

Why I love Monovariants: From Zombies to Conway's Soldiers to Fibonacci Games

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Goals of the Talk

- Research: What questions to ask? How? With whom?
- Explore: Look for the right perspective.
- Utilize: What are your tools and how can they be used?
- succeed: Control what you can: reports, talks,

Joint with many students and junior faculty over the years.

Research: What questions to ask? How? With whom?

- Build on what you know and can learn.
- What will be interesting?
- How will you work?
- Where are the questions? Classes, arXiv, conferences,

Explore: Look for the right perspective.

- Ask interesting questions.
- Look for connections.
- Be a bit of a jack-of-all trades.

Leads naturally into....

Utilize: What are your tools and how can they be used?

Law of the Hammer:

- Abraham Kaplan: I call it the law of the instrument, and it may be formulated as follows: Give a small boy a hammer, and he will find that everything he encounters needs pounding.
- Abraham Maslow: I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.
- Bernard Baruch: If all you have is a hammer, everything looks like a nail.



Succeed: Control what you can: reports, talks

- Write up your work: post on the arXiv, submit.
- Go to conferences: present and mingle (no spam and P&J).
- Turn things around fast: show progress, no more than 24 hours on mundane.
- Service: refereeing, MathSciNet,

Invariants / Monovariants

Invariant: a quantity that is unchanged throughout the process / operations. (Big application: Noether's theorem).

Monovariant: a quantity that only changes in one direction throughout the process / operations. See

<https://howardhalim.com/math/Invariants%20and%20Monovariants.pdf>

for a nice collection of problems.

Often a challenge to find a useful monovariant.

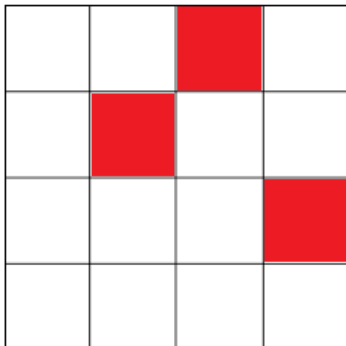
Zombies

Zombie Infection: Rules

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.

Zombie Infection: Rules

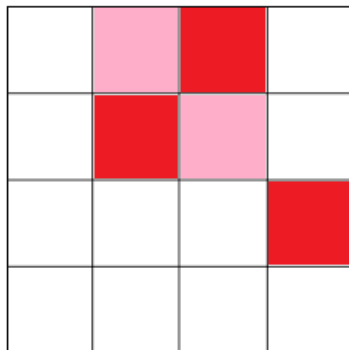
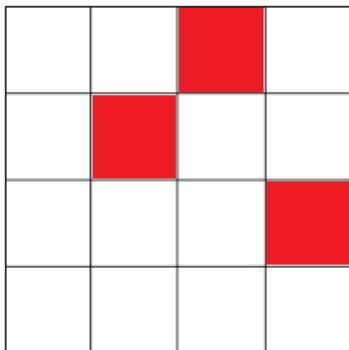
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Initial Configuration

Zombie Infection: Rules

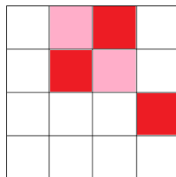
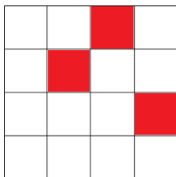
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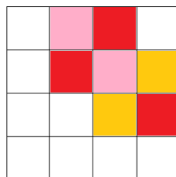
Initial Configuration One moment later

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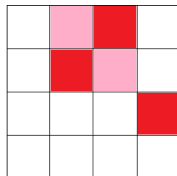
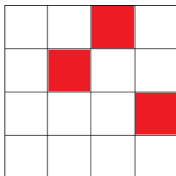
Initial Configuration One moment later



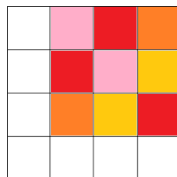
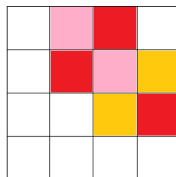
Two moments later

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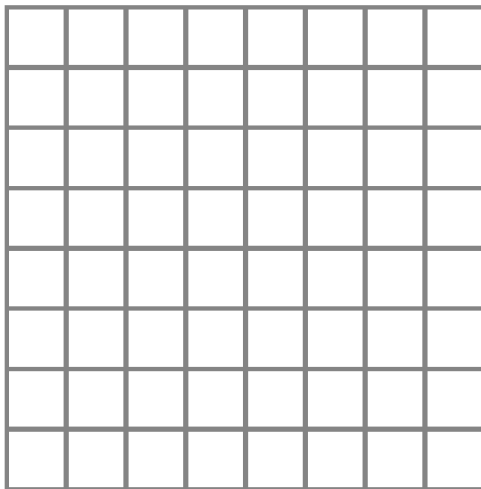
Initial Configuration One moment later



Two moments later Three moments later

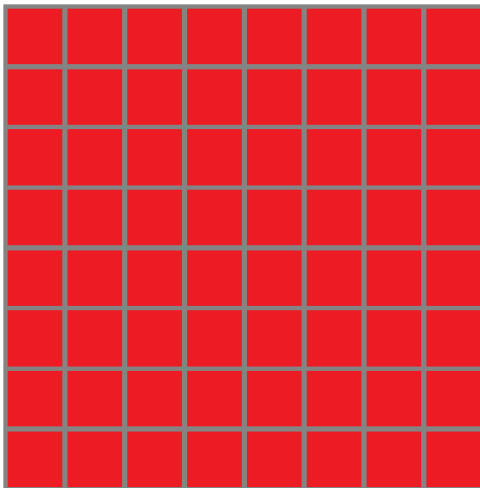
Zombie Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?



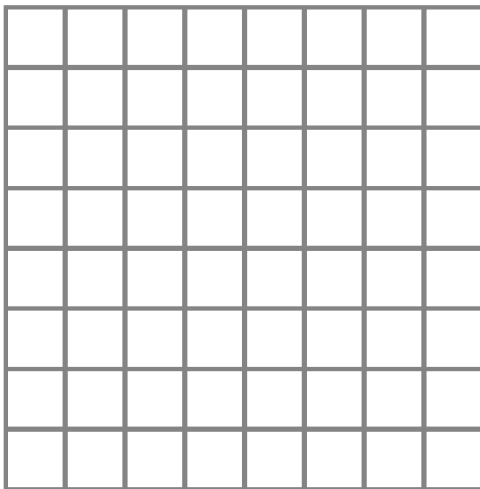
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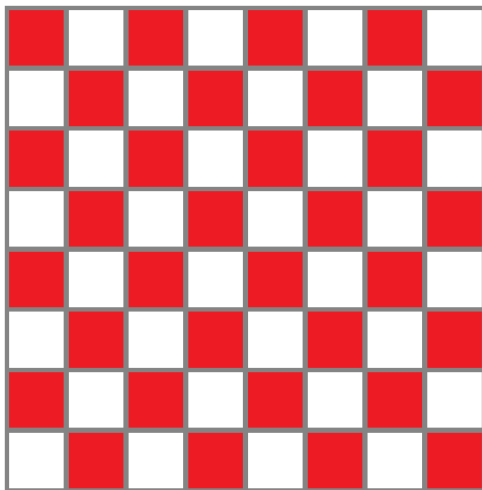
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Next simplest initial state ensuring all eventually infected...?



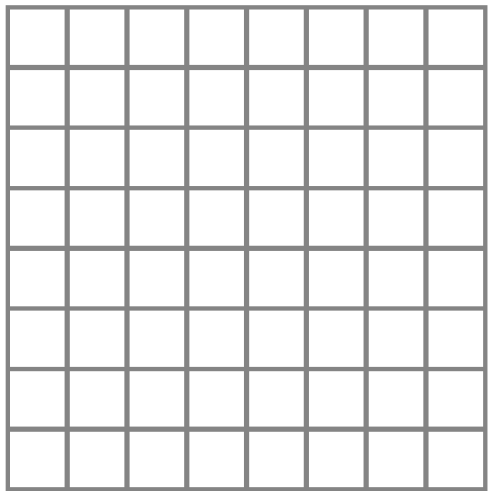
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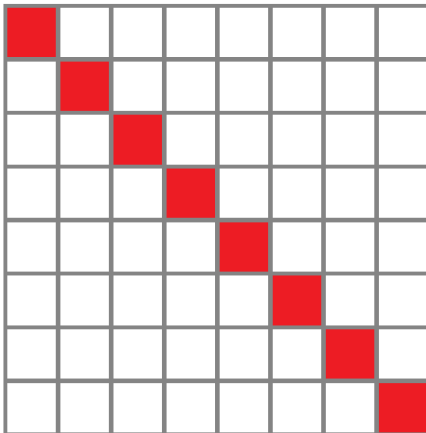
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Fewest number of initial infections needed to get all...?



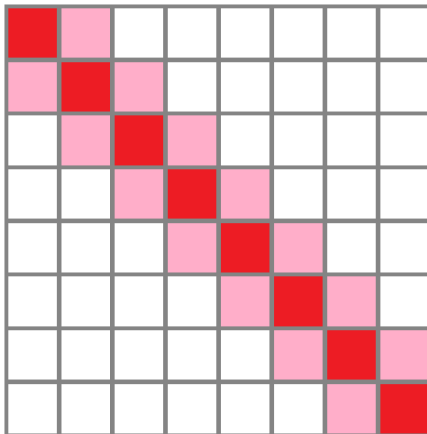
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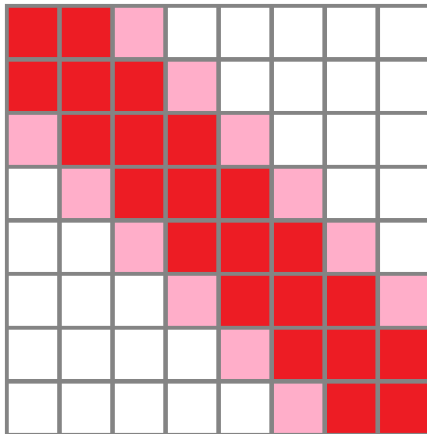
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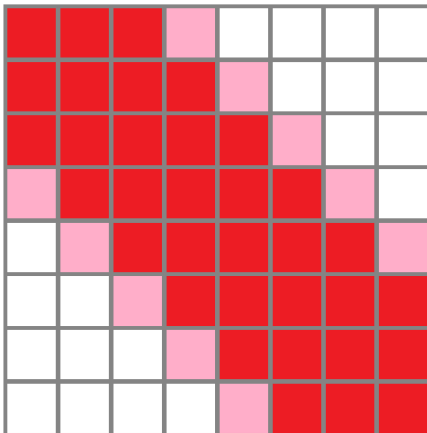
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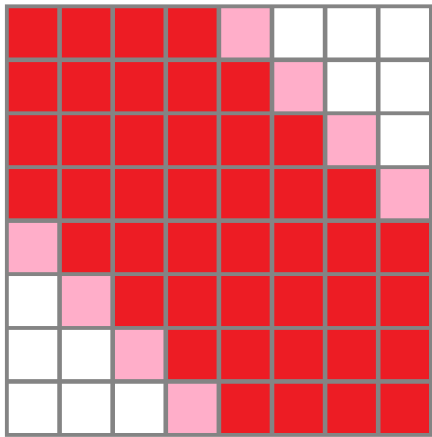
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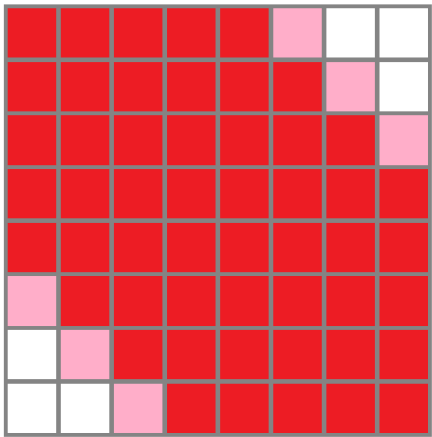
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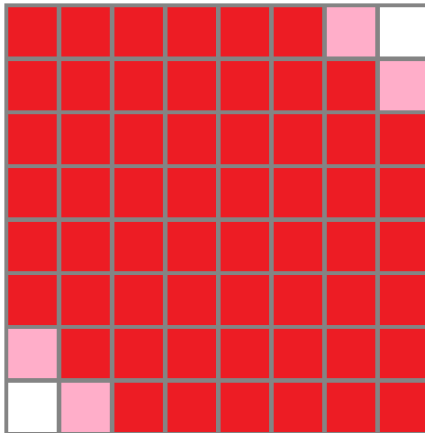
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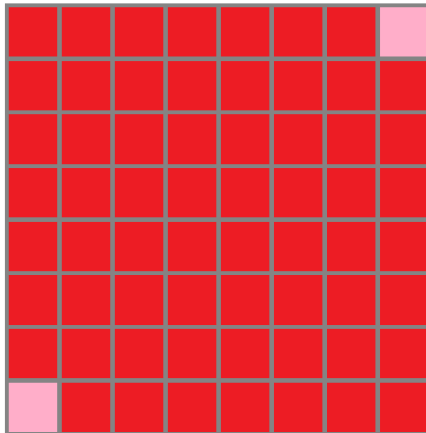
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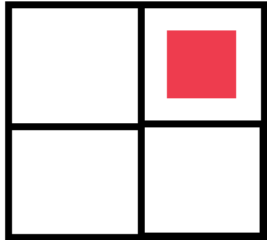
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Fewest number of initial infections needed to get all...?



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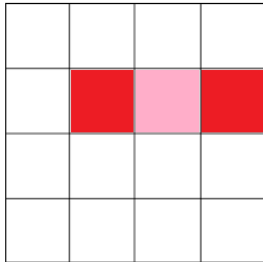
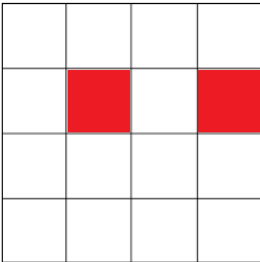


■	1	2
1	3	4
2	4	5

1	■	1
2	3	2
4	5	4

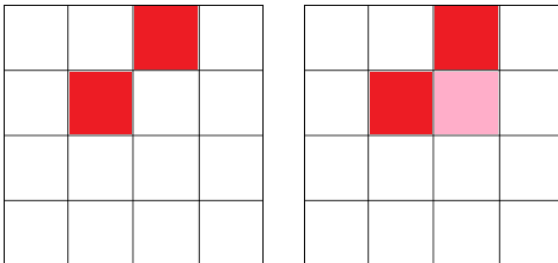
2	1	2
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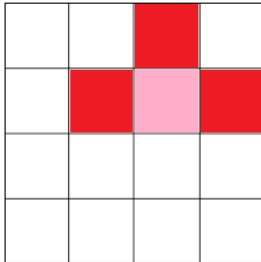
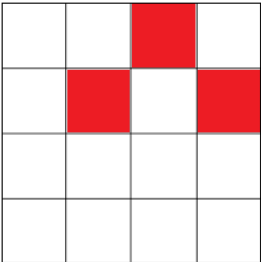
Perimeter of infection unchanged.

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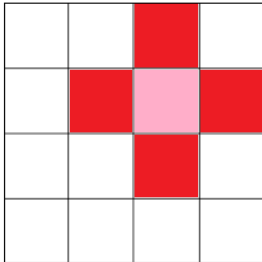
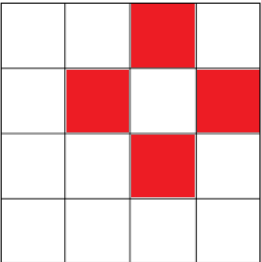
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Zombie Infection: Can $n - 1$ infect all on an $n \times n$ board?



Perimeter of infection decreases by 2.

Zombie Infection: Can $n - 1$ infect all on an $n \times n$ board?



Perimeter of infection decreases by 4.

Zombie Infection: $n - 1$ cannot infect all

- If $n - 1$ infected, maximum perimeter is $4(n - 1) = 4n - 4$.

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- Other questions?

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- Perimeter of $n \times n$ square is $4n$, so at least 1 square safe!
- How many must be safe?
- Other questions: Is a row safe? Higher dimensions? Other regions (torus)?

Conway's Soldiers

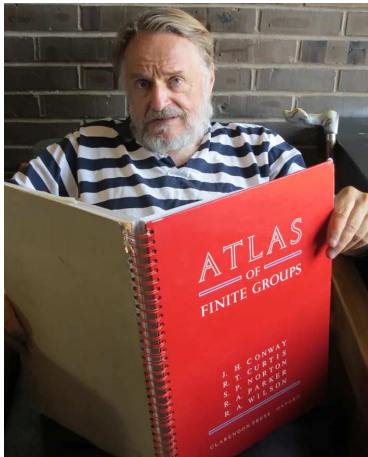


Figure: John Horton Conway: Image from The Guardian.

Conway's Soldiers / Checker Problem

Problem: Infinite checkerboard, pieces at all (x, y) with $y \leq 0$.
Using horizontal / vertical jumps (jumped piece gone forever),
how high can you move a piece?

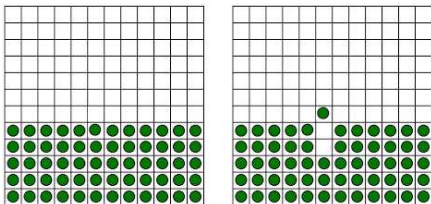


Figure: Left: A subset of the initial configuration. Right: moving a soldier / checker up 1.

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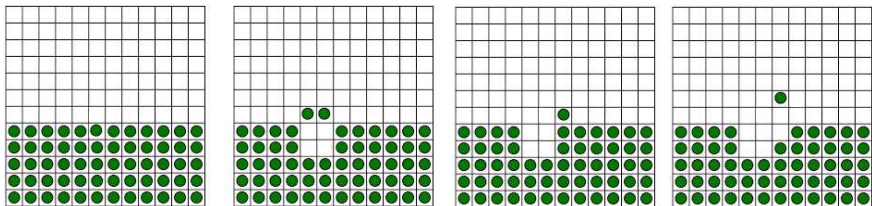


Figure: Left: A subset of the initial configuration. Right: moving a soldier / checker up 2. Can you do 3? 4? 5? Any height?

Conway's Soldiers

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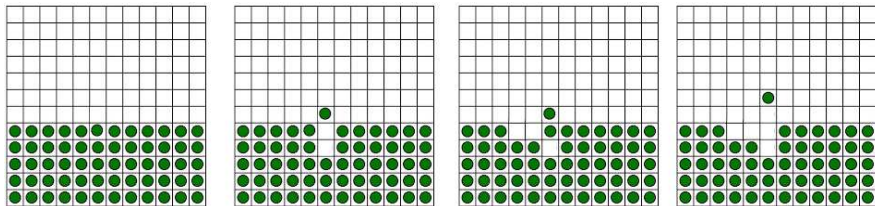


Figure: Left: A subset of the initial configuration. Right: Also moving a soldier / checker up 2. Can you do 3? 4? 5? Any height?

Conway's Soldiers: The Monovariant: I

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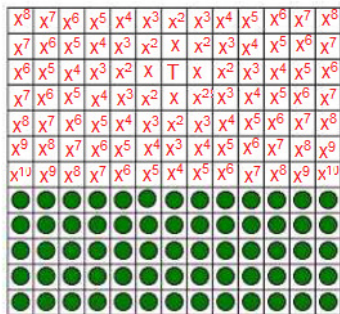


Figure: Conway's monovariant: What is it?

Conway's Soldiers: The Monovariant: II

Choose target $T = (0, 5)$.

Fix x (to be determined later) and attach x^{i+j} to a point that is i units horizontally and j units vertically from T .

x^8	x^7	x^6	x^5	x^4	x^3	x^2	x^3	x^4	x^5	x^6	x^7	x^8
x^7	x^6	x^5	x^4	x^3	x^2	x	x^2	x^3	x^4	x^5	x^6	x^7
x^6	x^5	x^4	x^3	x^2	x	T	x	x^2	x^3	x^4	x^5	x^6
x^7	x^6	x^5	x^4	x^3	x^2	x	x^2	x^3	x^4	x^5	x^6	x^7
x^8	x^7	x^6	x^5	x^4	x^3	x^2	x^3	x^4	x^5	x^6	x^7	x^8
x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^4	x^5	x^6	x^7	x^8	x^9
x^{1+j}	x^9	x^8	x^7	x^6	x^5	x^4	x^5	x^6	x^7	x^8	x^9	x^{1+j}
●	●	●	●	●	●	●	●	●	●	●	●	●
●	●	●	●	●	●	●	●	●	●	●	●	●
●	●	●	●	●	●	●	●	●	●	●	●	●
●	●	●	●	●	●	●	●	●	●	●	●	●
●	●	●	●	●	●	●	●	●	●	●	●	●

Conway's Soldiers: The Monovariant: III

Choose a target point T ; for us it is a point of height 5 above the checkers: $T = (0, 5)$.

Fix x (to be determined later) and attach x^{i+j} to a point that is i units horizontally and j units vertically from T .

What is the value of the initial board?

- Zeroth row: $\dots, x^7, x^6, x^5, x^6, x^7, \dots$: sum is

$$x^5 + 2 \sum_{k=6}^{\infty} x^k = x^5 + \frac{2x^6}{1-x} = \frac{(1+x)x^5}{1-x}.$$

Conway's Soldiers: The Monovariant: III

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- Each row is x times previous: Thus initial board value is

$$\frac{(1+x)x^5}{1-x} \sum_{n=0}^{\infty} x^n = \frac{(1+x)x^5}{(1-x)^2}.$$

Conway's Soldiers: The Monovariant: IV

Two moves: lose 2 pieces and add a piece further from T , or lose 2 pieces and add a piece closer to T .

First type of move clearly decreases value of board.

Conway's Soldiers: The Monovariant: IV

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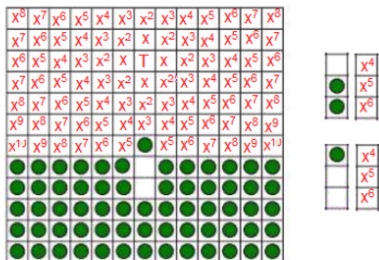


Figure: Moving pieces on x^6 and x^5 to on x^4 .
Change is $x^4 - x^5 - x^6 = x^4(1 - x - x^2)$, want this to be zero.

Conway's Soldiers: The Monovariant: IV

Two moves: lose 2 pieces and add a piece further from T , or lose 2 pieces and add a piece closer to T .

Second type replaces x^{n+2} and x^{n+1} with an x^n : change is $x^n - x^{n+1} - x^{n+2}$. Choose x so that this change is zero.

Thus $1 - x - x^2 = 0$ or $x = (-1 \pm \sqrt{5})/2$. Take positive root, $(-1 + \sqrt{5})/2 = \varphi - 1$ (φ the golden mean).

Monovariant: sum of the values of squares with checkers.

Conway's Soldiers: The Monovariant: V

Choose a target point T .

- Initial board value is

$$\frac{(1+x)x^5}{(1-x)^2} : \text{when } x = \frac{\sqrt{5}-1}{2} \text{ get } 1.$$

- Target at $(0, 4)$ contributes $x = \frac{\sqrt{5}-1}{2} \approx 0.618034$; as less than 1 possible (and can be done).
- Target at $(0, 5)$, board's value at least 1. Moves never increase value: **IMPOSSIBLE IN FINITE TIME!**¹

¹Possible in "infinite" game: <https://tartarus.org/gareth/maths/stuff/solarmy.pdf>.

New Results

Conway Checkers m -game: Start with m checkers on each gridpoint (original game is just 1), if jump over it lose one checker.

SMALL 2024

Given a Conway Checkers m -game, the maximum row attainable, n_m , satisfies

$$\lfloor \log_{\varphi}(m) + 4.67 \rfloor \leq n_m \leq \lfloor \log_{\varphi}(m) + 5 \rfloor$$

for sufficiently large m , where φ is the golden ratio $\frac{\sqrt{5}+1}{2}$.

Zeckendorf Minimality

Introduction: Summand Minimality

Fibonacci: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:

$$2024 = 1597 + 377 + 34 + 13 + 3 = F_{16} + F_{13} + F_8 + F_6 + F_3.$$

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1, 2

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Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8, 13...

Summand Minimality

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

Theorem

*The Zeckendorf decomposition is **summand minimal**.*

Overall Question

What other recurrences are summand minimal?

Zeckendorf Decomposition is Minimal

Theorem

*The Zeckendorf decomposition is **summand minimal**: no decomposition as a sum of Fibonacci numbers (1, 2, 3, 5, ...) has fewer summands than it.*

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More natural $\sum_k a_k k$ but square-root makes strictly decreasing.

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Bounded process: For fixed n , only indices up to certain point used, and $a_k \leq n$.

Zeckendorf Decomposition is Minimal: Proof

Show $\text{Index}(\mathcal{D})$ is a mono-variant, end in the Zeckendorf decomposition, number summands never increased.

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$$F_k \wedge F_{k+1} \rightarrow F_{k+2}:$$

- $\sqrt{k} + \sqrt{k+1} > \sqrt{k+2}$.

$$2F_k \rightarrow F_{k-2} + F_{k+1}:$$

- $k \geq 3: 2\sqrt{k} > \sqrt{k-2} + \sqrt{k+1}$

- $k = 2: 2\sqrt{2} > \sqrt{1} + \sqrt{3}$

- $k = 1: 2\sqrt{1} > \sqrt{2}$

Only finitely many values, each move lowers, continue till hit Zeckendorf, number of summands never increased.

Positive Linear Recurrence Sequences

Definition

A **positive linear recurrence sequence (PLRS)** is a sequence given by a recurrence $\{a_n\}$ with

$$a_n := c_1 a_{n-1} + \cdots + c_t a_{n-t}$$

and each $c_i \geq 0$ and $c_1, c_t > 0$. We use **ideal initial conditions** $a_{-(n-1)} = 0, \dots, a_{-1} = 0, a_0 = 1$ and call (c_1, \dots, c_t) the **signature of the sequence**.

Theorem (Cordwell, Hlavacek, Huynh, M., Peterson, Vu)

For a PLRS with signature (c_1, c_2, \dots, c_t) , the Generalized Zeckendorf Decompositions are summand minimal if and only if

$$c_1 \geq c_2 \geq \cdots \geq c_t.$$

Zeckendorf Games

Fibonacci Game: Rules

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 - ◇ If pieces at F_k and F_{k+1} remove and add one at F_{k+2} .

Questions:

- Does the game end? How long?
- For each N who has the winning strategy?
- What is the winning strategy?

Sample Game

Start with 10 pieces at F_1 , rest empty.

10	0	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1 , rest empty.

8	1	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1 , rest empty.

6	2	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $2F_2 = F_3 + F_1$

Sample Game

Start with 10 pieces at F_1 , rest empty.

7	0	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1 , rest empty.

5	1	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_2 + F_3 = F_4$.

Sample Game

Start with 10 pieces at F_1 , rest empty.

5	0	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1 , rest empty.

3	1	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1 , rest empty.

1	2	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_2 = F_3$.

Sample Game

Start with 10 pieces at F_1 , rest empty.

0	1	1	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_3 + F_4 = F_5$.

Sample Game

Start with 10 pieces at F_1 , rest empty.

0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

No moves left, Player One wins.

Sample Game

Player One won in 9 moves.

(1)	10	0	0	0	0
(2)	8	1	0	0	0
(1)	6	2	0	0	0
(2)	7	0	1	0	0
(1)	5	1	1	0	0
(2)	5	0	0	1	0
(1)	3	1	0	1	0
(2)	1	2	0	1	0
(1)	0	1	1	1	0
	0	1	0	0	1
	$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Sample Game

Player Two won in 10 moves.

(1)	10	0	0	0	0
(2)	8	1	0	0	0
(1)	6	2	0	0	0
(2)	7	0	1	0	0
(1)	5	1	1	0	0
(2)	5	0	0	1	0
(1)	3	1	0	1	0
(2)	1	2	0	1	0
(1)	2	0	1	1	0
(2)	0	1	1	1	0
	0	1	0	0	1
	$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Games end

Theorem

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive: $(\sqrt{k} + \sqrt{k+1}) - \sqrt{k+2} > 0$.
- Splitting: $2\sqrt{k} - (\sqrt{k+1} + \sqrt{k+1}) > 0$.
- Spitting 1's: $2\sqrt{1} - \sqrt{2} > 0$.
- Splitting 2's: $2\sqrt{2} - (\sqrt{3} + \sqrt{1}) > 0$.

Games Lengths: I

Upper bound: At most $3n - 3Z(n) - I(n) + 1$ moves

- $Z(n)$ is the number of terms in the Zeckendorf decomposition,
- $I(n)$ is the sum of the indices.

Fastest game: $n - Z(n)$ moves ($Z(n)$ is the number of summands in n 's Zeckendorf decomposition).

From always moving on the largest summand possible (deterministic).

Games Lengths: II

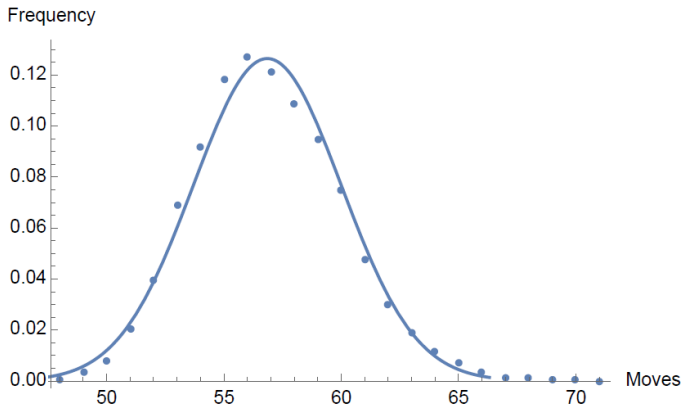


Figure: Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when $n = 60$ vs a Gaussian. **Natural conjecture....**

Winning Strategy

Theorem

Player Two Has a Winning Strategy

Idea is to show if not, Player Two could steal Player One's strategy.

Non-constructive!

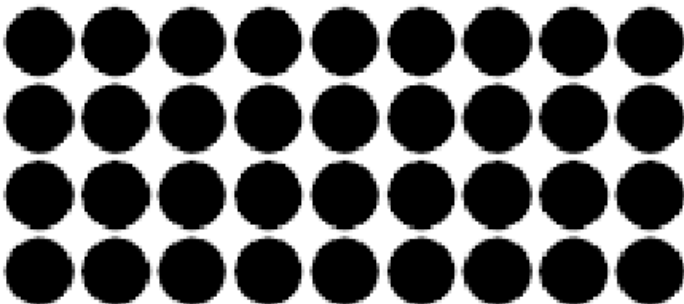
Will highlight idea with a simpler game.

Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at (i, j) and coloring every dot (m, n) with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Prove Player 1 has a winning strategy!

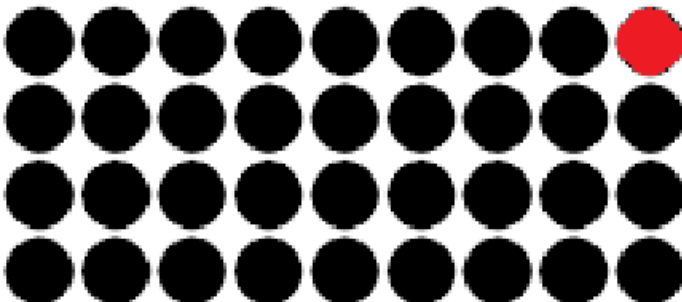


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Proof Player 1 has a winning strategy. If have, play; if not, steal.

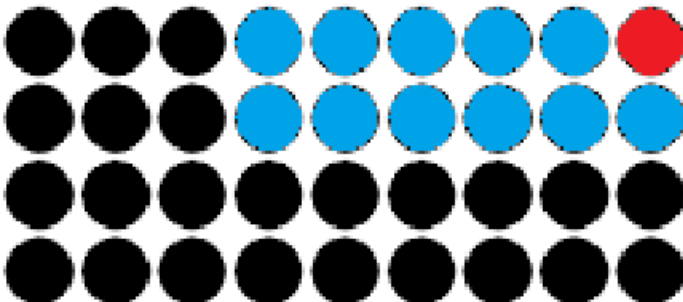


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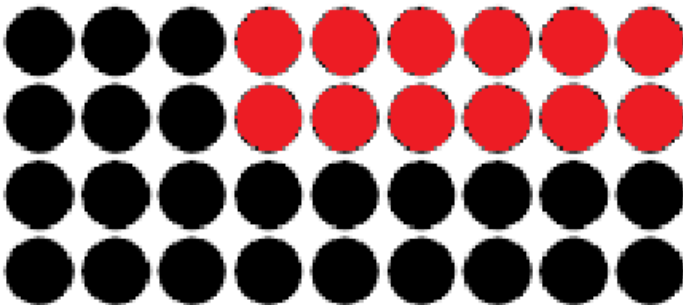


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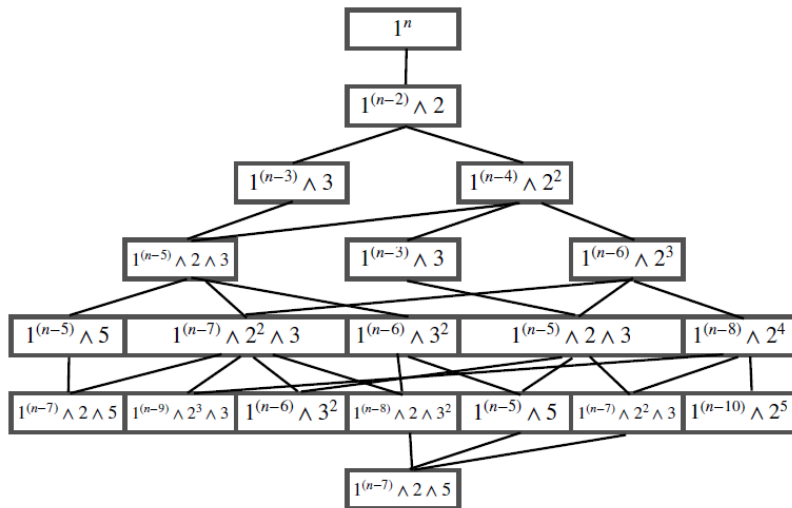
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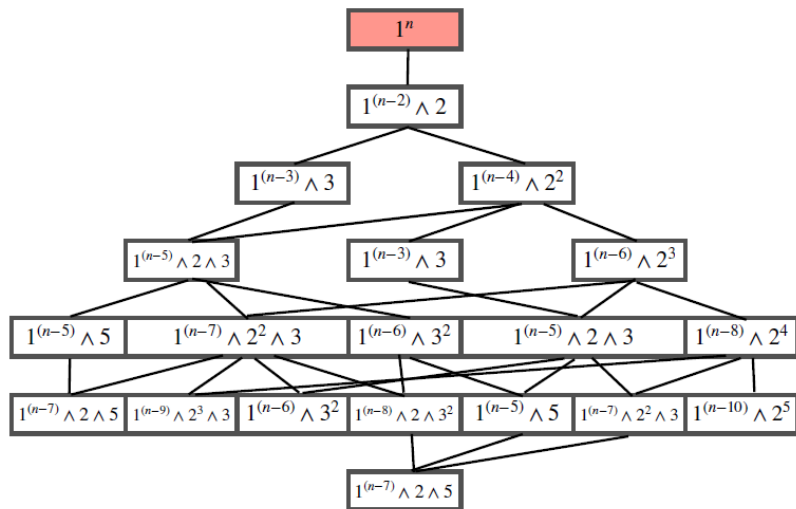
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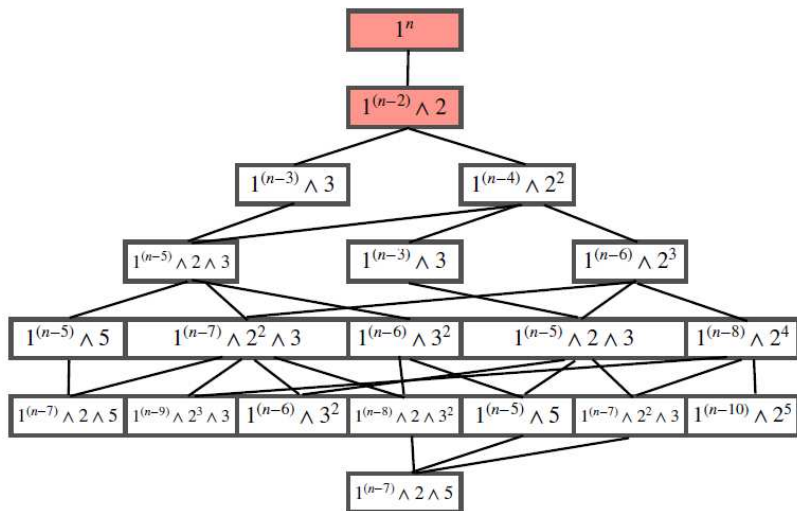
Sketch of Proof for Player Two's Winning Strategy



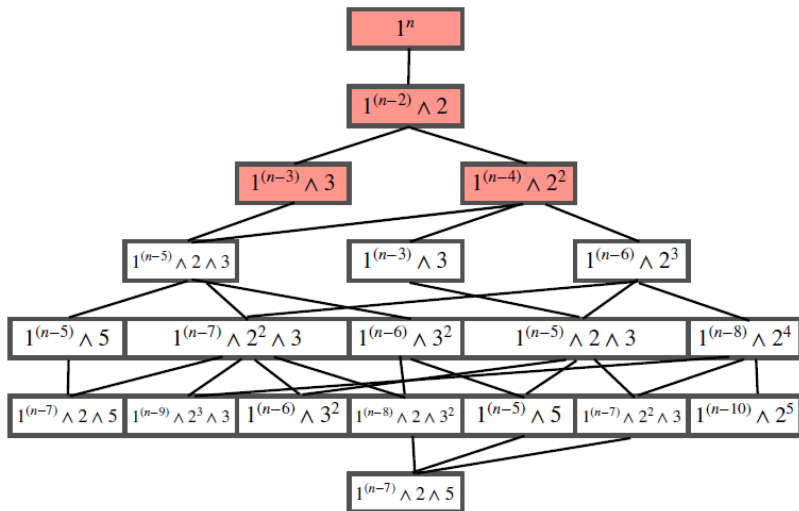
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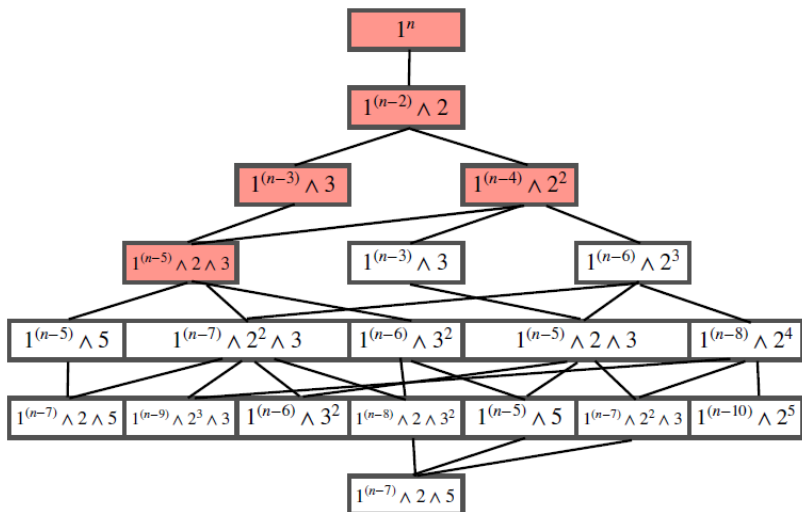
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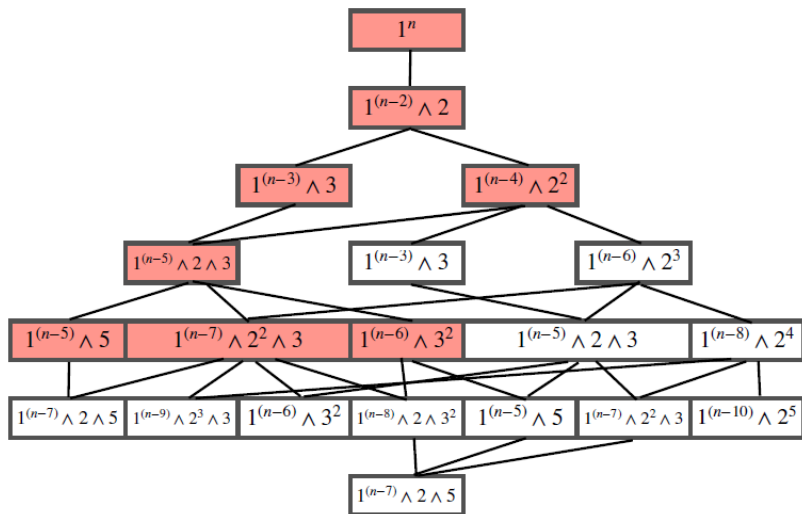
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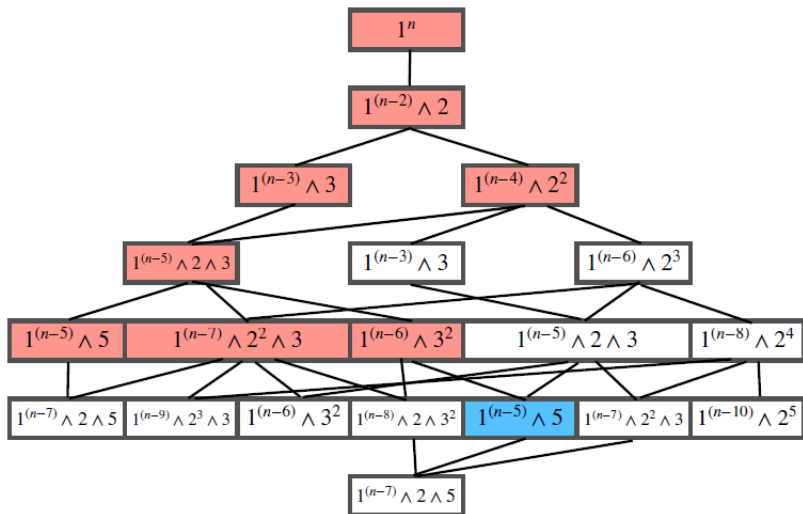
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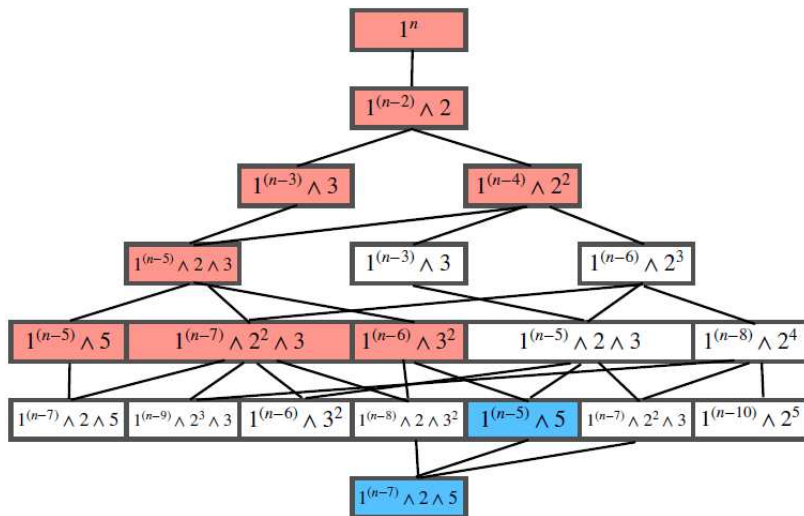
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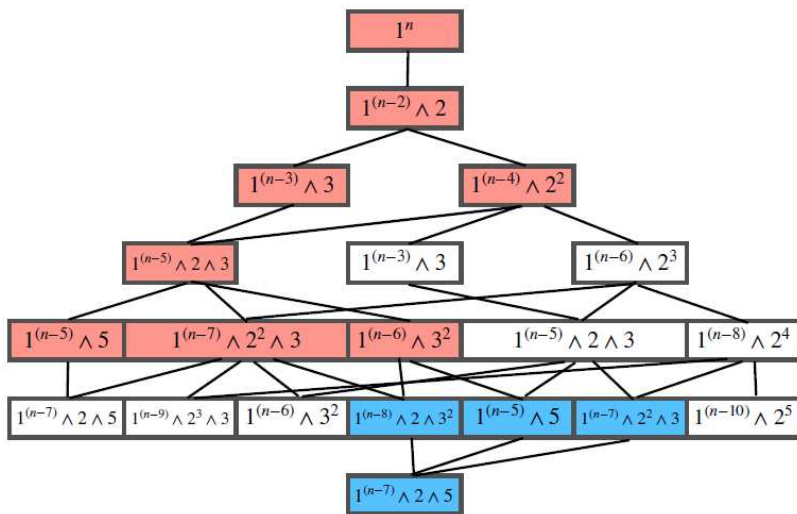
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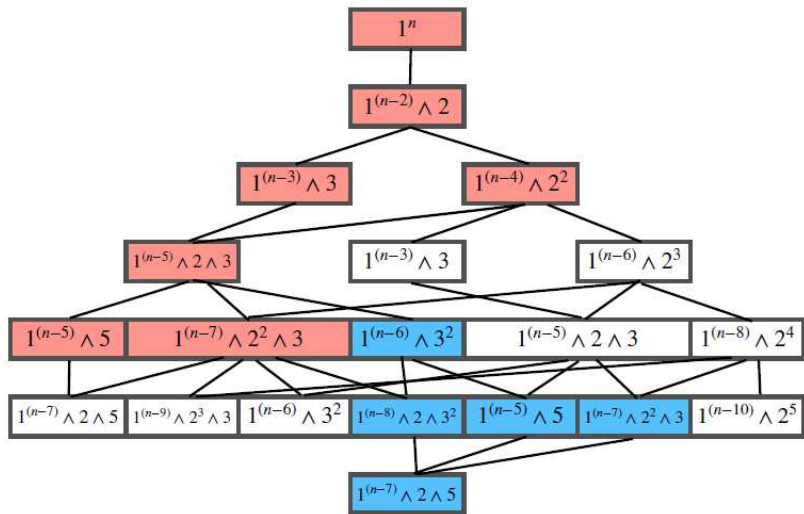
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Sketch of Proof for Player Two's Winning Strategy



The Bergman Game

Definition

The **Bergman Game** is played with the standard split/combine moves from the Zeckendorf game, but on a two-sided infinite tape instead of a one-sided infinite tape.

It produces base- φ decompositions ($\varphi = (1 + \sqrt{5})/2$).

Example

0	0	4	0	0
1	0	2	1	0
1	0	1	0	1

$$4 = \varphi^{-2} + \varphi^0 + \varphi^2.$$

The Bergman Game

Theorem (Baily, Dell, Durmić, Fleischmann, Jackson, Mijares, M., Pesikoff, Reifenberg, Reina, Yang)

The longest Bergman Game with n summands terminates in $\Theta(n^2)$ time regardless of where the summands are placed. The shortest possible Bergman Game terminates in $\Theta(n)$ time.

Natural Question: Who has the winning strategy?

- Not currently known.
- Game tree explodes, escaping a strategy steal.

The Frodnekcez Game (Reverse Zeckendorf Game): Rules

- The Zeckendorf game in reverse, last to move wins.
- Bins F_1, F_2, F_3, \dots , for some natural number N , start with one piece in bin F_k if F_k is in the Zeckendorf decomposition of N , and have other bins empty.
- A turn is one of the following moves:
 - ◇ If one piece at F_{k+1} and one at F_{k-2} , can remove and add two pieces on F_k .
 - ◇ If piece at F_{k+2} , remove and add one piece at both F_k and F_{k+1} .

$(F_1 \text{ and } F_3 \text{ becomes } 2F_2, \text{ and } F_2 \text{ becomes } 2F_1)$

Problem created and analyzed by PANTHERs 2023 from the 2023 SMALL REU: Zoe Batterman, Aditya Jambhale, Akash Narayanan, Kishan Sharma, Andrew Yang, Chris Yao.

Winning Strategy?

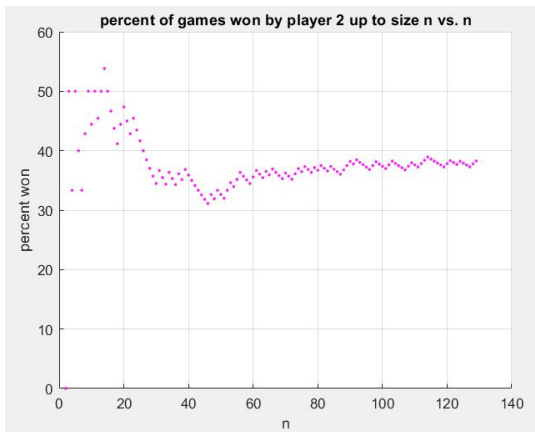


Figure: In the forward Zeckendorf game, Player 2 wins for all $N > 2$. The reverse game is more interesting. **Natural conjecture...**

Current / Future Work

- What if $p \geq 3$ people play the Fibonacci game? Some multi-player results.
- Does the number of moves in random games converge to a Gaussian? Evidence....
- How long do games take? Proved closed interval.
- Accelerated games: do as many of one move as wish....
- What of other recurrences?

\$500 Prize: Determine the winning strategy.

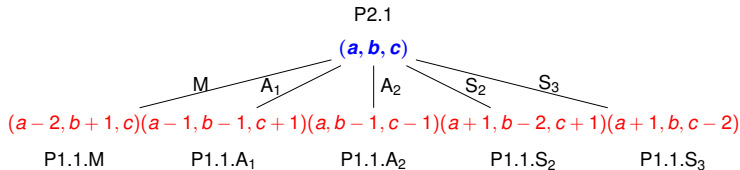
Black Hole Zeckendorf Game (Ongoing Work: SMALL 2024)

How can we simplify the game?

F_m Black Hole Variation

Any pieces placed in a column F_i for $i \geq m$ are permanently removed from gameplay.

For the F_4 case, this allows for the following moves:



Thanks / References

Thanks

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- The Generalized Bergman Game (with Benjamin Baily, Justine Dell, Irfan Durmic, Henry Fleischmann, Faye Jackson, Isaac Mijares, Ethan Pesikoff, Luke Reifenberg, Alicia Smith Reina, Yingzi Yang). <https://arxiv.org/abs/2109.00117>
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Thank you!

The Cookie Problem and Zeckendorf's Theorem

The Cookie Problem

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Cookie Monster eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do.

Divides the cookies into P sets.

Example: 8 cookies and 5 people ($C = 8, P = 5$):



Preliminaries: The Cookie Problem: Reinterpretation

Reinterpreting the Cookie Problem

The number of solutions to $x_1 + \dots + x_P = C$ with $x_i \geq 0$ is $\binom{C+P-1}{P-1}$.

Let $p_{n,k} = \# \{N \in [F_n, F_{n+1}) : \text{the Zeckendorf decomposition of } N \text{ has exactly } k \text{ summands}\}$.

For $N \in [F_n, F_{n+1})$, the **largest summand is F_n** .

$$N = F_{i_1} + F_{i_2} + \dots + F_{i_{k-1}} + F_n,$$

$$1 \leq i_1 < i_2 < \dots < i_{k-1} < i_k = n, \quad i_j - i_{j-1} \geq 2.$$

$$d_1 := i_1 - 1, \quad d_j := i_j - i_{j-1} - 2 \quad (j > 1).$$

$$d_1 + d_2 + \dots + d_k = n - 2k + 1, \quad d_j \geq 0.$$

Cookie counting $\Rightarrow p_{n,k} = \binom{n-2k+1-k-1}{k-1} = \binom{n-k}{k-1}$.