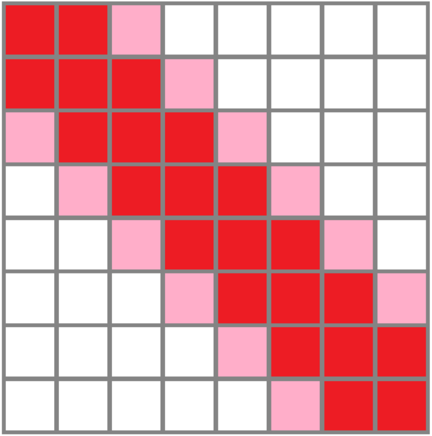

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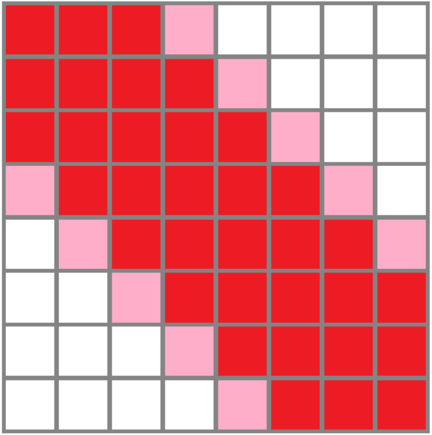

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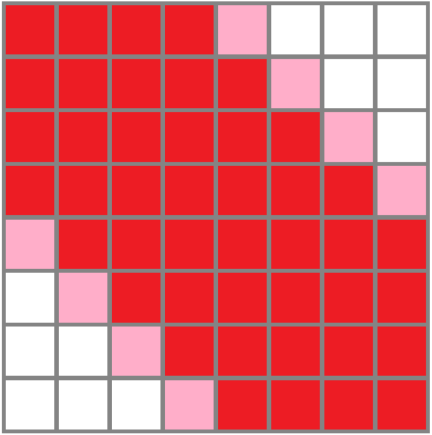

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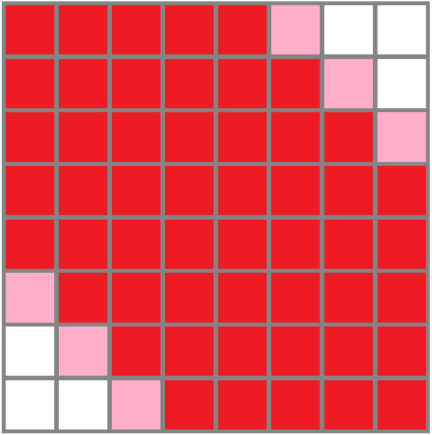

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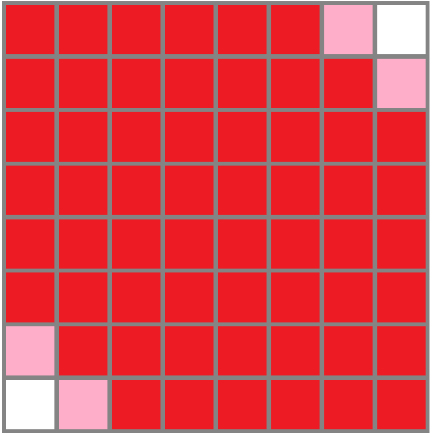

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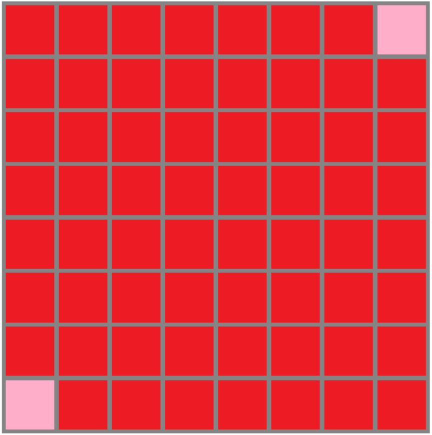

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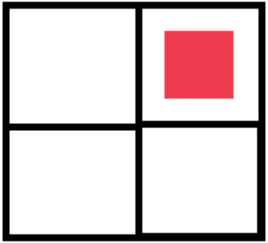

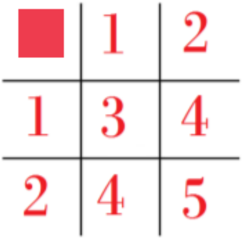

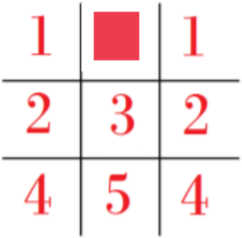

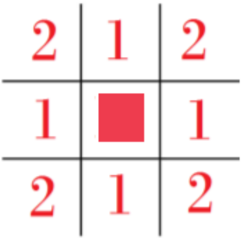

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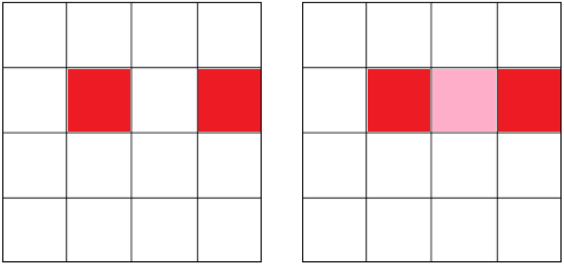

Perimeter of infection unchanged.

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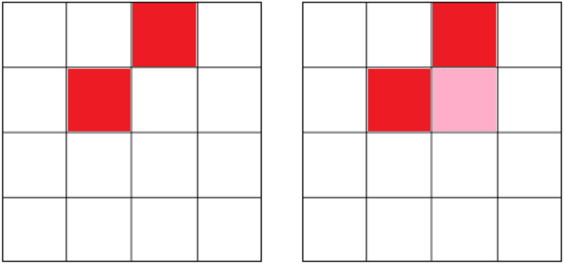

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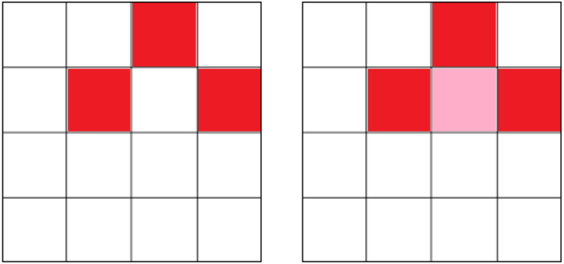

Perimeter of infection decreases by 2.

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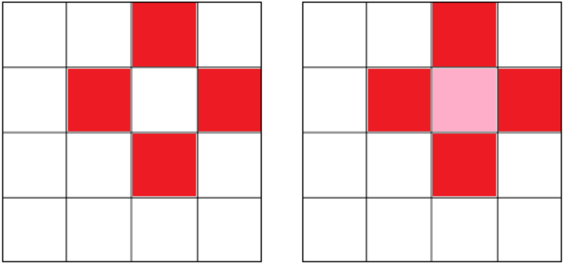

Perimeter of infection decreases by 4.

Zombie Infection: $n-1$ cannot infect all

- If $n-1$ infected, maximum perimeter is $4(n-1) = 4n-4$.

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- How many must be safe?
- Other questions?

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- How many must be safe?
- Other questions: Is a row safe? Higher dimensions? Other regions (torus)?

Conway's Soldiers

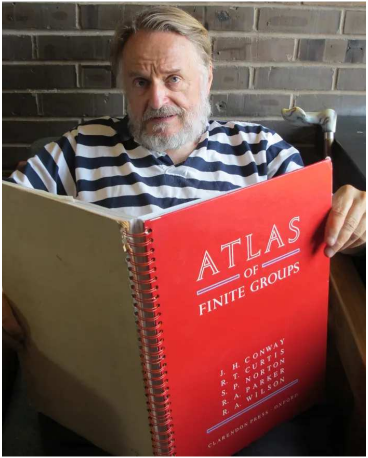

Figure: John Horton Conway: Image from The Guardian.

Conway's Soldiers / Checker Problem

Problem: Infinite checkerboard, pieces at all (x, y) with $y \leq 0$. Using horizontal / vertical jumps (jumped piece gone forever), how high can you move a piece?

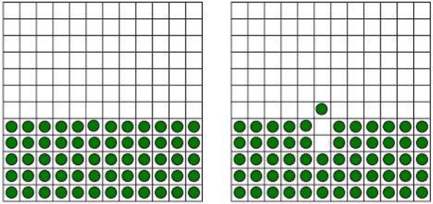

Figure: Left: A subset of the initial configuration. Right: moving a soldier / checker up 1.

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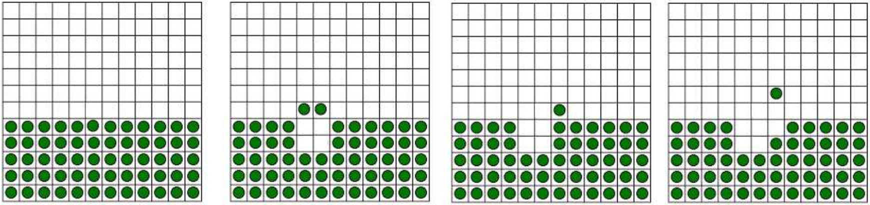

Figure: Left: A subset of the initial configuration. Right: moving a soldier / checker up 2. Can you do 3? 4? 5? Any height?

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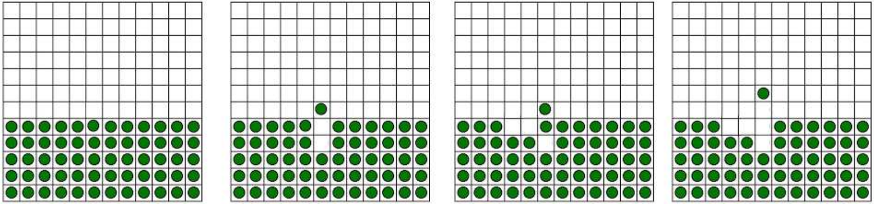

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Conway's Soldiers: The Monovariant: I

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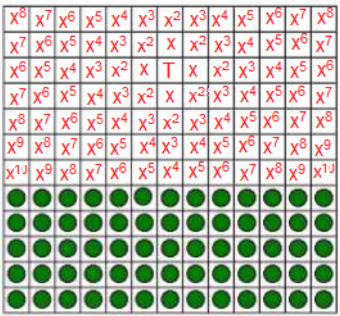
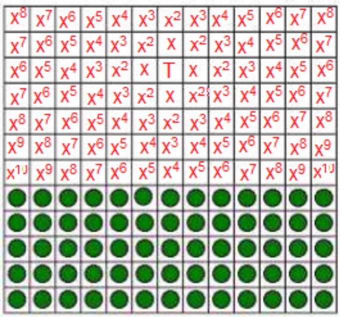


Figure: Conway's monovariant: What is it?

Conway's Soldiers: The Monovariant: II

Choose target $T = (0, 5)$.

Fix x (to be determined later) and attach x^{i+j} to a point that is i units horizontally and j units vertically from T.



Conway's Soldiers: The Monovariant: III

Choose a target point T; for us it is a point of height 5 above the checkers: $T = (0,5)$.

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What is the value of the initial board?

- Zeroth row: $\dots, x^7, x^6, x^5, x^6, x^7, \dots$: sum is

$$x^5 + 2\sum_{k=6}^{\infty} x^k \ = \ x^5 + \frac{2x^6}{1-x} \ = \ \frac{(1+x)x^5}{1-x}.$$

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- Each row is x times previous: Thus initial board value is

$$\frac{(1+x)x^5}{1-x} \sum_{n=0}^{\infty} x^n \;=\; \frac{(1+x)x^5}{(1-x)^2}.$$

Conway's Soldiers: The Monovariant: IV

Two moves: lose 2 pieces and add a piece further from T, or lose 2 pieces and add a piece closer to T.

First type of move clearly decreases value of board.

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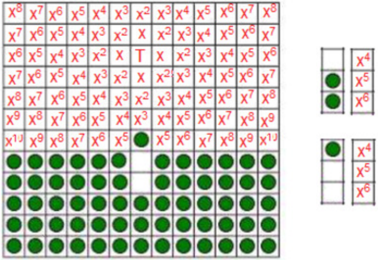

Figure: Moving pieces on x^6 and x^5 to on x^4.
Change is $x^4 - x^5 - x^6 = x^4(1 - x - x^2)$, want this to be zero.

Conway's Soldiers: The Monovariant: IV

Two moves: lose 2 pieces and add a piece further from T, or lose 2 pieces and add a piece closer to T.

Second type replaces x^{n+2} and x^{n+1} with an x^n: change is $x^n - x^{n+1} - x^{n+2}$. Choose x so that this change is zero.

Thus $1 - x - x^2 = 0$ or $x = (-1 \pm \sqrt{5})/2$. Take positive root, $(-1 + \sqrt{5})/2 = \varphi - 1$ (φ the golden mean).

Monovariant: sum of the values of squares with checkers.

Conway's Soldiers: The Monovariant: V

Choose a target point T.

- Initial board value is
$$\frac{(1+x)x^5}{(1-x)^2} \text{ : when } x = \frac{\sqrt{5}-1}{2} \text{ get } 1.$$
- Target at $(0,4)$ contributes $x = \frac{\sqrt{5}-1}{2} \approx 0.618034$; as less than 1 possible (and can be done).
- Target at $(0,5)$, board's value at least 1. Moves never increase value: IMPOSSIBLE IN FINITE TIME![1]

[1] Possible in "infinite" game: https://tartarus.org/gareth/maths/stuff/solarmy.pdf.

New Results

Conway Checkers m-game: Start with m checkers on each gridpoint (original game is just 1), if jump over it lose one checker.

SMALL 2024

Given a Conway Checkers m-game, the maximum row attainable, n_m, satisfies

$$\lfloor \log_{\varphi}(m) + 4.67 \rfloor \ \leq \ n_m \ \leq \ \lfloor \log_{\varphi}(m) + 5 \rfloor$$

for sufficiently large m, where φ is the golden ratio $\frac{\sqrt{5}+1}{2}$.

Zeckendorf Minimality

Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
$2024 \;=\; 1597 + 377 + 34 + 13 + 3 \;=\; F_{16} + F_{13} + F_8 + F_6 + F_3.$

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Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8, 13...

Summand Minimality

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

Theorem

*The Zeckendorf decomposition is **summand minimal**.*

Overall Question

What other recurrences are summand minimal?

Zeckendorf Decomposition is Minimal

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*The Zeckendorf decomposition is **summand minimal**: no decomposition as a sum of Fibonacci numbers (1, 2, 3, 5, ...) has fewer summands than it.*

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If $n = \sum_k a_k F_k$ (with a_k non-negative integers), define the weighted index attached to this decomposition $\mathcal{D}$ to be $\text{Index}(\mathcal{D}) = \sum_k a_k \sqrt{k}$.

More natural $\sum_k a_k k$ but square-root makes strictly decreasing.

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Bounded process: For fixed n, only indices up to certain point used, and $a_k \leq n$.

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Show $\mathrm{Index}(\mathcal{D})$ is a mono-variant, end in the Zeckendorf decomposition, number summands never increased.

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$F_k \wedge F_{k+1} \ \rightarrow \ F_{k+2}$:

- $\sqrt{k} + \sqrt{k+1} > \sqrt{k+2}$.

$2F_k \ \rightarrow \ F_{k-2} + F_{k+1}$:

- $k \geq 3$: $2\sqrt{k} > \sqrt{k-2} + \sqrt{k+1}$
- $k = 2$: $2\sqrt{2} > \sqrt{1} + \sqrt{3}$
- $k = 1$: $2\sqrt{1} > \sqrt{2}$

Only finitely many values, each move lowers, continue till hit Zeckendorf, number of summands never increased.

Positive Linear Recurrence Sequences

Definition

A **positive linear recurrence sequence (PLRS)** is a sequence given by a recurrence $\{a_n\}$ with

$$a_n \ := \ c_1 a_{n-1} + \cdots + c_t a_{n-t}$$

and each $c_i \geq 0$ and $c_1, c_t > 0$. We use **ideal initial conditions** $a_{-(n-1)} = 0, \ldots, a_{-1} = 0, a_0 = 1$ and call $(c_1, \ldots, c_t)$ the **signature of the sequence.**

Theorem (Cordwell, Hlavacek, Huynh, M., Peterson, Vu)

For a PLRS with signature $(c_1, c_2, \ldots, c_t)$*, the Generalized Zeckendorf Decompositions are summand minimal if and only if*

$$c_1 \geq c_2 \geq \cdots \geq c_t.$$

Zeckendorf Games

Fibonacci Game: Rules

- Two player game, alternate turns, last to move wins.

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- Bins F_1, F_2, F_3, $\dots$, start with N pieces in F_1 and others empty.

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  - ◇ If pieces at F_k and F_{k+1} remove and add one at F_{k+2}.

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  - ◇ If pieces at F_k and F_{k+1} remove and add one at F_{k+2}.

Questions:

- Does the game end? How long?
- For each N who has the winning strategy?
- What is the winning strategy?

Sample Game

Start with 10 pieces at F_1, rest empty.

10	0	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

8	1	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

6	2	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $2F_2 = F_3 + F_1$

Sample Game

Start with 10 pieces at F_1, rest empty.

7	0	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

5	1	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_2 + F_3 = F_4$.

Sample Game

Start with 10 pieces at F_1, rest empty.

5	0	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

3	1	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

1	2	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_2 = F_3$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	1	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_3 + F_4 = F_5$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

No moves left, Player One wins.

Sample Game

Player One won in 9 moves.

(1)	10	0	0	0	0
(2)	8	1	0	0	0
(1)	6	2	0	0	0
(2)	7	0	1	0	0
(1)	5	1	1	0	0
(2)	5	0	0	1	0
(1)	3	1	0	1	0
(2)	1	2	0	1	0
(1)	0	1	1	1	0
	0	1	0	0	1
	$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Sample Game

Player Two won in 10 moves.

(1)	10	0	0	0	0
(2)	8	1	0	0	0
(1)	6	2	0	0	0
(2)	7	0	1	0	0
(1)	5	1	1	0	0
(2)	5	0	0	1	0
(1)	3	1	0	1	0
(2)	1	2	0	1	0
(1)	2	0	1	1	0
(2)	0	1	1	1	0
	0	1	0	0	1
	$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Games end

Theorem

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive: $\left(\sqrt{k}+\sqrt{k+1}\right)-\sqrt{k+2}>0$.
- Splitting: $2\sqrt{k}-\left(\sqrt{k+1}+\sqrt{k+1}\right)>0$.
- Spitting 1's: $2\sqrt{1}-\sqrt{2}>0$.
- Splitting 2's: $2\sqrt{2}-\left(\sqrt{3}+\sqrt{1}\right)>0$.

Games Lengths: I

Upper bound: At most $3n - 3Z(n) - I(n) + 1$ moves

- $Z(n)$ is the number of terms in the Zeckendorf decomposition,
- $I(n)$ is the sum of the indices.

Fastest game: $n - Z(n)$ moves ($Z(n)$ is the number of summands in n's Zeckendorf decomposition).

From always moving on the largest summand possible (deterministic).

Games Lengths: II

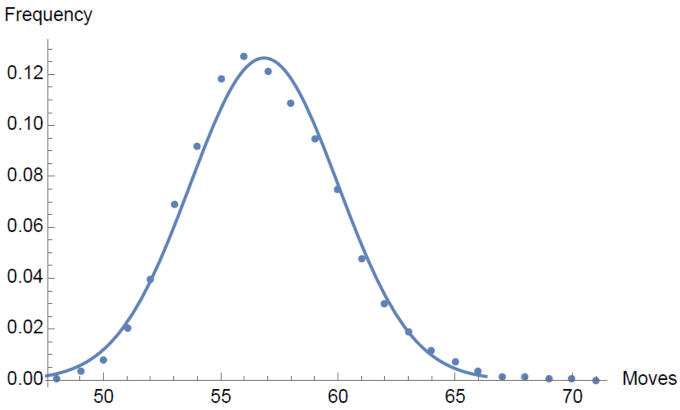


Figure: Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when $n = 60$ vs a Gaussian. Natural conjecture....

Winning Strategy

Theorem

Player Two Has a Winning Strategy

Idea is to show if not, Player Two could steal Player One's strategy.

Non-constructive!

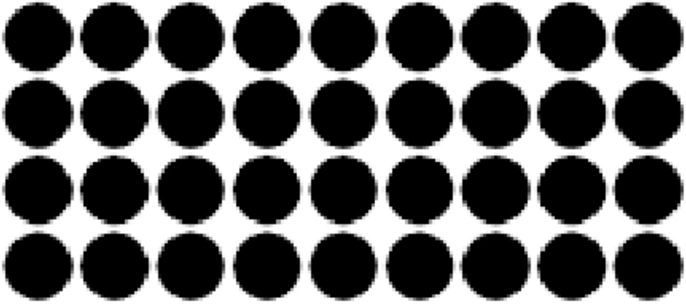
Will highlight idea with a simpler game.

Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at (i, j) and coloring every dot (m, n) with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Prove Player 1 has a winning strategy!



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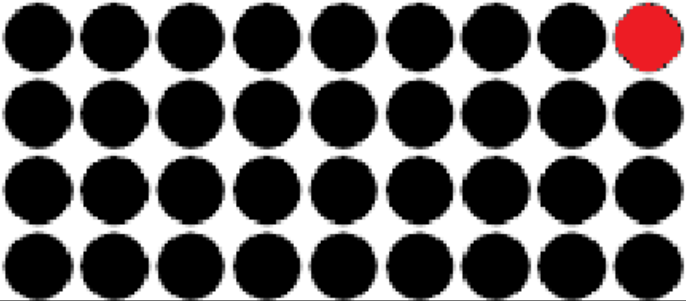
Proof Player 1 has a winning strategy. If have, play; if not, steal.



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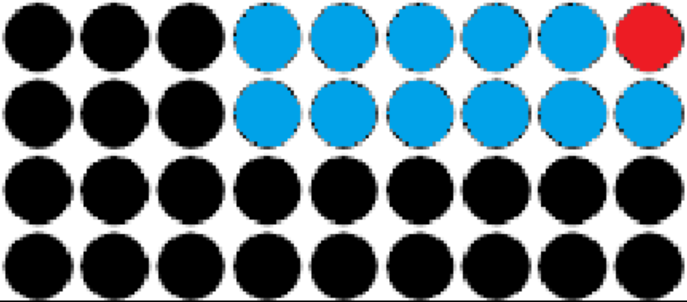

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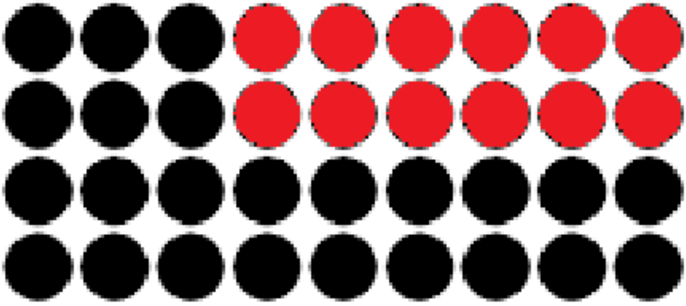
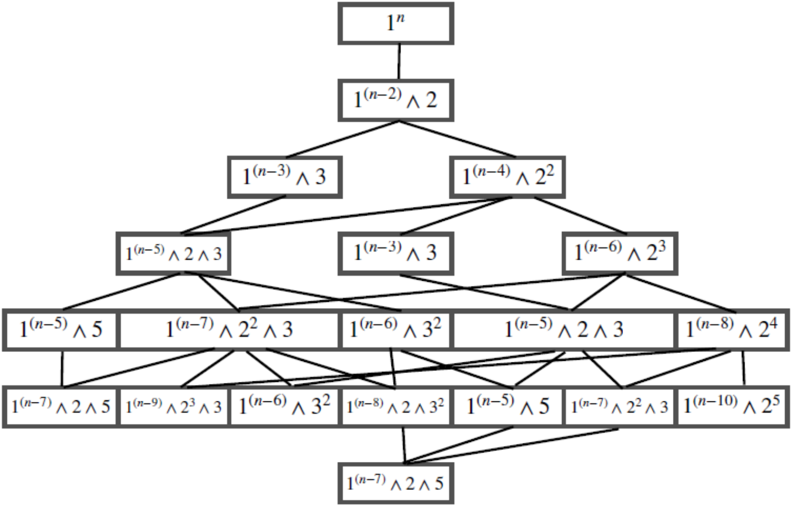


Sketch of Proof for Player Two's Winning Strategy



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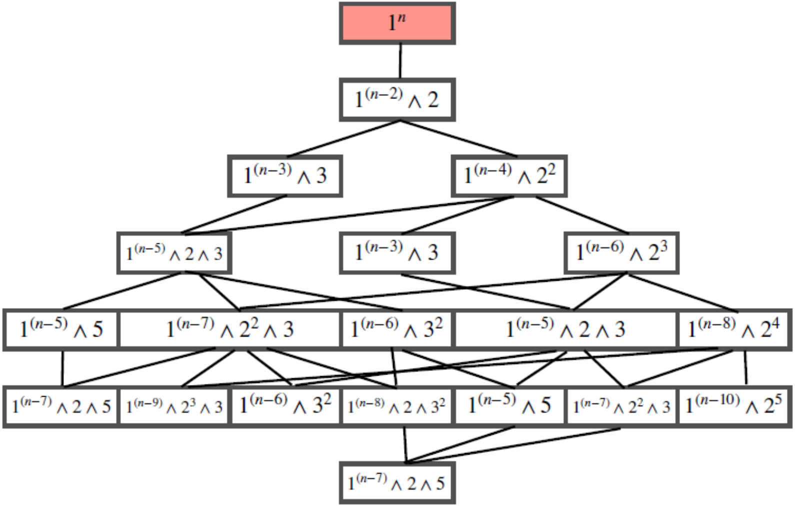

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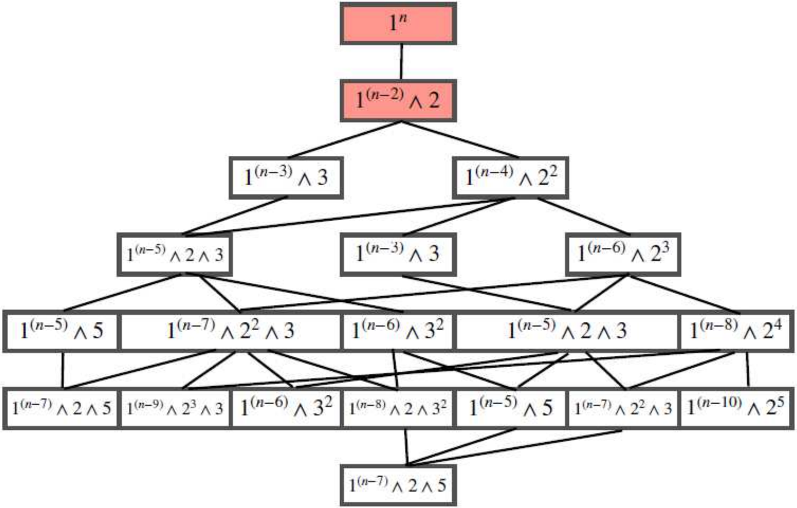

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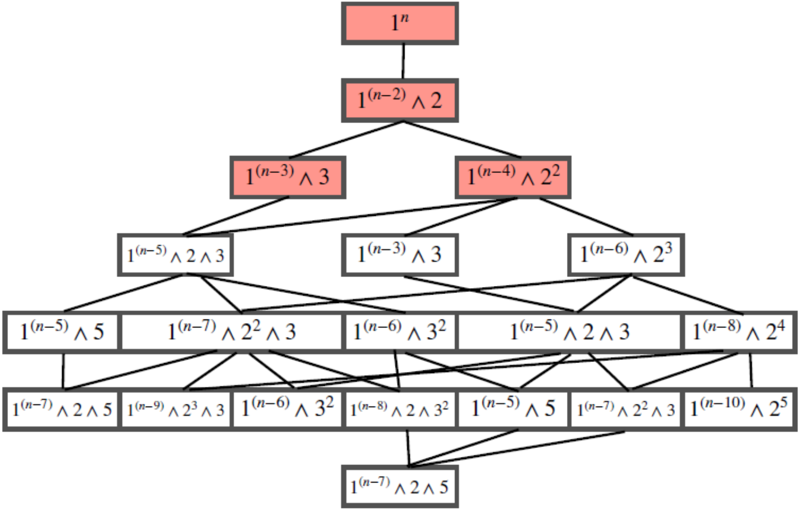

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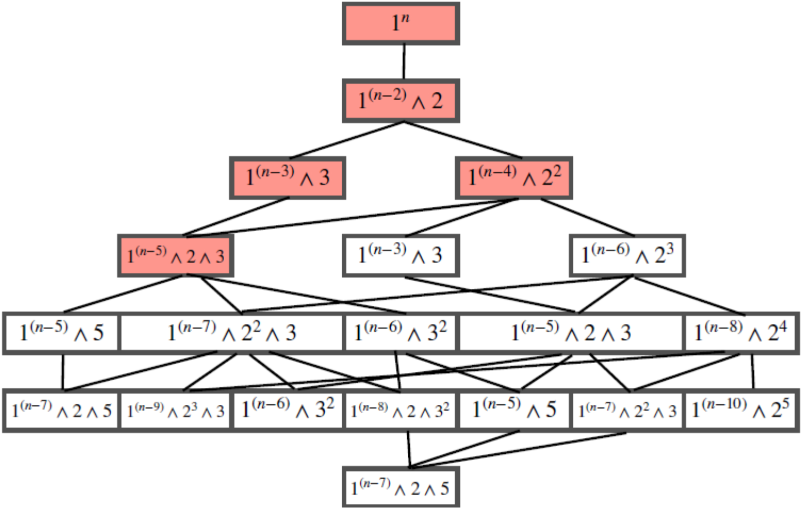

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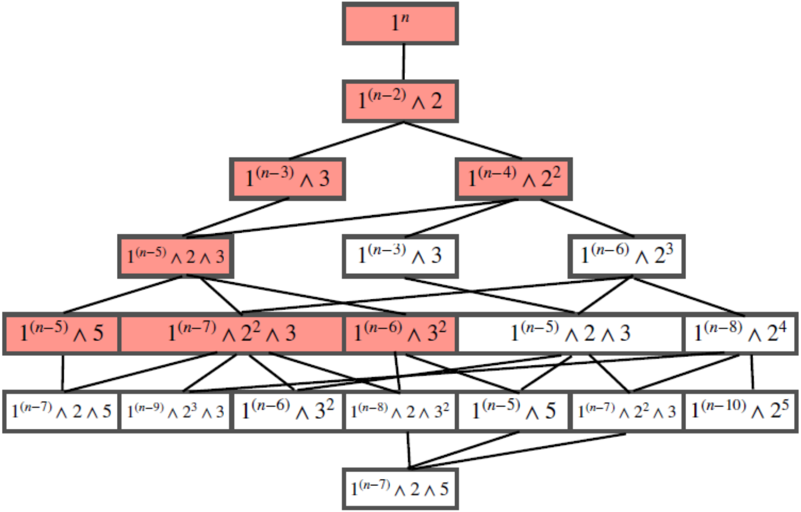

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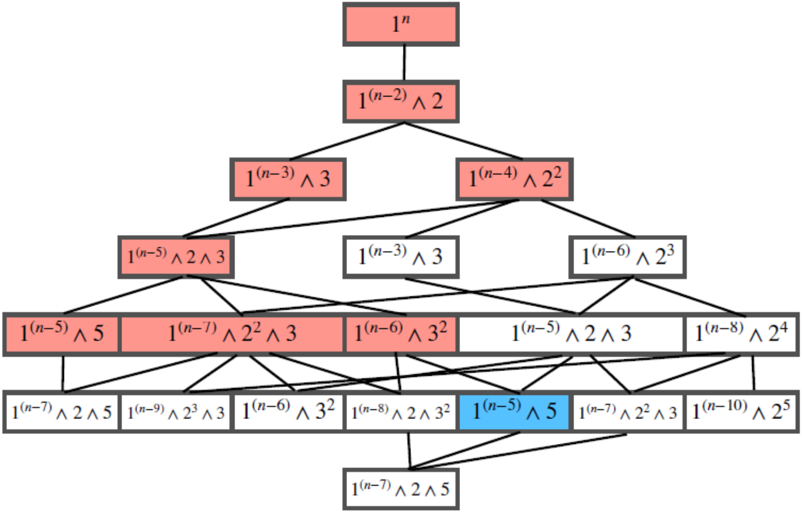

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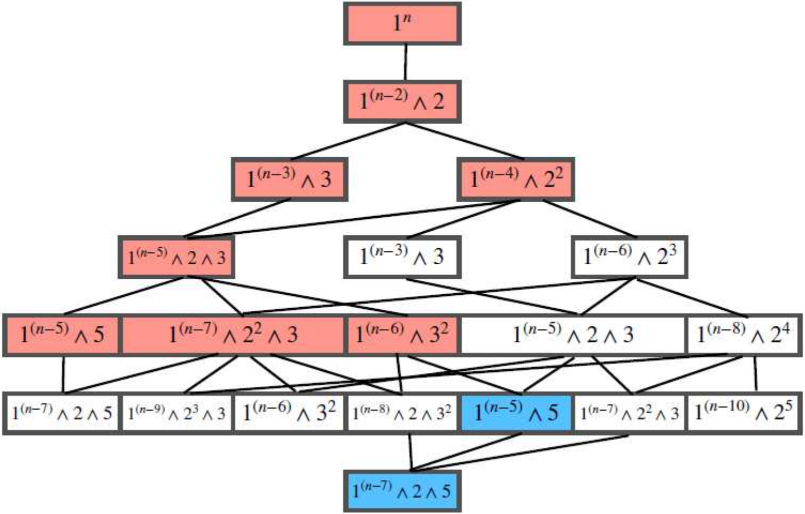

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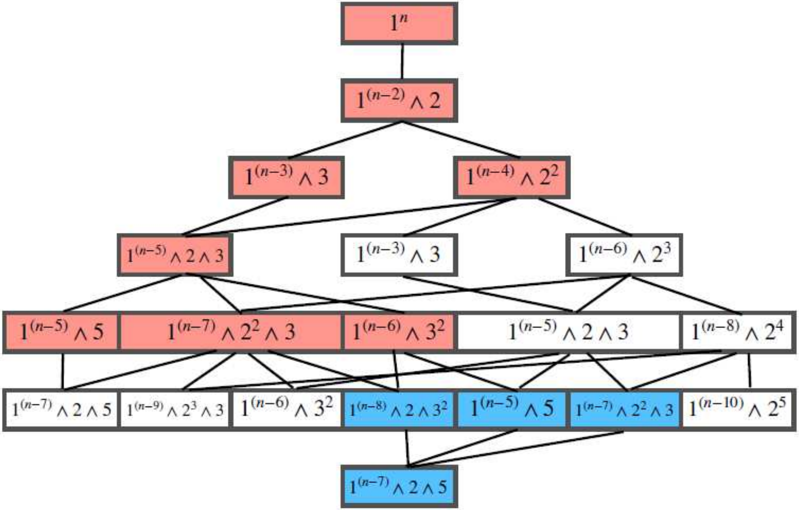

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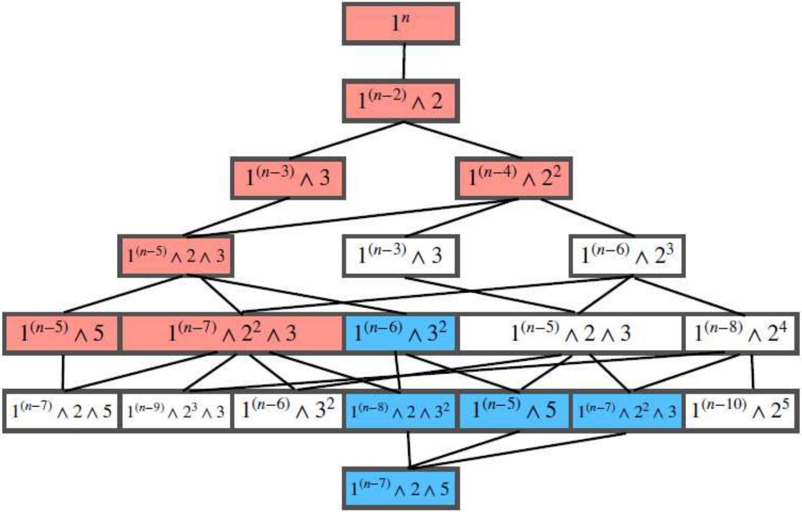


The Bergman Game

Definition

The **Bergman Game** is played with the standard split/combine moves from the Zeckendorf game, but on a two-sided infinite tape instead of a one-sided infinite tape.
It produces base-φ decompositions ($\varphi = (1+\sqrt{5})/2$).

Example

$$\begin{matrix} 0 & 0 & 4 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{matrix}$$

$4 = \varphi^{-2} + \varphi^{0} + \varphi^{2}$.

The Bergman Game

Theorem (Baily, Dell, Durmić, Fleischmann, Jackson, Mijares, M., Pesikoff, Reifenberg, Reina, Yang)

The longest Bergman Game with n summands terminates in $\Theta(n^2)$ time regardless of where the summands are placed. The shortest possible Bergman Game terminates in $\Theta(n)$ time.

Natural Question: Who has the winning strategy?

- Not currently known.
- Game tree explodes, escaping a strategy steal.

The Frodnekcez Game (Reverse Zeckendorf Game): Rules

- The Zeckendorf game in reverse, last to move wins.
- Bins $F_1, F_2, F_3, \dots$, for some natural number N, start with one piece in bin F_k if F_k is in the Zeckendorf decomposition of N, and have other bins empty.
- A turn is one of the following moves:
  - ◇ If one piece at F_{k+1} and one at F_{k-2}, can remove and add two pieces on F_k.
  - ◇ If piece at F_{k+2}, remove and add one piece at both F_k and F_{k+1}.

    (F_1 and F_3 becomes $2F_2$, and F_2 becomes $2F_1$)

Problem created and analyzed by PANTHers 2023 from the 2023 SMALL REU: Zoe Batterman, Aditya Jambhale, Akash Narayanan, Kishan Sharma, Andrew Yang, Chris Yao.

Winning Strategy?

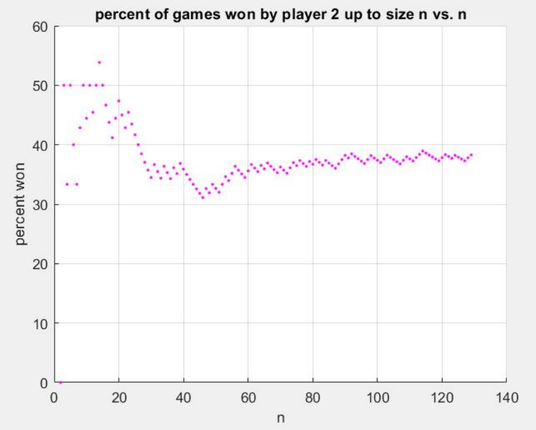


Figure: In the forward Zeckendorf game, Player 2 wins for all $N > 2$. The reverse game is more interesting. Natural conjecture...

Current / Future Work

- What if $p \geq 3$ people play the Fibonacci game? Some multi-player results.
- Does the number of moves in random games converge to a Gaussian? Evidence....
- How long do games take? Proved closed interval.
- Accelerated games: do as many of one move as wish....
- What of other recurrences?

$500 Prize: Determine the winning strategy.

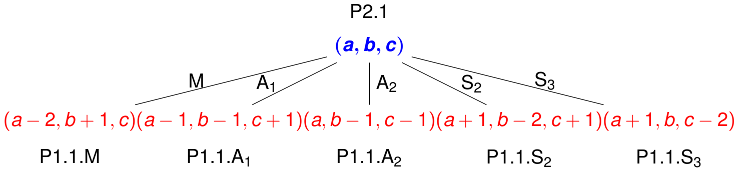
Black Hole Zeckendorf Game (Ongoing Work: SMALL 2024)

How can we simplify the game?

F_m Black Hole Variation

Any pieces placed in a column F_i for $i \geq m$ are permanently removed from gameplay.

For the F_4 case, this allows for the following moves:



Thanks / References

Thanks

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Thank you!

The Cookie Problem and Zeckendorf's Theorem

The Cookie Problem

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The number of ways of dividing C identical cookies among P distinct people is $\binom{C+P-1}{P-1}$.

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Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

Let $p_{n,k} = \#\{N \in [F_n, F_{n+1})$: the Zeckendorf decomposition of N has exactly k summands$\}$.

For $N \in [F_n, F_{n+1})$, the largest summand is F_n.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \; i_j - i_{j-1} \geq 2.$$

$$d_1 := i_1 - 1, \; d_j := i_j - i_{j-1} - 2 \; (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1, \; d_j \geq 0.$$

Cookie counting $\Rightarrow p_{n,k} = \binom{n-2k+1-k-1}{k-1} = \binom{n-k}{k-1}$.