Why I love Monovariants: From Zombies to Conway's Soldiers to Fibonacci Games

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Invariants / Monovariants

Invariant: a quantity that is unchanged throughout the process / operations. (Big application: Noether's theorem).

Monovariant: a quantity that only changes in one direction throughout the process / operations. See

https://howardhalim.com/math/Invariants%20and%20Monovariants.pdf

for a nice collection of problems.

Often a challenge to find a useful monovariant.

Zombies

Zombie Infection: Rules

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.

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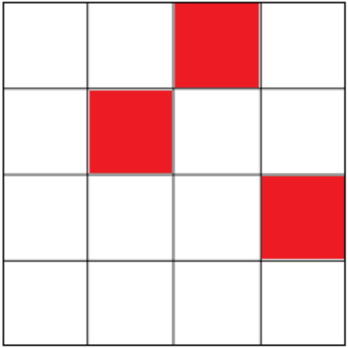

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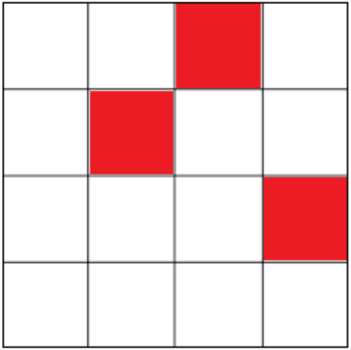

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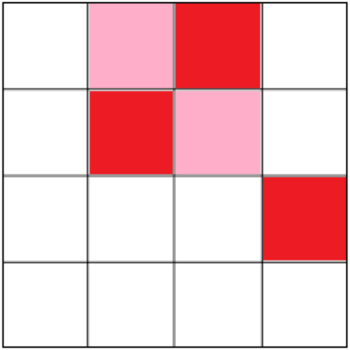

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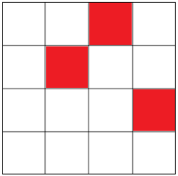

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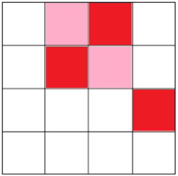

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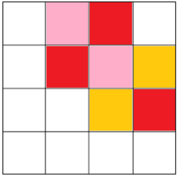

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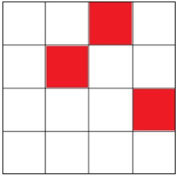

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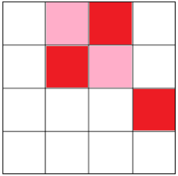

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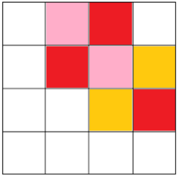

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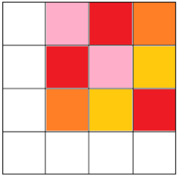
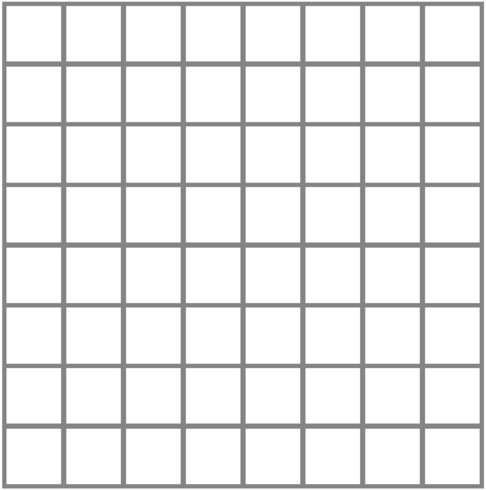


Threemoments later

Zombie Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?



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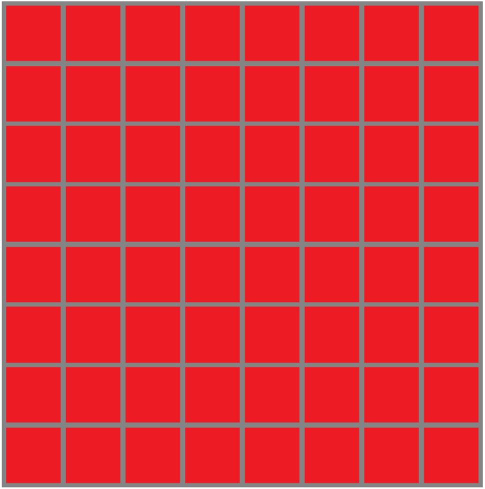
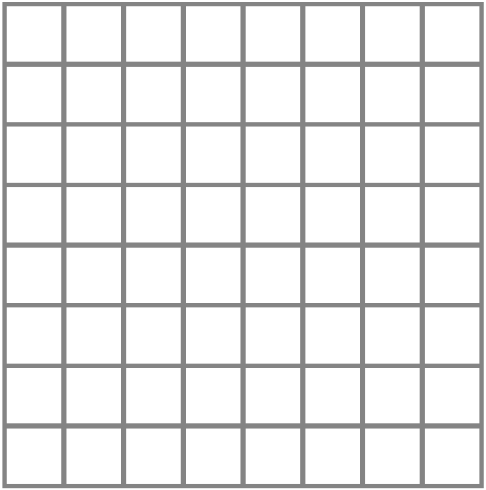


Zombie Infection: Conquering The World

Next simplest initial state ensuring all eventually infected...?



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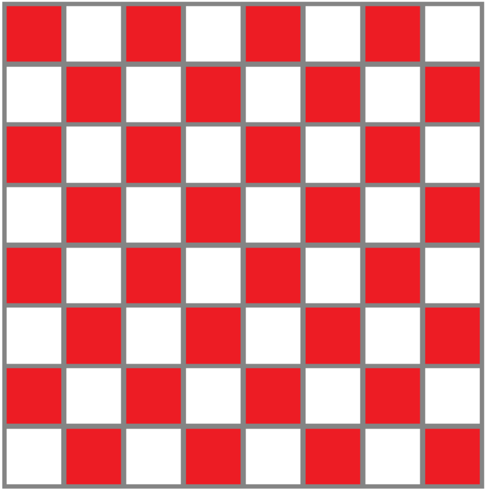
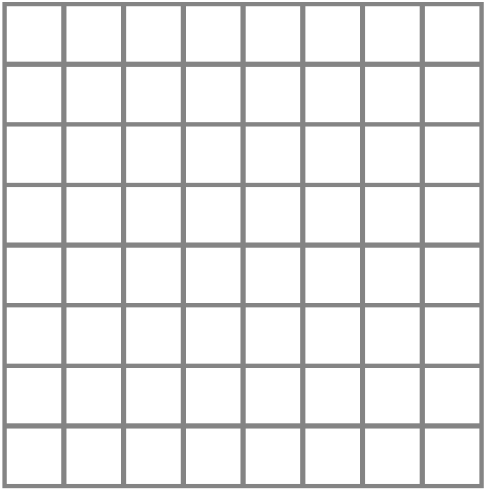


Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?



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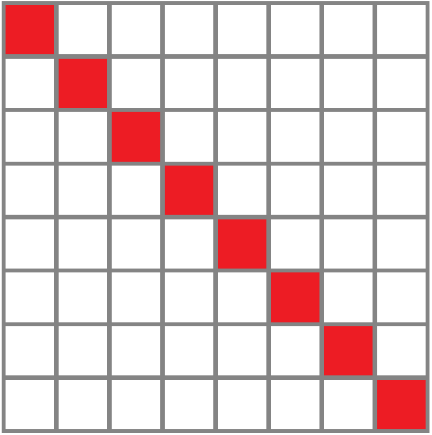

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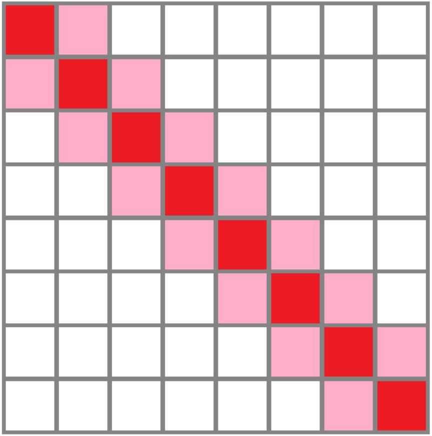

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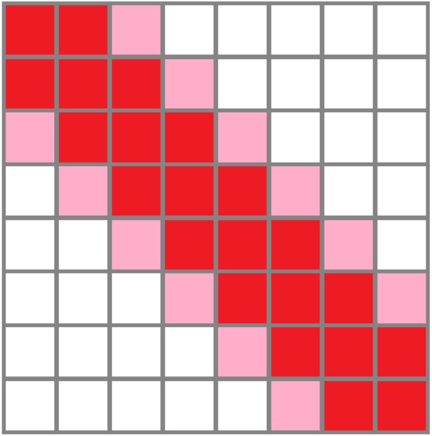

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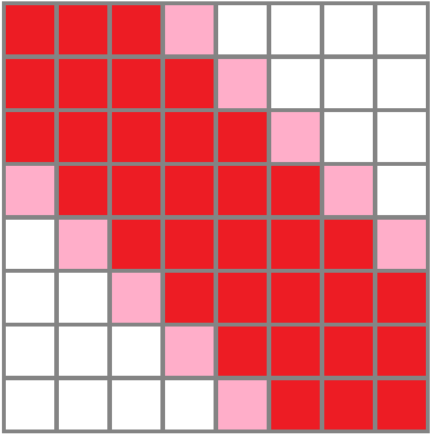

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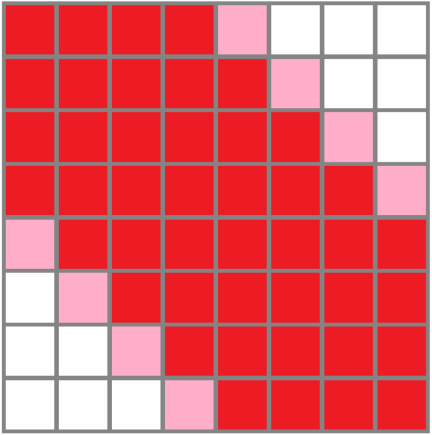

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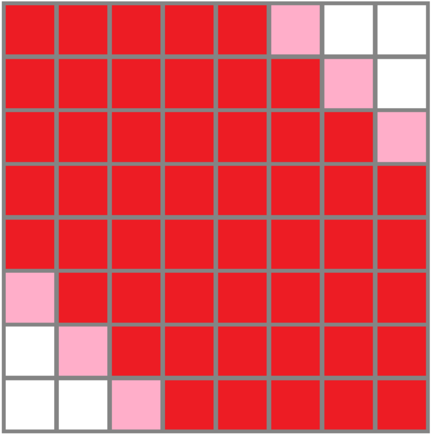

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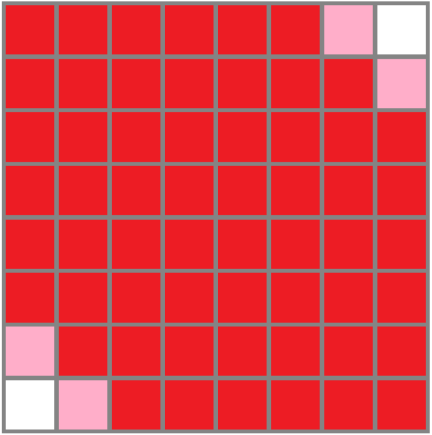

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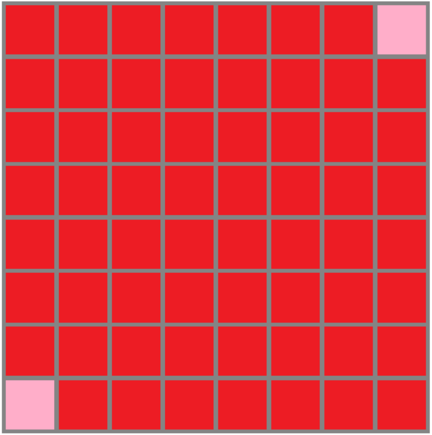

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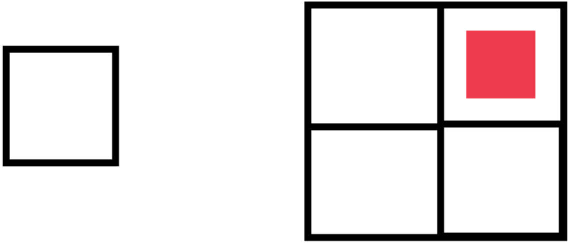

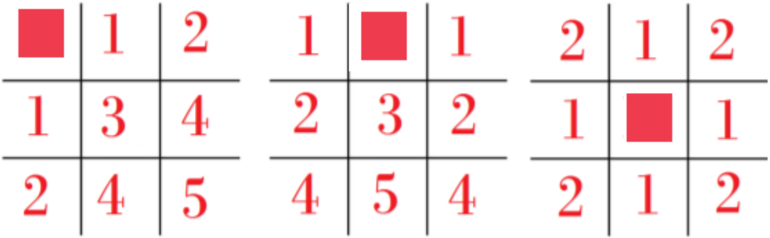

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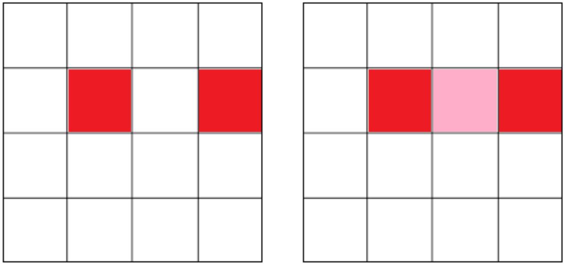

Perimeter of infection unchanged.

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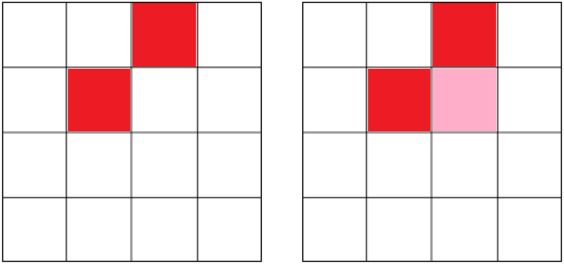

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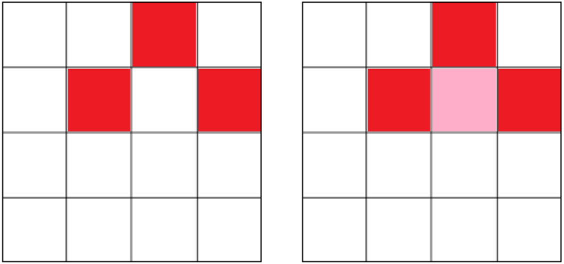
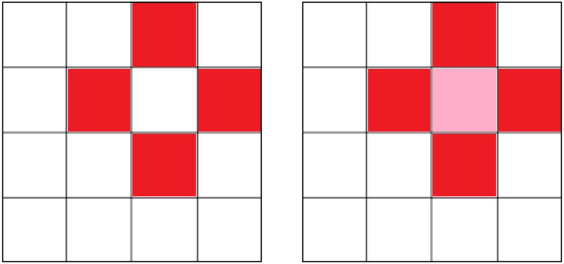


Perimeter of infection decreases by 2.

Zombie Infection: Can $n-1$ infect all on an $n \times n$ board?



Perimeter of infection decreases by 4.

Zombie Infection: $n-1$ cannot infect all

- If $n-1$ infected, maximum perimeter is $4(n-1) = 4n-4$.

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- Other questions?

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- How many must be safe?
- Other questions: Is a row safe? Higher dimensions? Other regions (torus)?

Conway's Soldiers

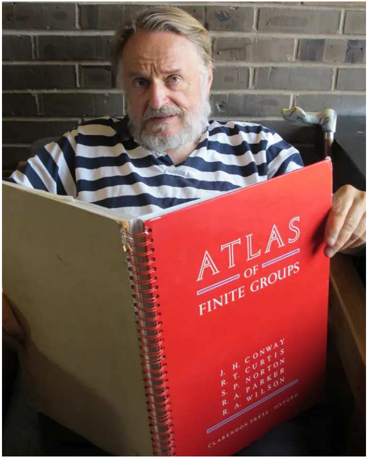

Figure: John Horton Conway: Image from The Guardian.

Conway's Soldiers / Checker Problem

Problem: Infinite checkerboard, pieces at all (x, y) with $y \leq 0$. Using horizontal / vertical jumps (jumped piece gone forever), how high can you move a piece?

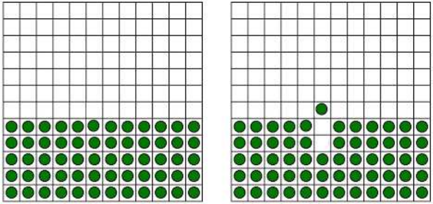

Figure: Left: A subset of the initial configuration. Right: moving a soldier / checker up 1.

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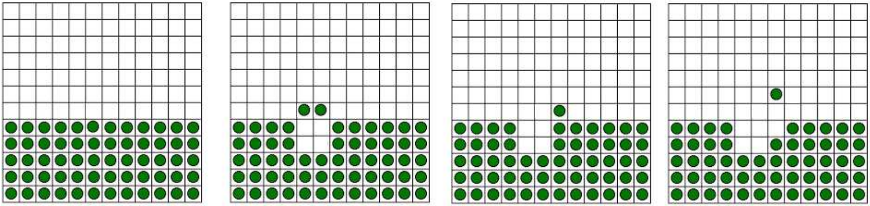

Figure: Left: A subset of the initial configuration. Right: moving a soldier / checker up 2. Can you do 3? 4? 5? Any height?

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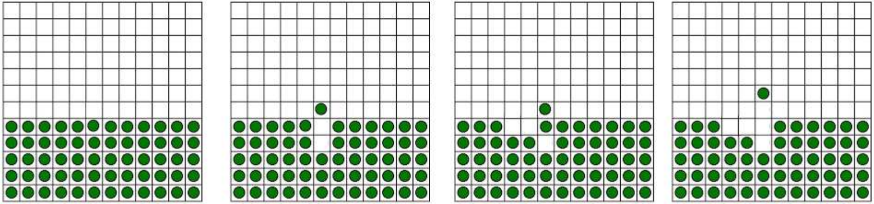

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Conway's Soldiers: The Monovariant: I

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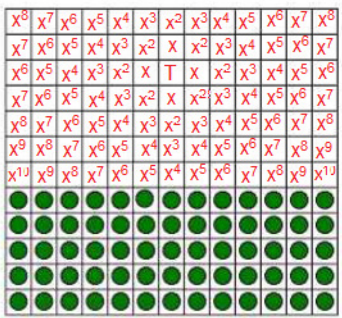
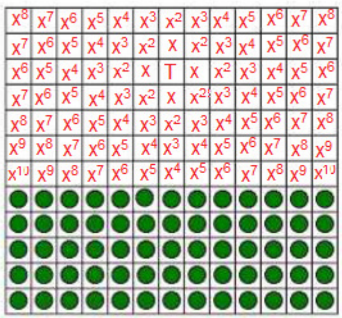


Figure: Conway's monovariant: What is it?

Conway's Soldiers: The Monovariant: II

Choose target $T = (0, 5)$.

Fix x (to be determined later) and attach x^{i+j} to a point that is i units horizontally and j units vertically from T.



Conway's Soldiers: The Monovariant: III

Choose a target point T; for us it is a point of height 5 above the checkers: $T = (0, 5)$.

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What is the value of the initial board?

- Zeroth row: $\dots, x^7, x^6, x^5, x^6, x^7, \dots$: sum is

$$x^5 + 2\sum_{k=6}^{\infty} x^k \;=\; x^5 + \frac{2x^6}{1-x} \;=\; \frac{(1+x)x^5}{1-x}.$$

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- Each row is x times previous: Thus initial board value is

$$\frac{(1+x)x^5}{1-x}\sum_{n=0}^{\infty} x^n \ = \ \frac{(1+x)x^5}{(1-x)^2}.$$

Conway's Soldiers: The Monovariant: IV

Two moves: lose 2 pieces and add a piece further from T, or lose 2 pieces and add a piece closer to T.

First type of move clearly decreases value of board.

Conway's Soldiers: The Monovariant: IV

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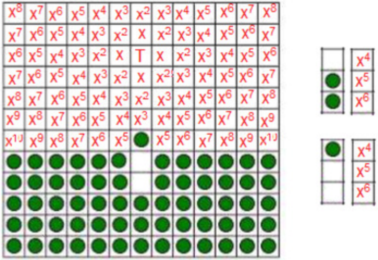

Figure: Moving pieces on x^6 and x^5 to on x^4.
Change is $x^4 - x^5 - x^6 = x^4(1 - x - x^2)$, want this to be zero.

Conway's Soldiers: The Monovariant: IV

Two moves: lose 2 pieces and add a piece further from T, or lose 2 pieces and add a piece closer to T.

Second type replaces x^{n+2} and x^{n+1} with an x^n: change is $x^n - x^{n+1} - x^{n+2}$. Choose x so that this change is zero.

Thus $1 - x - x^2 = 0$ or $x = (-1 \pm \sqrt{5})/2$. Take positive root, $(-1 + \sqrt{5})/2 = \varphi - 1$ (φ the golden mean).

Monovariant: sum of the values of squares with checkers.

Conway's Soldiers: The Monovariant: V

Choose a target point T.

- Initial board value is

$$\frac{(1+x)x^5}{(1-x)^2} \text{ : when } x = \frac{\sqrt{5}-1}{2} \text{ get } 1.$$

- Target at $(0,4)$ contributes $x = \frac{\sqrt{5}-1}{2} \approx 0.618034$; as less than 1 possible (and can be done).

- Target at $(0,5)$, board's value at least 1. Moves never increase value: IMPOSSIBLE IN FINITE TIME![1]

[1] Possible in "infinite" game: https://tartarus.org/gareth/maths/stuff/solarmy.pdf.

New Results

Conway Checkers *m*-game: Start with *m* checkers on each gridpoint (original game is just 1), if jump over it lose one checker.

SMALL 2024

Given a Conway Checkers *m*-game, the maximum row attainable, n_m, satisfies

$$\lfloor \log_\varphi(m) + 4.67 \rfloor \;\leq\; n_m \;\leq\; \lfloor \log_\varphi(m) + 5 \rfloor$$

for sufficiently large *m*, where φ is the golden ratio $\frac{\sqrt{5}+1}{2}$.

Zeckendorf Minimality

Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
$2024 \;=\; 1597 + 377 + 34 + 13 + 3 \;=\; F_{16} + F_{13} + F_8 + F_6 + F_3$.

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Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8, 13...

Summand Minimality

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

Theorem

The Zeckendorf decomposition is ***summand minimal****.*

Overall Question

What other recurrences are summand minimal?

Zeckendorf Decomposition is Minimal

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If $n = \sum_k a_k F_k$ (with a_k non-negative integers), define the weighted index attached to this decomposition $\mathcal{D}$ to be $\mathrm{Index}(\mathcal{D}) = \sum_k a_k \sqrt{k}$.

More natural $\sum_k a_k k$ but square-root makes strictly decreasing.

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More natural $\sum_k a_k k$ but square-root makes strictly decreasing.

Bounded process: For fixed n, only indices up to certain point used, and $a_k \leq n$.

Zeckendorf Decomposition is Minimal: Proof

Show $\text{Index}(\mathcal{D})$ is a mono-variant, end in the Zeckendorf decomposition, number summands never increased.

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Zeckendorf Decomposition is Minimal: Proof

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If $\mathcal{D}$ is not the Zeckendorf, have $2F_k$ or $F_k \wedge F_{k+1}$.

$F_k \wedge F_{k+1} \;\rightarrow\; F_{k+2}$:

- $\sqrt{k} + \sqrt{k+1} > \sqrt{k+2}$.

$2F_k \;\rightarrow\; F_{k-2} + F_{k+1}$:

- $k \geq 3$: $2\sqrt{k} > \sqrt{k-2} + \sqrt{k+1}$
- $k = 2$: $2\sqrt{2} > \sqrt{1} + \sqrt{3}$
- $k = 1$: $2\sqrt{1} > \sqrt{2}$

Only finitely many values, each move lowers, continue till hit Zeckendorf, number of summands never increased.

Positive Linear Recurrence Sequences

Definition

A **positive linear recurrence sequence (PLRS)** is a sequence given by a recurrence $\{a_n\}$ with

$$a_n \ := \ c_1 a_{n-1} + \cdots + c_t a_{n-t}$$

and each $c_i \geq 0$ and $c_1, c_t > 0$. We use **ideal initial conditions** $a_{-(n-1)} = 0, \dots, a_{-1} = 0, a_0 = 1$ and call $(c_1, \dots, c_t)$ the **signature of the sequence.**

Theorem (Cordwell, Hlavacek, Huynh, M., Peterson, Vu)

For a PLRS with signature $(c_1, c_2, \dots, c_t)$, the Generalized Zeckendorf Decompositions are summand minimal if and only if

$$c_1 \ \geq \ c_2 \ \geq \ \dots \ \geq \ c_t.$$

New Results: SMALL 2025

Definition

Given a sequence $\{a_n\}_{n=1}^{\infty}$, a set $C \subset \mathbb{R}$, a representation of $n \in \mathbb{Z}$ as a finite sum $n = \sum_{i=1}^{k} c_i a_i$ with each $c_i \in C$ is ***n*-summand minimal** if no other finite representation $n = \sum_{i=1}^{j} c_i' a_i$ satisfies $\sum_{i=1}^{j} |c_i'| < \sum_{i=1}^{k} |c_i|$.

The Zeckendorf decomposition is n-summand minimal for every $n \in \mathbb{N}$ when $C = \{0, 1\}$.

Question: What restrictions are necessary to ensure summand minimality depending on the choice of C and $\{a_n\}_{n=1}^{\infty}$?

Example. The Zeckendorf decomposition $4 = 3 + 1 = F_3 + F_1$ is not 4-summand minimal for $C = \{0, \frac{1}{2}, 1\}$ as $4 = \frac{1}{2}(8) = \frac{1}{2}F_5$.

Example. The Zeckendorf decomposition $4 = 3 + 1 = F_3 + F_1$ is not 4-summand minimal for $C = \{0, \frac{1}{2}, 1\}$ as $4 = \frac{1}{2}(8) = \frac{1}{2}F_5$.

Theorem (SMALL 2025: Duvivier, Kennon, M–, Rysmakhanov, Watson)

Every $n \in \mathbb{N}$ has a unique representation $n = \sum_{j=1}^{k} a_j$ with each a_j satisfying either $a_j = F_i$ for some i or $a_j = \frac{1}{2}F_i$ for some $i \neq 2$ such that

- *If $a_s = F_i$ and $a_t = F_j$ then $i \geq j + 3$,*
- *If $a_s = \frac{1}{2}F_i$ and $a_t = \frac{1}{2}F_j$ then $i \geq j + 4$,*
- *If $a_s = \frac{1}{2}F_i$ and $a_t = F_j$ with $i > j$ then $i \geq j + 5$,*
- *If $a_s = F_i$ and $a_t = \frac{1}{2}F_j$ with $i > j$ then $i \geq j + 2$.*

This representation is n-summand minimal for all n.

Zeckendorf Games

Fibonacci Game: Rules

- Two player game, alternate turns, last to move wins.

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- Bins $F_1, F_2, F_3, \dots$, start with N pieces in F_1 and others empty.

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    (if $k = 1$ then $2F_1$ becomes $1F_2$)
  - ◇ If pieces at F_k and F_{k+1} remove and add one at F_{k+2}.

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  - ◇ If pieces at F_k and F_{k+1} remove and add one at F_{k+2}.

Questions:

- Does the game end? How long?
- For each N who has the winning strategy?
- What is the winning strategy?

Sample Game

Start with 10 pieces at F_1, rest empty.

10	0	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

8	1	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

6	2	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $2F_2 = F_3 + F_1$

Sample Game

Start with 10 pieces at F_1, rest empty.

7	0	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

5	1	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_2 + F_3 = F_4$.

Sample Game

Start with 10 pieces at F_1, rest empty.

5	0	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

3	1	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

1	2	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_2 = F_3$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	1	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_3 + F_4 = F_5$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

No moves left, Player One wins.

Sample Game

Player One won in 9 moves.

(1)	10	0	0	0	0
(2)	8	1	0	0	0
(1)	6	2	0	0	0
(2)	7	0	1	0	0
(1)	5	1	1	0	0
(2)	5	0	0	1	0
(1)	3	1	0	1	0
(2)	1	2	0	1	0
(1)	0	1	1	1	0
	0	1	0	0	1
	$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Sample Game

Player Two won in 10 moves.

(1)	10	0	0	0	0
(2)	8	1	0	0	0
(1)	6	2	0	0	0
(2)	7	0	1	0	0
(1)	5	1	1	0	0
(2)	5	0	0	1	0
(1)	3	1	0	1	0
(2)	1	2	0	1	0
(1)	2	0	1	1	0
(2)	0	1	1	1	0
	0	1	0	0	1
	$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Games end

Theorem

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive: $\left(\sqrt{k}+\sqrt{k+1}\right)-\sqrt{k+2}>0$.
- Splitting: $2\sqrt{k}-\left(\sqrt{k+1}+\sqrt{k+1}\right)>0$.
- Spitting 1's: $2\sqrt{1}-\sqrt{2}>0$.
- Splitting 2's: $2\sqrt{2}-\left(\sqrt{3}+\sqrt{1}\right)>0$.

Games Lengths: I

Upper bound: At most $3n - 3Z(n) - I(n) + 1$ moves

- $Z(n)$ is the number of terms in the Zeckendorf decomposition,
- $I(n)$ is the sum of the indices.

Fastest game: $n - Z(n)$ moves ($Z(n)$ is the number of summands in n's Zeckendorf decomposition).

From always moving on the largest summand possible (deterministic).

Games Lengths: II

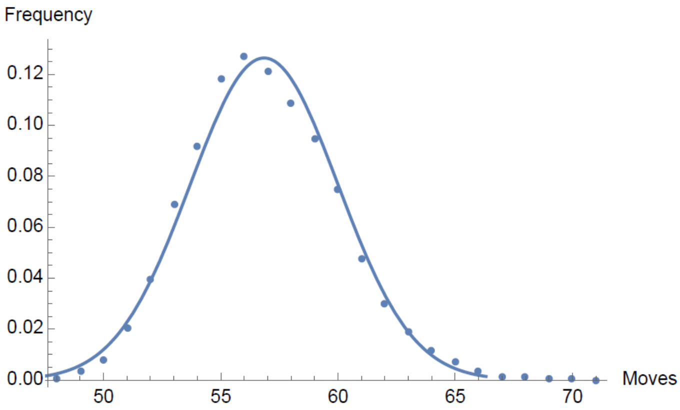


Figure: Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when $n = 60$ vs a Gaussian. Natural conjecture....

Winning Strategy

Theorem

Player Two Has a Winning Strategy

Idea is to show if not, Player Two could steal Player One's strategy.

Non-constructive!

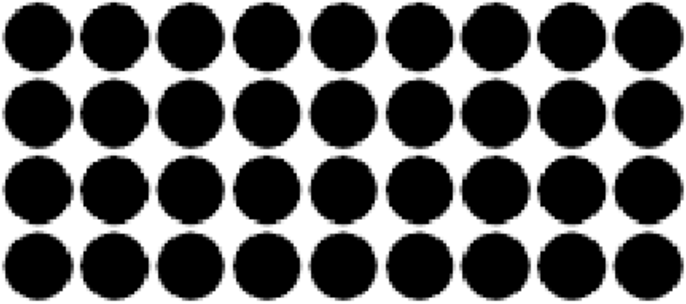
Will highlight idea with a simpler game.

Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at (i, j) and coloring every dot (m, n) with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Prove Player 1 has a winning strategy!



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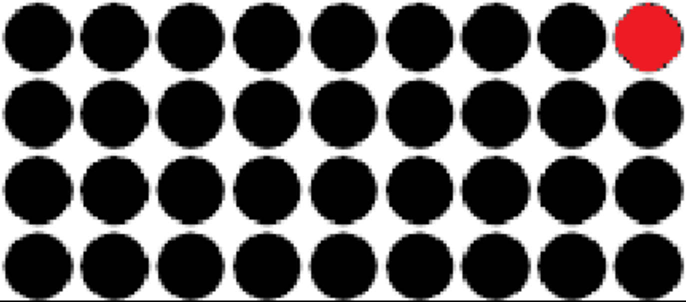
Proof Player 1 has a winning strategy. If have, play; if not, steal.



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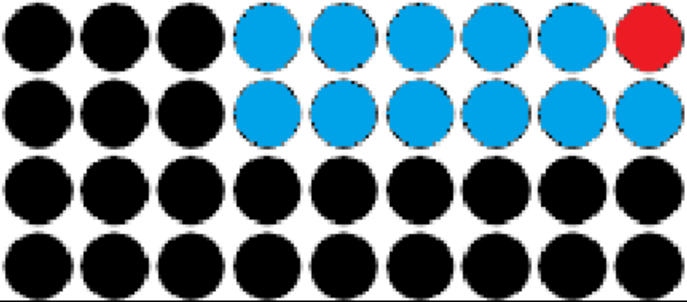

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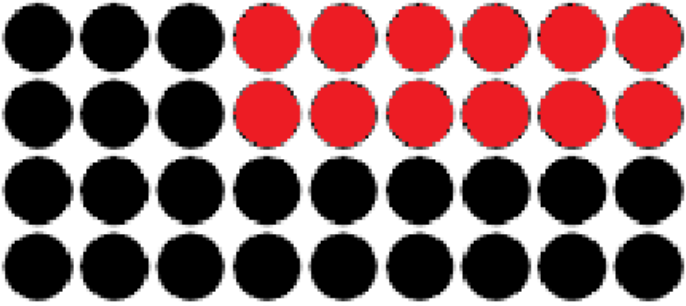
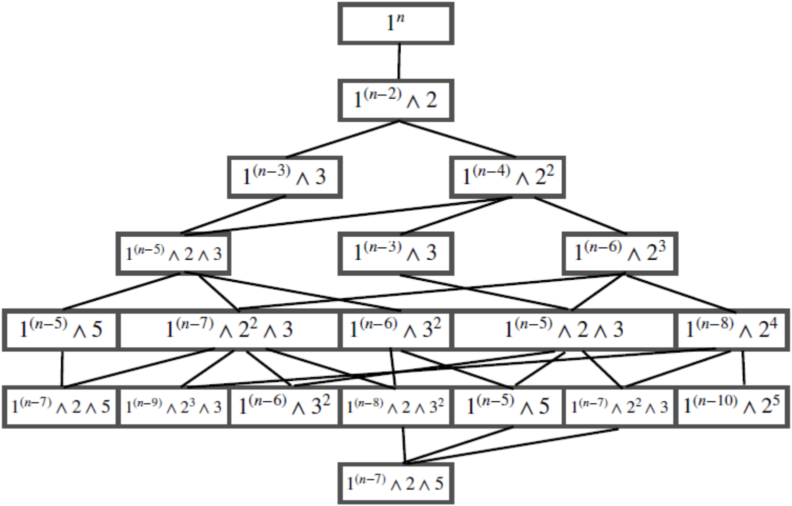


Sketch of Proof for Player Two's Winning Strategy



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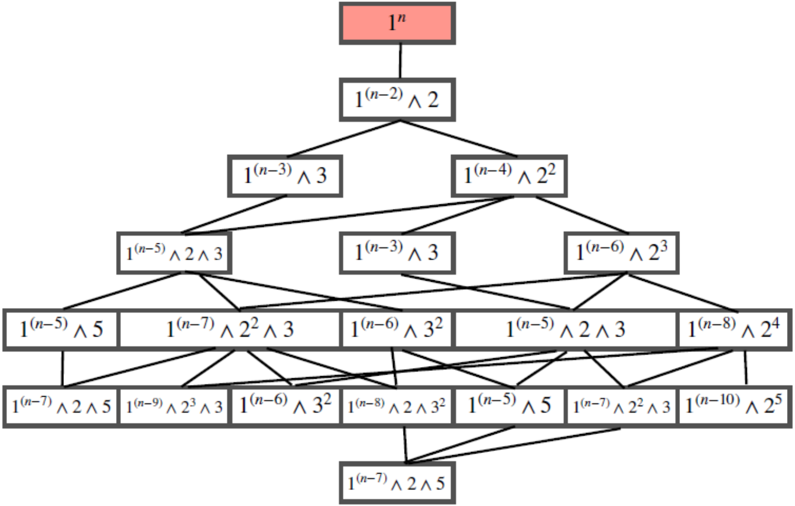

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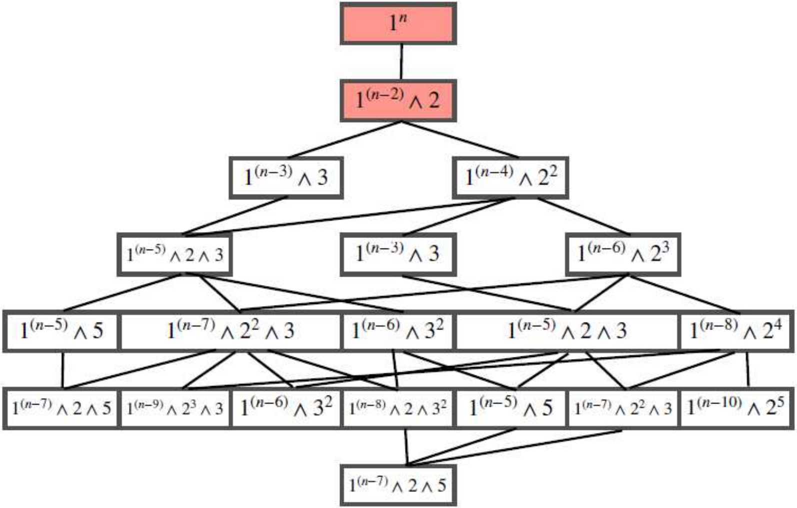

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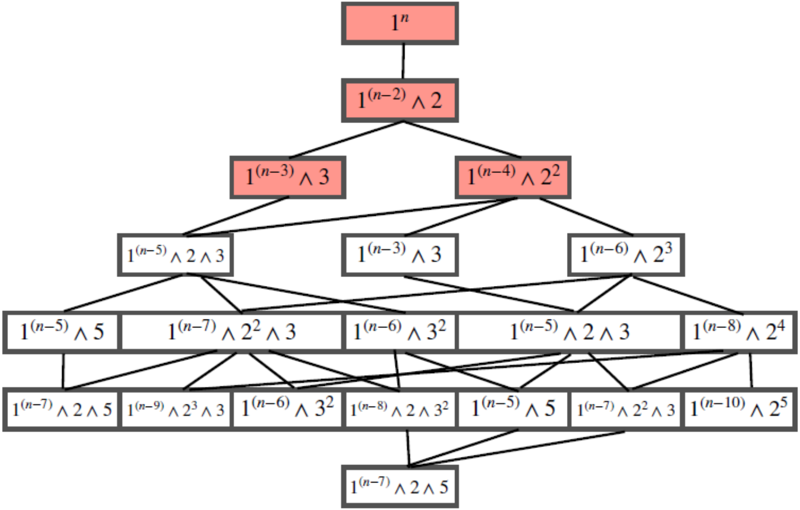

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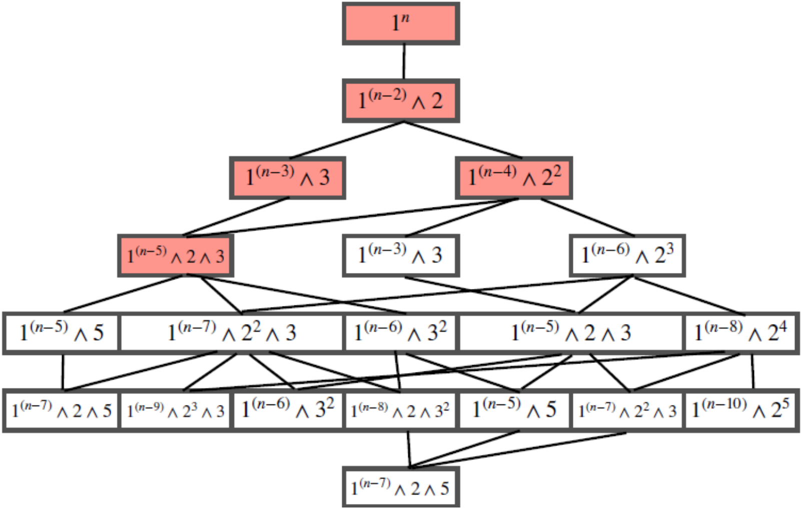

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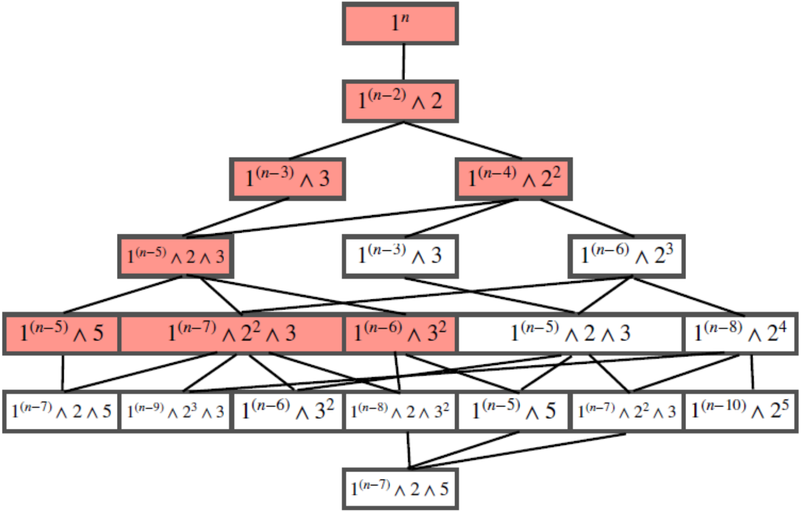

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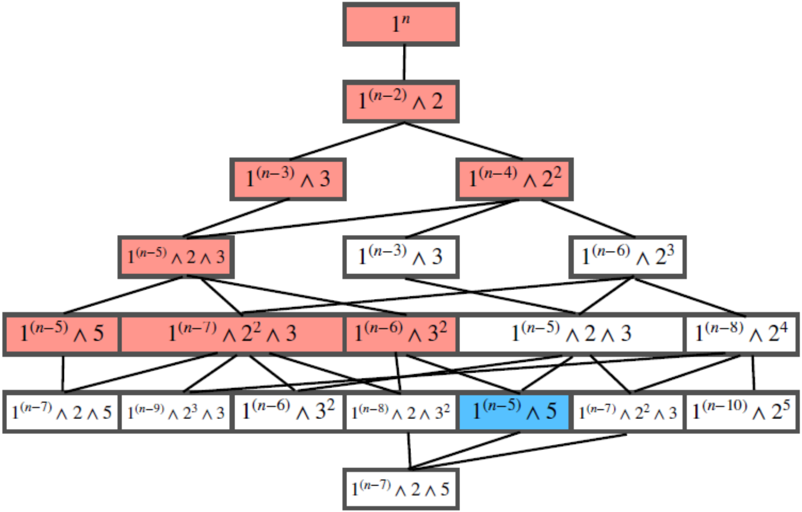

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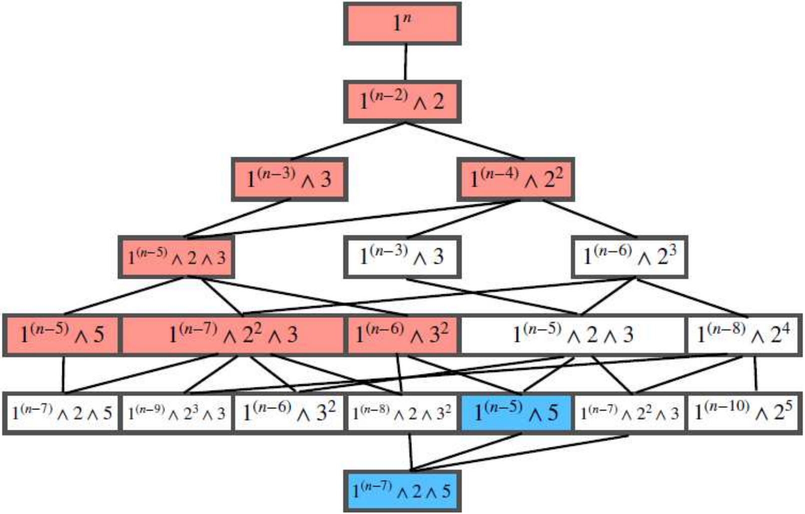

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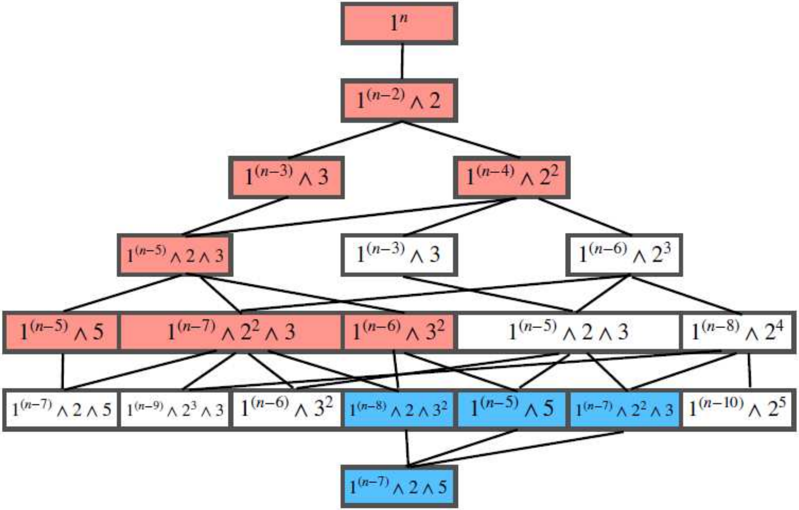

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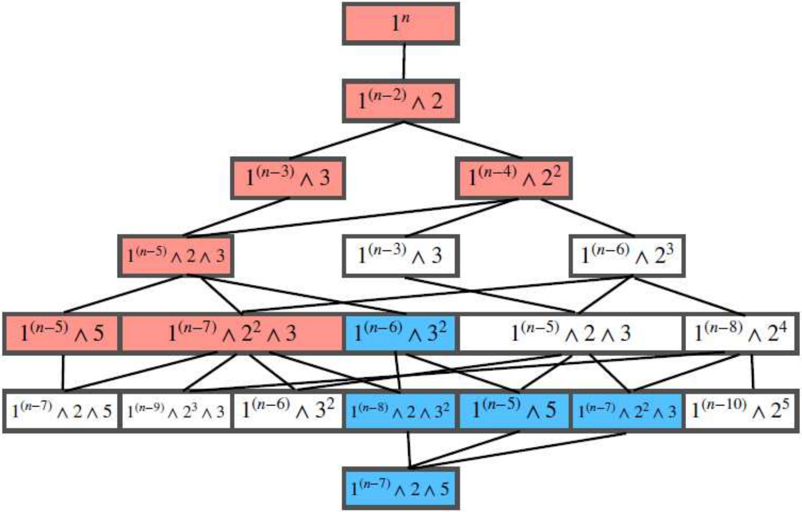


The Bergman Game

Definition

The **Bergman Game** is played with the standard split/combine moves from the Zeckendorf game, but on a two-sided infinite tape instead of a one-sided infinite tape.
It produces base-φ decompositions ($\varphi = (1+\sqrt{5})/2$).

Example

0	0	4	0	0
1	0	2	1	0
1	0	1	0	1

$4 = \varphi^{-2} + \varphi^{0} + \varphi^{2}$.

The Bergman Game

Theorem (Baily, Dell, Durmić, Fleischmann, Jackson, Mijares, M., Pesikoff, Reifenberg, Reina, Yang)

The longest Bergman Game with n summands terminates in $\Theta(n^2)$ time regardless of where the summands are placed. The shortest possible Bergman Game terminates in $\Theta(n)$ time.

Natural Question: Who has the winning strategy?

- Not currently known.
- Game tree explodes, escaping a strategy steal.

The Frodnekcez Game (Reverse Zeckendorf Game): Rules

- The Zeckendorf game in reverse, last to move wins.
- Bins $F_1, F_2, F_3, \dots,$ for some natural number N, start with one piece in bin F_k if F_k is in the Zeckendorf decomposition of N, and have other bins empty.
- A turn is one of the following moves:
  - ◇ If one piece at F_{k+1} and one at F_{k-2}, can remove and add two pieces on F_k.
  - ◇ If piece at F_{k+2}, remove and add one piece at both F_k and F_{k+1}.

    (F_1 and F_3 becomes $2F_2$, and F_2 becomes $2F_1$)

Problem created and analyzed by PANTHers 2023 from the 2023 SMALL REU: Zoe Batterman, Aditya Jambhale, Akash Narayanan, Kishan Sharma, Andrew Yang, Chris Yao.

Winning Strategy?

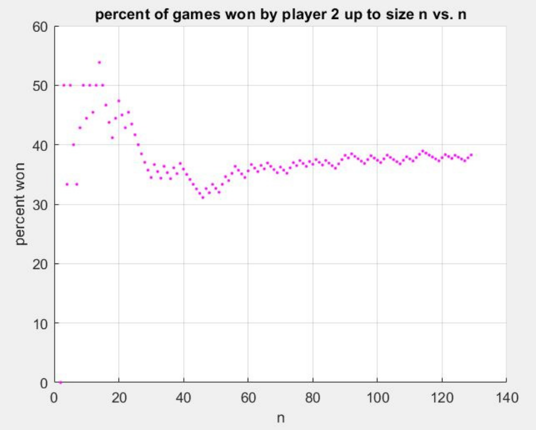


Figure: In the forward Zeckendorf game, Player 2 wins for all $N > 2$. The reverse game is more interesting. Natural conjecture...

Current / Future Work

- What if $p \geq 3$ people play the Fibonacci game? Some multi-player results.
- Does the number of moves in random games converge to a Gaussian? Evidence....
- How long do games take? Proved closed interval.
- Accelerated games: do as many of one move as wish....
- What of other recurrences?

$500 Prize: Determine the winning strategy.

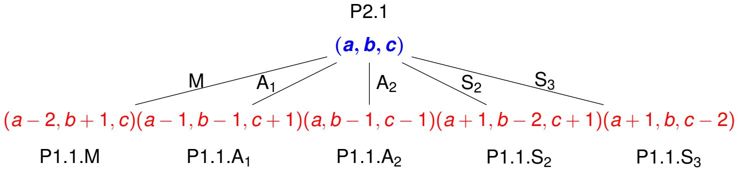
Black Hole Zeckendorf Game (Ongoing Work: SMALL 2024)

How can we simplify the game?

F_m Black Hole Variation

Any pieces placed in a column F_i for $i \geq m$ are permanently removed from gameplay.

For the F_4 case, this allows for the following moves:



SMALL 2025: Bortnovskyi, Kennon, Lucas, M–, Pan, Vranesko.

Definition

(The Two Player Ordered Zeckendorf Game)

- Start with an ordered list of n copies of F_1.
- Possible moves:
  1. **Merging:** $(F_i, F_{i+1}) \to F_{i+2}$
  2. **Splitting:** $(F_i, F_i) \to (F_{i-2}, F_{i+1})$ for $i > 2$
  3. **Splitting Twos:** $(F_2, F_2) \to (F_1, F_3)$
  4. **Splitting Ones:** $(F_1, F_1) \to F_2$
  5. **Switching:** $(F_i, F_j) \to (F_j, F_i)$ if $i > j$
- The last player to move wins.

Sample Game

$n = 5$

- $(F_1 \quad F_1 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_1 \quad F_1 \quad F_1)$
- $(F_2 \quad F_2 \quad F_1)$
- $(F_1 \quad F_3 \quad F_1)$
- $(F_1 \quad F_1 \quad F_3)$
- $(F_2 \quad F_3)$
- (F_4)

Player Two won in 6 moves.

Game Lengths

Question: Does the game always terminate?

Theorem

The ordered Zeckendorf game always terminates. The final state is the Zeckendorf decomposition of n with the elements placed in ascending order.

Game Lengths

Question: Does the game always terminate?

Theorem

The ordered Zeckendorf game always terminates. The final state is the Zeckendorf decomposition of n with the elements placed in ascending order.

Question: How long does it take?

Theorem

The shortest game has length $n - Z(n)$, where $Z(n)$ is the number of terms in the Zeckendorf decomposition of n. The longest game has length $O(n^2)$.

Thanks / References

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Papers

- The Zeckendorf Game (with Paul Baird-Smith, Alyssa Epstein and Kristen Flint), Proceedings of CANT 2018. https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/ZeckGameCANT10.pdf
- On Summand Minimality of Generalized Zeckendorf Decompositions (with Katherine Cordwell, Max Hlavacek, Chi Huynh, Carsten Peterson, and Yen Nhi Truong Vu), Research in Number Theory (4 (2018), no. 43, https://doi.org/10.1007/s40993-018-0137-7)/ https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/ZeckMinimalSummands61.pdf
- The Generalized Zeckendorf Game (with Paul Baird-Smith, Alyssa Epstein and Kristen Flint), Fibonacci Quarterly (57 (2019) no. 5, 1-14). https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/ZeckGameGeneral_FibQ10.pdf
- The Fibonacci Quilt Game (with Alexandra Newlon), Fibonacci Quarterly (2 (2020), 157-168). https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/FQgame30.pdf
- Extending Zeckendorf's Theorem to a Non-constant Recurrence and the Zeckendorf Game on this Non-constant Recurrence Relation (with Ela Boldyriew, Anna Cusenza, Linglong Dai, Pei Ding, Aidan Dunkelberg, John Haviland, Kate Huffman, Dianhui Ke, Daniel Kleber, Jason Kuretski, John Lentfer, Tianhao Luo, Clayton Mizgerd, Vashisth Tiwari, Jingkai Ye, Yunhao Zhang, Xiaoyan Zheng, and Weiduo Zhu), Fibonacci Quarterly. (5 (2020), 55-76). https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/ZeckExtendingZeckNonConstantGame40.pdf
- Deterministic Zeckendorf Games (with Ruoci Li, Xiaonan Li, Clay Mizgerd, Chenyang Sun, Dong Xia, And Zhyi Zhou), Fibonacci Quarterly. (58 (2020), no. 5, 152-160). https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/zeckgamedeterministic31.pdf

Papers

- Winning Strategy for the Multiplayer and Multialliance Zeckendorf Games (with Anna Cusenza, Aiden Dunkelberg, Kate Huffman, Dianhui Ke, Daniel Kleber, Clayton Mizgerd, Vashisth Tiwari, Jingkai Ye, Xiaoyan Zheng), Fibonacci Quarterly. (59 (2021), 308–318). https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/ZeckGameWinningAlliancePolymath20.pdf
- Bounds on Zeckendorf Games (with Anna Cusenza, Aiden Dunkelberg, Kate Huffman, Dianhui Ke, Daniel Kleber, Micah McClatchey, Clayton Mizgerd, Vashisth Tiwari, Jingkai Ye, Xiaoyan Zheng), Fibonacci Quarterly (1 (2022), no. 1, 57–71). https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/ZeckGame_BoundLength_2020polyreu_10.pdf
- Completeness of Positive Linear Recurrence Sequences (with Ela Boldyriew, John Haviland, Phuc Lam, John Lentfer, Fernando Trejos Suarez), submitted to CANT Conference Proceedings. https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/ZeckCompletePLRS02.pdf
- The Generalized Bergman Game (with Benjamin Baily, Justine Dell, Irfan Durmic, Henry Fleischmann, Faye Jackson, Isaac Mijares, Ethan Pesikoff, Luke Reifenberg, Alicia Smith Reina, Yingzi Yang). https://arxiv.org/abs/2109.00117
- Winning Strategies for the Generalized Zeckendorf Game (Steven J. Miller, Eliel Sosis, Jingkai Ye), submitted to the Fibonacci Quarterly. https://web.williams.edu/Mathematics/sjmiller/public_html/math/papers/ZeckGame_WinStrategies_Polymath2022.pdf
- Accelerated Zeckendorf Games (with Diego Garcia-Fernandezsesma, Thomas Rascon, Risa Vandegrift, Ajmain Yamin), preprint.

Thank you!

The Cookie Problem
and Zeckendorf's Theorem

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Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \cdots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

Let $p_{n,k} = \#\{N \in [F_n, F_{n+1})$: the Zeckendorf decomposition of N has exactly k summands$\}$.

For $N \in [F_n, F_{n+1})$, the largest summand is F_n.

$$N = F_{i_1} + F_{i_2} + \cdots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \cdots < i_{k-1} < i_k = n, \; i_j - i_{j-1} \geq 2.$$

$$d_1 := i_1 - 1, \; d_j := i_j - i_{j-1} - 2 \; (j > 1).$$

$$d_1 + d_2 + \cdots + d_k = n - 2k + 1, \; d_j \geq 0.$$

Cookie counting $\Rightarrow p_{n,k} = \binom{n-2k+1-k-1}{k-1} = \binom{n-k}{k-1}$.