Why I love Monovariants: From Zombies to Conway's Soldiers via the Golden Mean

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http://www.williams.edu/Mathematics/sjmiller/public_html

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Zombies

General Advice: What are your tools and how can they be used?

Law of the Hammer:

- Abraham Kaplan: I call it the law of the instrument, and it may be formulated as follows: Give a small boy a hammer, and he will find that everything he encounters needs pounding.
- Abraham Maslow: I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.
- Bernard Baruch: If all you have is a hammer, everything looks like a nail.

Zombie Infection: Rules

- If share walls with 2 or more infected, become infected.
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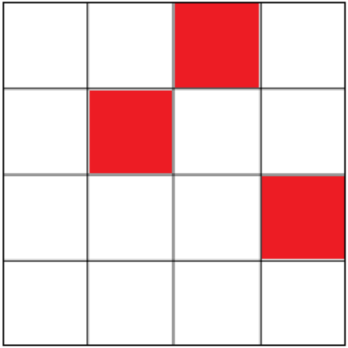

Initial Configuration

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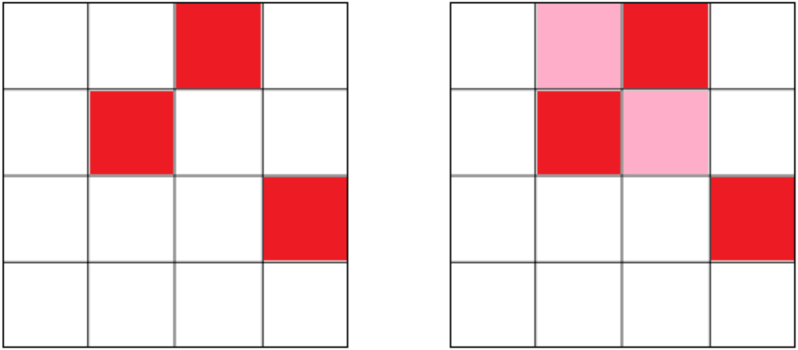

Initial Configuration *One moment later*

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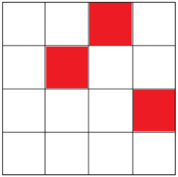

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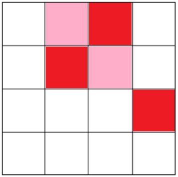
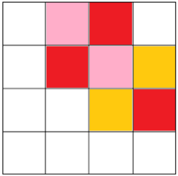


One moment later



Two moments later

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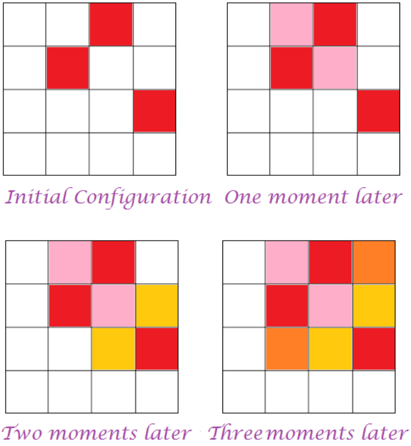
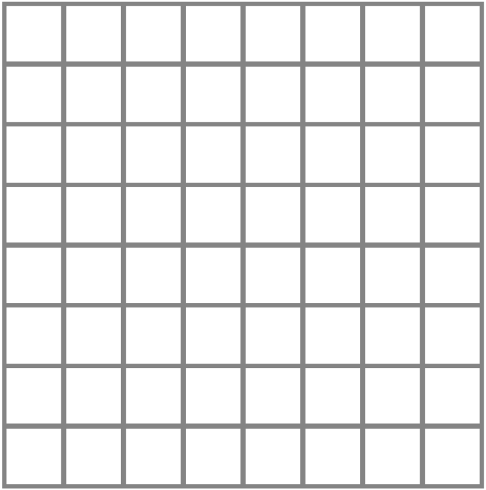


Zombie Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?



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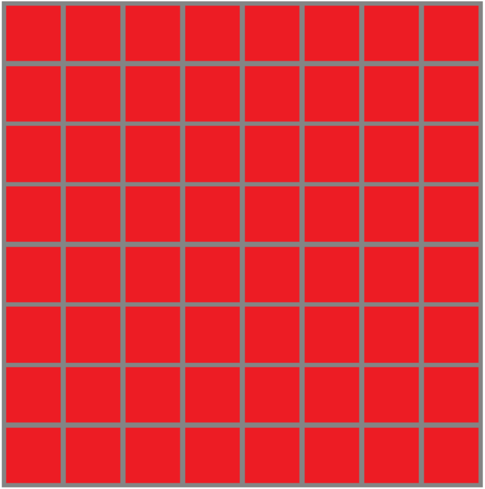

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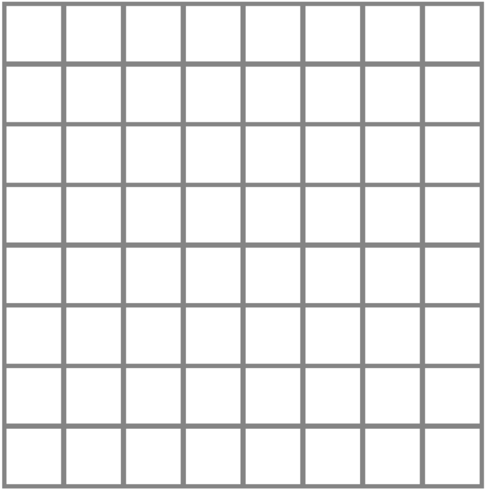
Next simplest initial state ensuring all eventually infected...?



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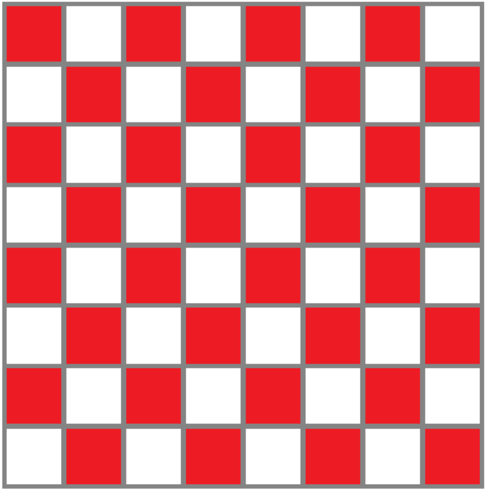

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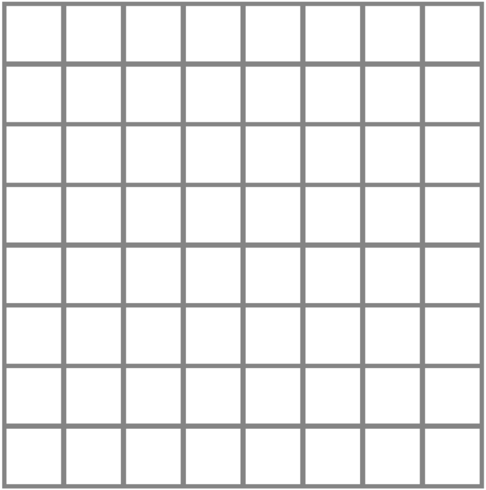

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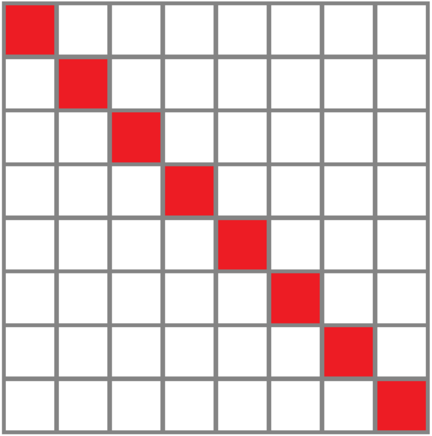

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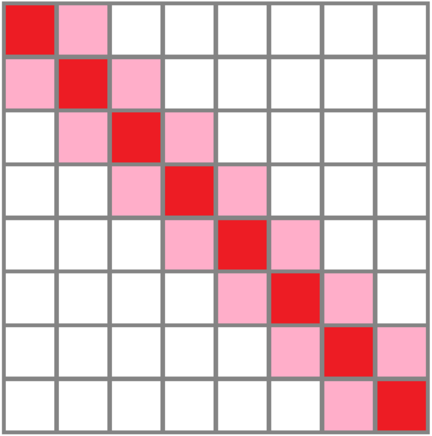

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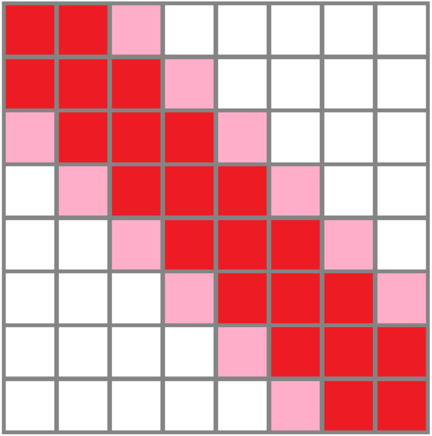

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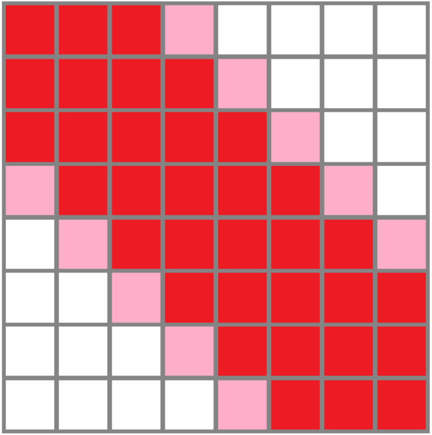

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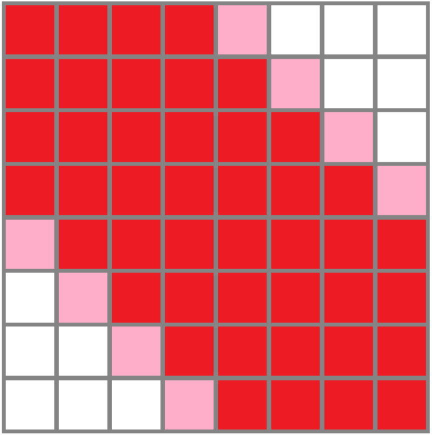

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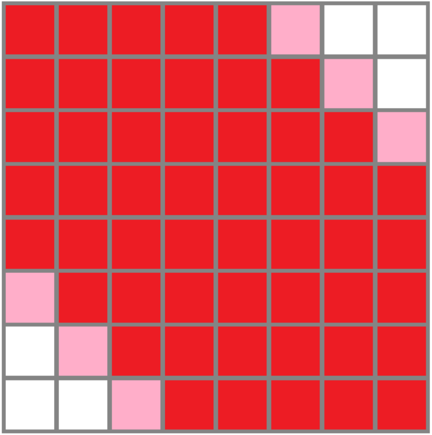

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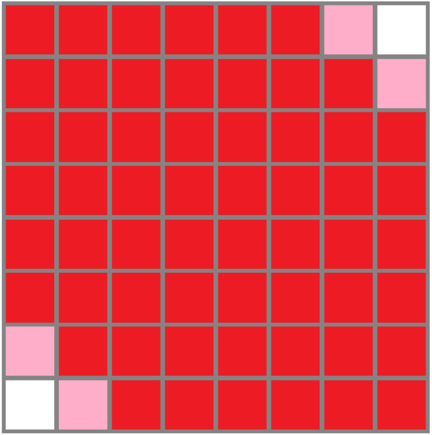

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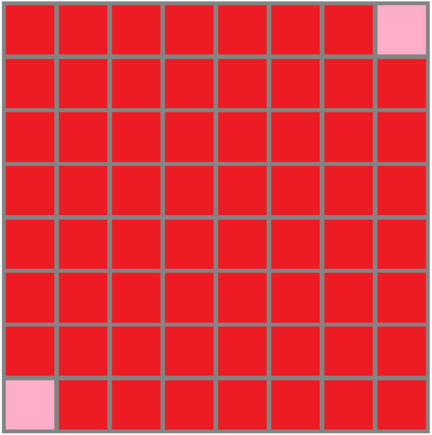
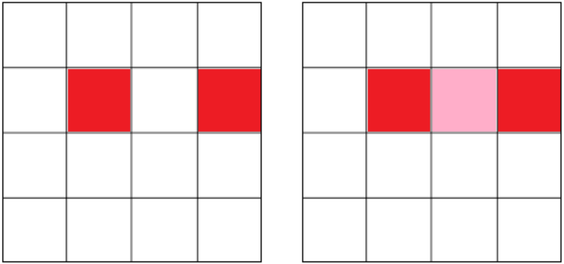


Zombie Infection: Can $n-1$ infect all on an $n \times n$ board?

Zombie Infection: Can $n - 1$ infect all on an $n \times n$ board?



Perimeter of infection unchanged.

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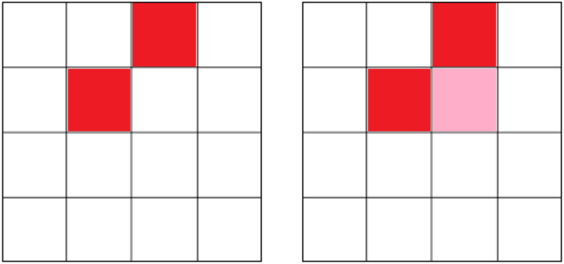

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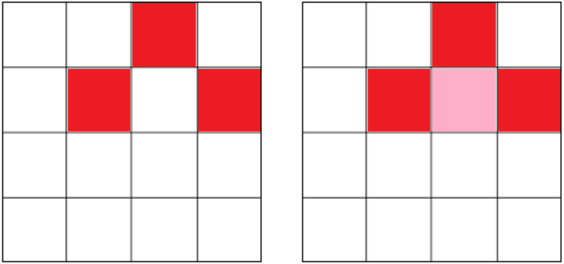

Perimeter of infection decreases by 2.

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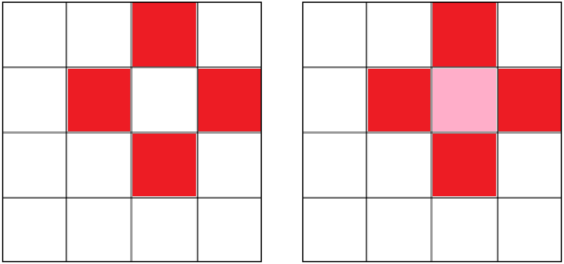

Perimeter of infection decreases by 4.

Zombie Infection: $n - 1$ cannot infect all

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- Mono-variant: As time passes, perimeter of infection never increases.

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- If $n-1$ infected, maximum perimeter is $4(n-1) = 4n-4$.
- Mono-variant: As time passes, perimeter of infection never increases.
- Perimeter of $n \times n$ square is $4n$, so at least 1 square safe!

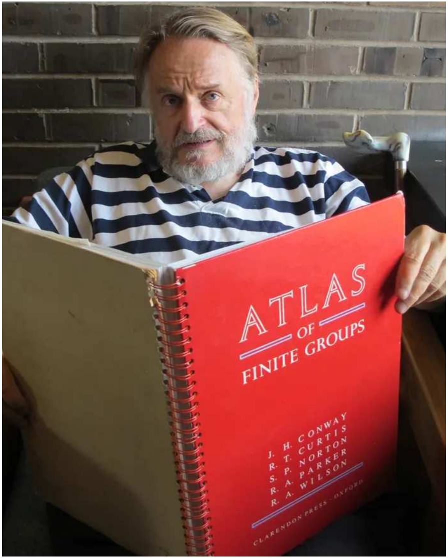

Figure: John Horton Conway: Image from The Guardian.

John Horton Conway

So many items to discuss....

- Game of life: https://en.wikipedia.org/wiki/Conway%27s_Game_of_Life
- Monster group: https://en.wikipedia.org/wiki/Monster_group
- Audioactive decay: https://en.wikipedia.org/wiki/Look-and-say_sequence
- 15 and 290 theorems: https://en.wikipedia.org/wiki/15_and_290_theorems

Conway's Soldiers

Problem: Infinite checkerboard, pieces at all (x, y) with $y \leq 0$. Using horizontal / vertical jumps (jumped piece gone forever), how high can you move a piece?

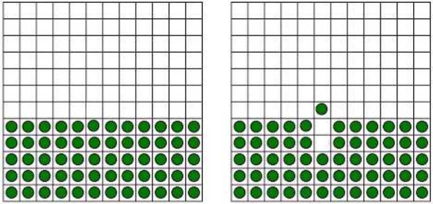

Figure: Left: A subset of the initial configuration. Right: moving a soldier / checker up 1. Can you do 2?

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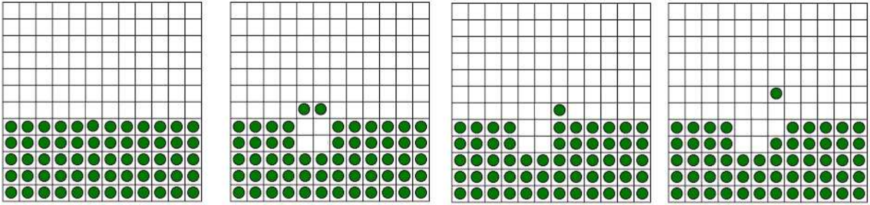

Figure: Left: A subset of the initial configuration. Right: moving a soldier / checker up 2. Can you do 3? 4? 5? Any height?

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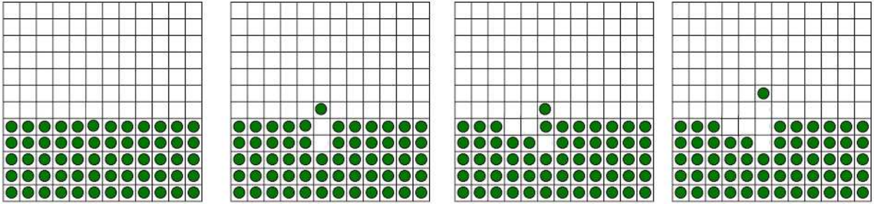

Figure: Left: A subset of the initial configuration. Right: Also moving a soldier / checker up 2. Can you do 3? 4? 5? Any height?

Conway's Soldiers: The Monovariant: I

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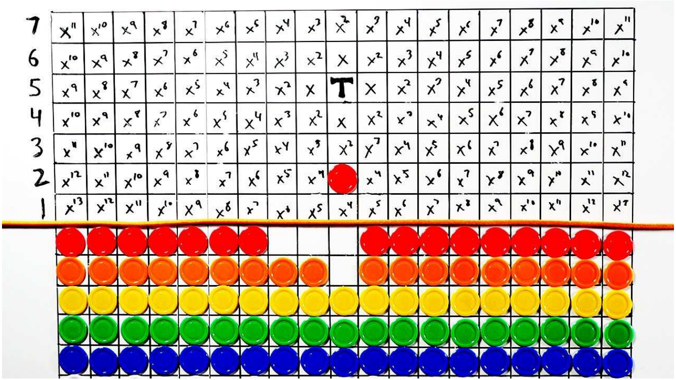
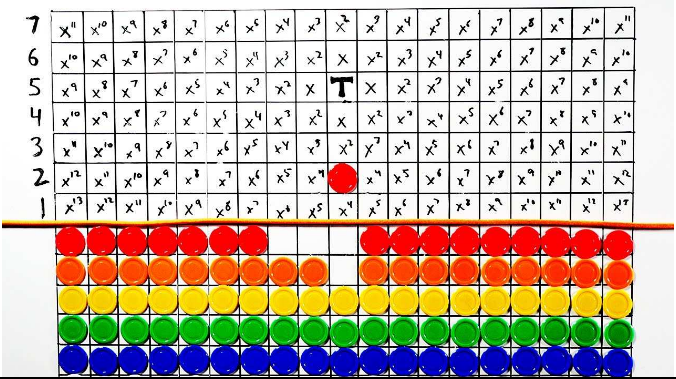


Figure: Conway's monovariant: What is it?

Conway's Soldiers: The Monovariant: II

Choose a target point T; for us it is a point of height 5 above the checkers: $T = (0, 5)$.

Fix x (to be determined later) and attach x^{i+j} to a point that is i units horizontally and j units vertically from T.



Conway's Soldiers: The Monovariant: III

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What is the value of the initial board?

- Zeroth row: $\dots, x^7, x^6, x^5, x^6, x^7, \dots$: sum is

$$x^5 + 2\sum_{k=6}^{\infty} x^k \;=\; x^5 + \frac{2x^6}{1-x} \;=\; \frac{(1+x)x^5}{1-x}.$$

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- Each row is x times previous: Thus initial board value is

$$(1+x)x^5 \sum_{n=0}^{\infty} \frac{x^n}{1-x} \;=\; \frac{(1+x)x^5}{(1-x)^2}.$$

Conway's Soldiers: The Monovariant: IV

Two moves: lose 2 pieces and add a piece further from T, or lose 2 pieces and add a piece closer to target.

First type of move clearly decreases value of board.

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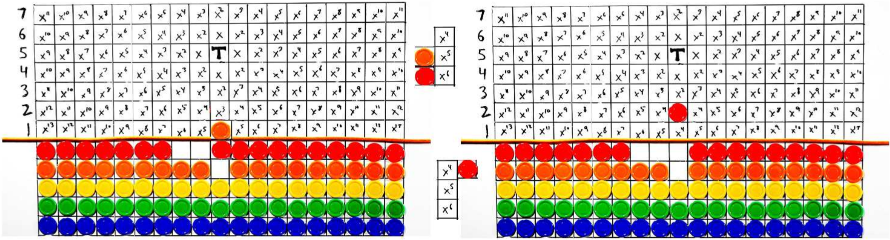

Figure: Moving from pieces on x^6 and x^5 to a piece on x^4. Change is $x^4 - x^5 - x^6$, want this to be zero.

Conway's Soldiers: The Monovariant: IV

Two moves: lose 2 pieces and add a piece further from T, or lose 2 pieces and add a piece closer to target.

Second type replaces x^{n+2} and x^{n+1} with an x^n: change is $x^n - x^{n+1} - x^{n+2}$. Choose x so that this change is zero.

Thus $1 - x - x^2 = 0$ or $x = (-1 \pm \sqrt{5})/2$. Take positive root, $(-1 + \sqrt{5})/2 = 1 - \varphi$ (φ the golden mean).

Monovariant: sum of the values of squares with checkers.

Conway's Soldiers: The Monovariant: V

Choose a target point T; for us it is a point of height 5 above the checkers: $T = (0, 5)$.

- Initial board value is

$$\frac{(1+x)x^5}{(1-x)^2} : \text{when } x = \frac{\sqrt{5}-1}{2} \text{ get } 3 - \sqrt{5} \approx 0.763932.$$

- If have a checker at $(0, 4)$ contributes $x = \frac{\sqrt{5}-1}{2} \approx 0.618034$; as less than 0.763932 possible (and can be done).

- If have a checker at the target $T = (0, 5)$ that board's value is at least 1. As moves never increase board value: IMPOSSIBLE!

Conway's Soldiers: Generalizations

How could you generalize Conway's game?

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How could you generalize Conway's game?

- What if we had a different lattice, say a hexagon?
- What if we tried to do in higher dimensions?
- What if we have some new moves (diagonal jump, jumping over two pieces)? Similar to hyper-radicals....
- What if you choose if you want to remove the jumped piece?

Goal is to get in the habit of asking questions!

Zeckendorf Games
with Cordwell, Epstein, Flynt, Hlavacek, Huynh, Peterson, Vu

Introduction: Summand Minimality

Fibonaccis: $F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n$.

Zeckendorf's Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:
$2020 = 1597+377+34+8+3+1 = F_{16}+F_{13}+F_8+F_5+F_3+F_1$.

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1, 2, 3

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Conversely, we can construct the Fibonacci sequence using this property:

1, 2, 3, 5, 8, 13...

Summand Minimality

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

Theorem

*The Zeckendorf decomposition is **summand minimal**.*

Overall Question

What other recurrences are summand minimal?

Positive Linear Recurrence Sequences

Definition

A **positive linear recurrence sequence (PLRS)** is a sequence given by a recurrence $\{a_n\}$ with

$$a_n \ := \ c_1 a_{n-1} + \cdots + c_t a_{n-t}$$

and each $c_i \geq 0$ and $c_1, c_t > 0$. We use **ideal initial conditions** $a_{-(n-1)} = 0, \dots, a_{-1} = 0, a_0 = 1$ and call $(c_1, \dots, c_t)$ the **signature of the sequence.**

Theorem (Cordwell, Hlavacek, Huynh, M., Peterson, Vu)

For a PLRS with signature $(c_1, c_2, \dots, c_t)$, the Generalized Zeckendorf Decompositions are summand minimal if and only if

$$c_1 \geq c_2 \geq \cdots \geq c_t.$$

Proof for Fibonacci Case

Idea of proof:

- $\mathcal{D} = b_1 F_1 + \cdots + b_n F_n$ decomposition of N, set $\mathrm{Ind}(\mathcal{D}) = b_1 \cdot 1 + \cdots + b_n \cdot n$.

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- Move to $\mathcal{D}'$ by
  - $\diamond$ $2F_k = F_{k+1} + F_{k-2}$ (and $2F_2 = F_3 + F_1$).
  - $\diamond$ $F_k + F_{k+1} = F_{k+2}$ (and $F_1 + F_1 = F_2$).

- Monovariant: Note $\mathrm{Ind}(\mathcal{D}') \leq \mathrm{Ind}(\mathcal{D})$.
  - $\diamond$ $2F_k = F_{k+1} + F_{k-2}$: $2k$ vs $2k - 1$.
  - $\diamond$ $F_k + F_{k+1} = F_{k+2}$: $2k + 1$ vs $k + 2$.

- If not at Zeckendorf decomposition can continue, if at Zeckendorf cannot. Better: $\mathrm{Ind}'(\mathcal{D}) = b_1\sqrt{1} + \cdots + b_n\sqrt{n}$.

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  - $\diamond$ If pieces at F_k and F_{k+1} remove and add one at F_{k+2}.

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  - ⋄ If pieces at F_k and F_{k+1} remove and add one at F_{k+2}.

Questions:

- Does the game end? How long?
- For each N who has the winning strategy?
- What is the winning strategy?

Sample Game

Start with 10 pieces at F_1, rest empty.

10	0	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

8	1	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

6	2	0	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $2F_2 = F_3 + F_1$

Sample Game

Start with 10 pieces at F_1, rest empty.

7	0	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$

Sample Game

Start with 10 pieces at F_1, rest empty.

5	1	1	0	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_2 + F_3 = F_4$.

Sample Game

Start with 10 pieces at F_1, rest empty.

5	0	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

3	1	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_1 + F_1 = F_2$.

Sample Game

Start with 10 pieces at F_1, rest empty.

1	2	0	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 2: $F_1 + F_2 = F_3$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	1	1	0
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Next move: Player 1: $F_3 + F_4 = F_5$.

Sample Game

Start with 10 pieces at F_1, rest empty.

0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

No moves left, Player One wins.

Sample Game

Player One won in 9 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
0	1	1	1	0
0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Sample Game

Player Two won in 10 moves.

10	0	0	0	0
8	1	0	0	0
6	2	0	0	0
7	0	1	0	0
5	1	1	0	0
5	0	0	1	0
3	1	0	1	0
1	2	0	1	0
2	0	1	1	0
0	1	1	1	0
0	1	0	0	1
$[F_1 = 1]$	$[F_2 = 2]$	$[F_3 = 3]$	$[F_4 = 5]$	$[F_5 = 8]$

Games end

Theorem

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: $\left(\sqrt{k}+\sqrt{k}\right)-\sqrt{k+2}<0$.
- Splitting: $2\sqrt{k}-\left(\sqrt{k+1}+\sqrt{k+1}\right)<0$.
- Adding 1's: $2\sqrt{1}-\sqrt{2}<0$.
- Splitting 2's: $2\sqrt{2}-\left(\sqrt{3}+\sqrt{1}\right)<0$.

Games Lengths: I

Upper bound: At most $n \log_\phi \left(n\sqrt{5} + 1/2\right)$ moves.

Fastest game: $n - Z(n)$ moves ($Z(n)$ is the number of summands in n's Zeckendorf decomposition).

From always moving on the largest summand possible (deterministic).

Games Lengths: II

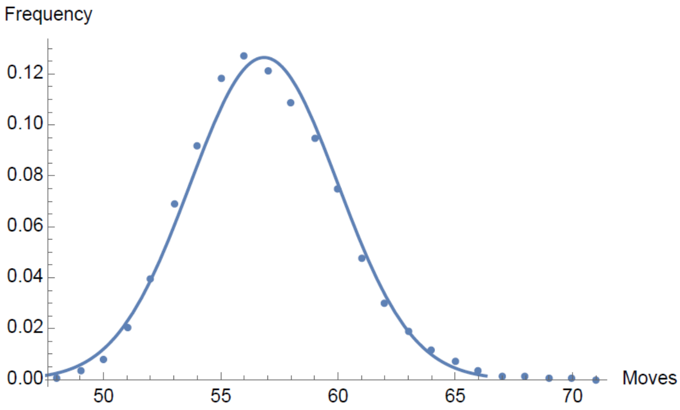


Figure: Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when $n = 60$ vs a Gaussian. Natural conjecture....

Winning Strategy

Theorem

Player Two Has a Winning Strategy

Idea is to show if not, Player Two could steal Player One's strategy.

Non-constructive!

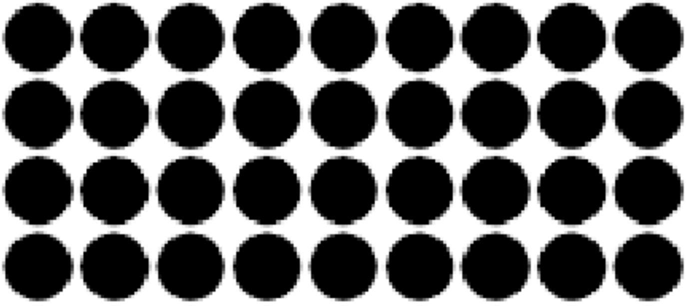
Will highlight idea with a simpler game.

Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at (i, j) and coloring every dot (m, n) with $i \leq m$ and $j \leq n$.

Once all dots colored game ends; whomever goes last loses.

Prove Player 1 has a winning strategy!



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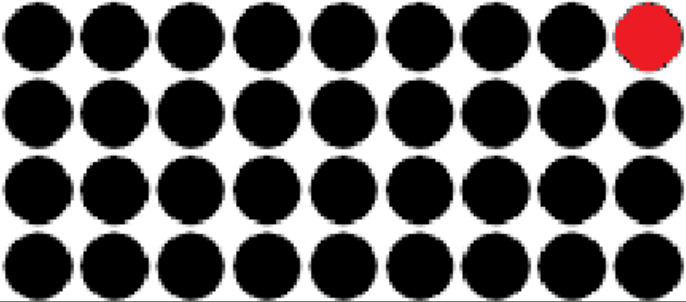
Proof Player 1 has a winning strategy. If have, play; if not, steal.



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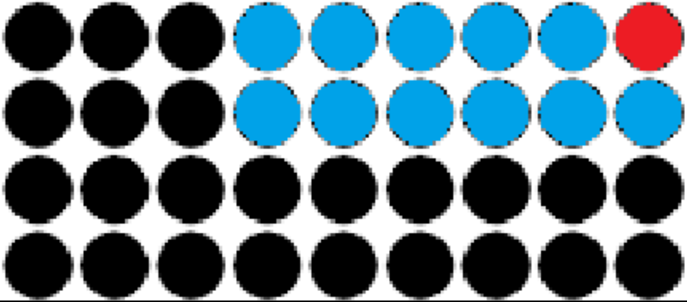

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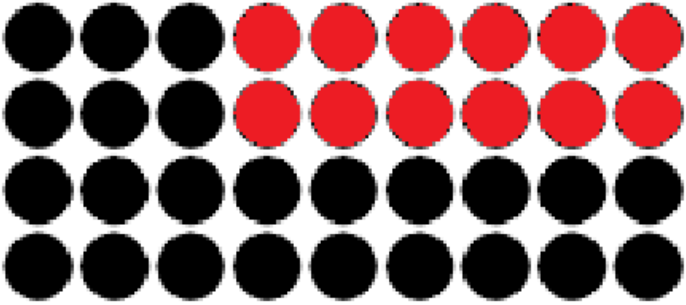
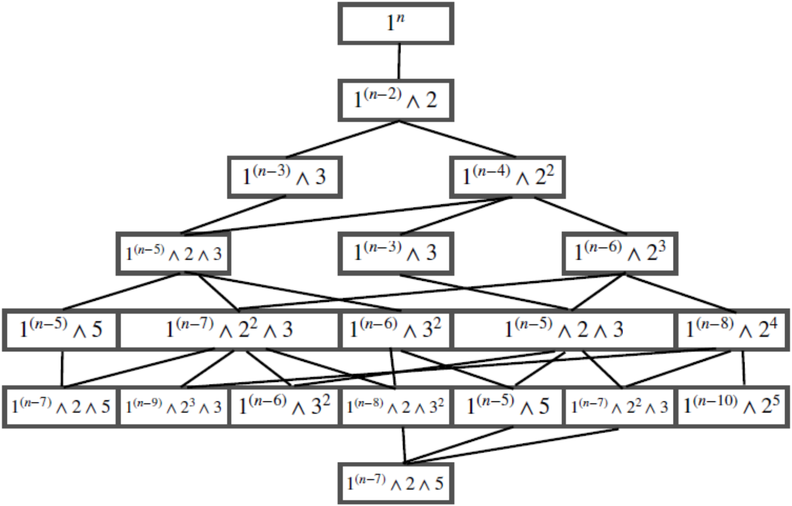


Sketch of Proof for Player Two's Winning Strategy



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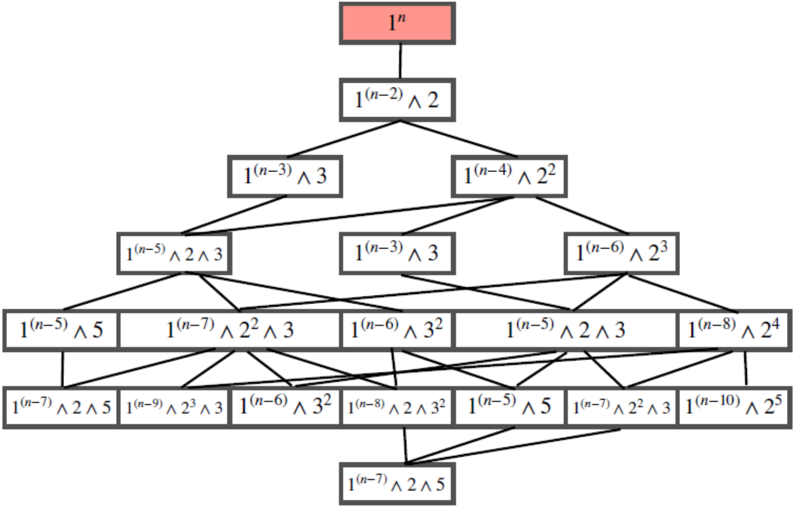

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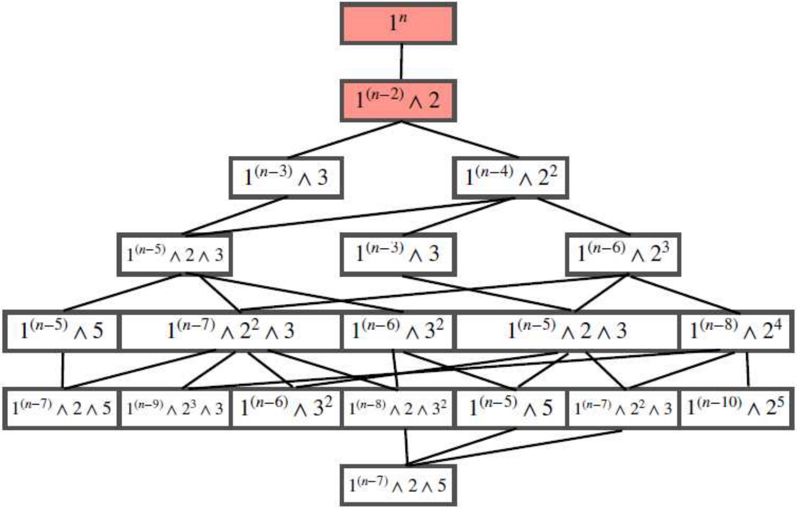

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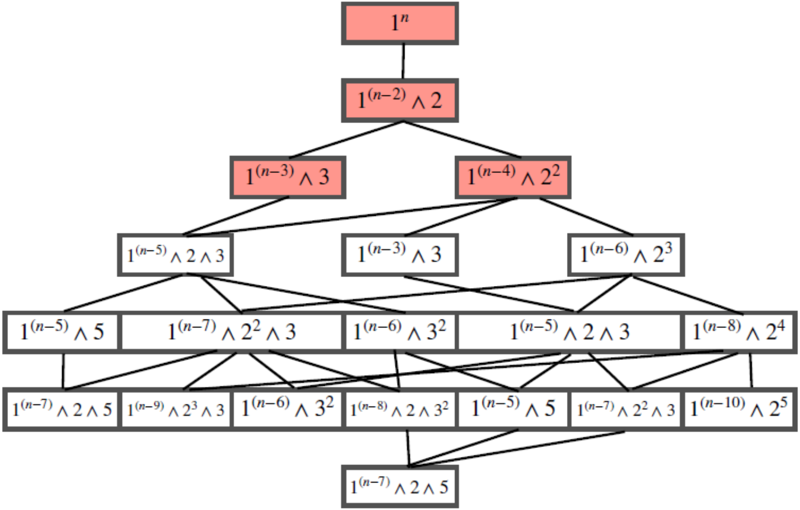

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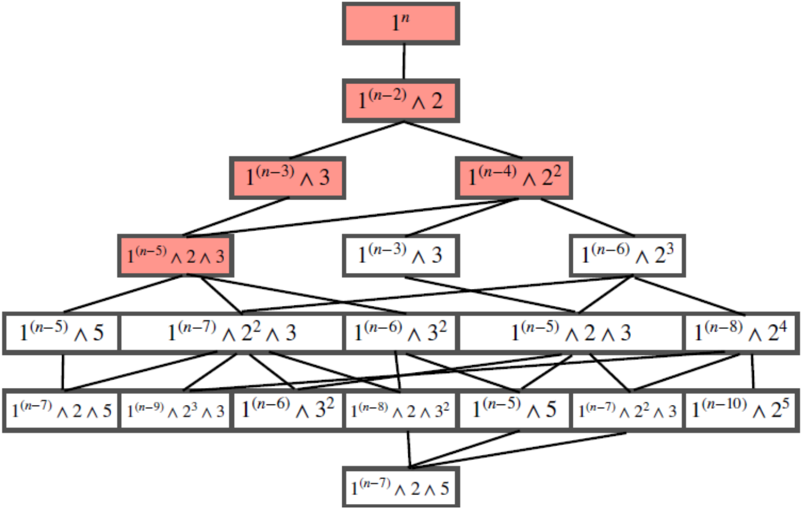

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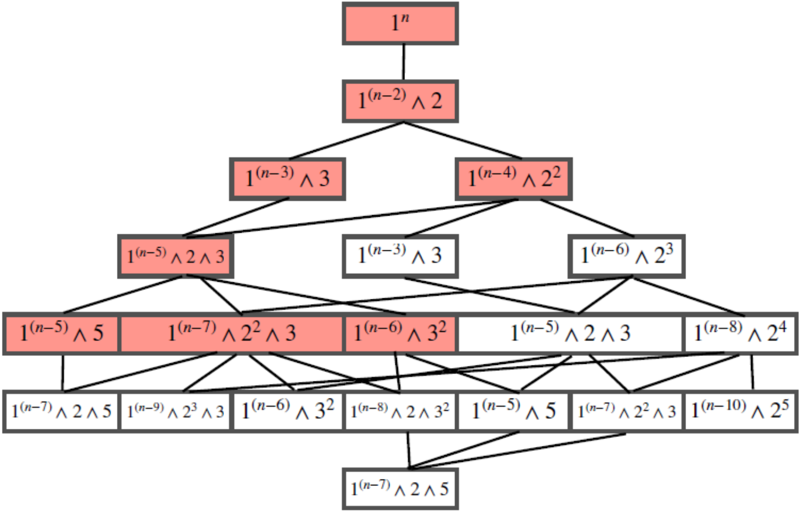

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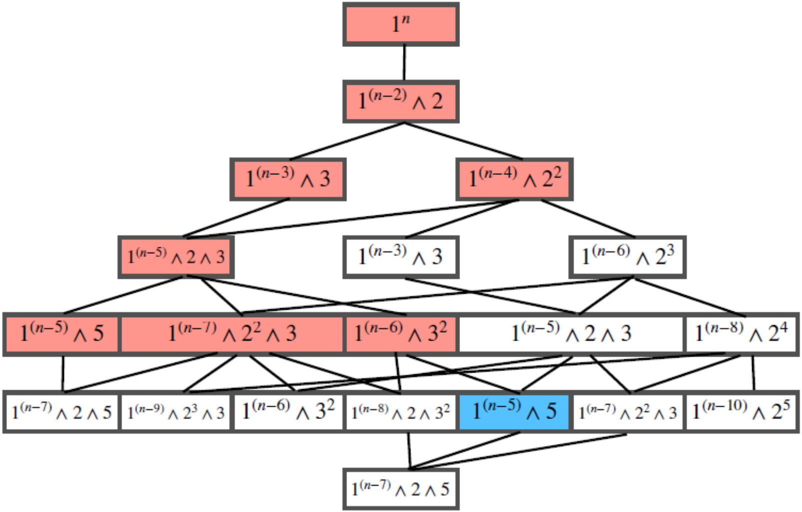

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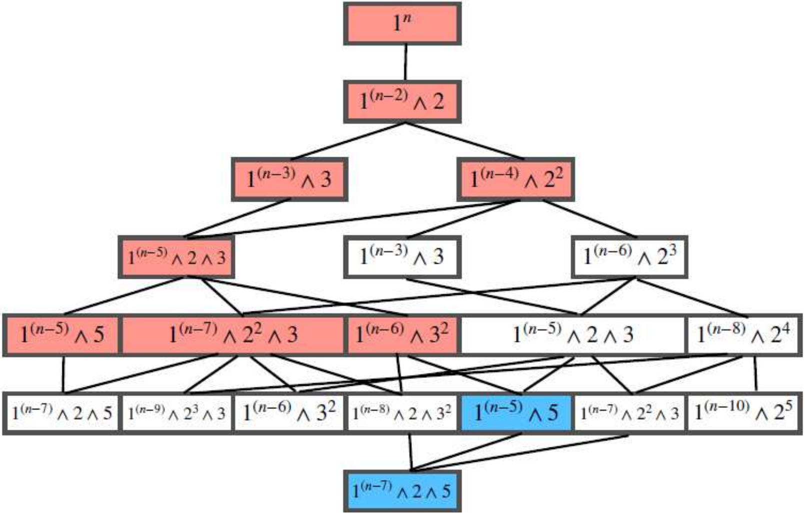

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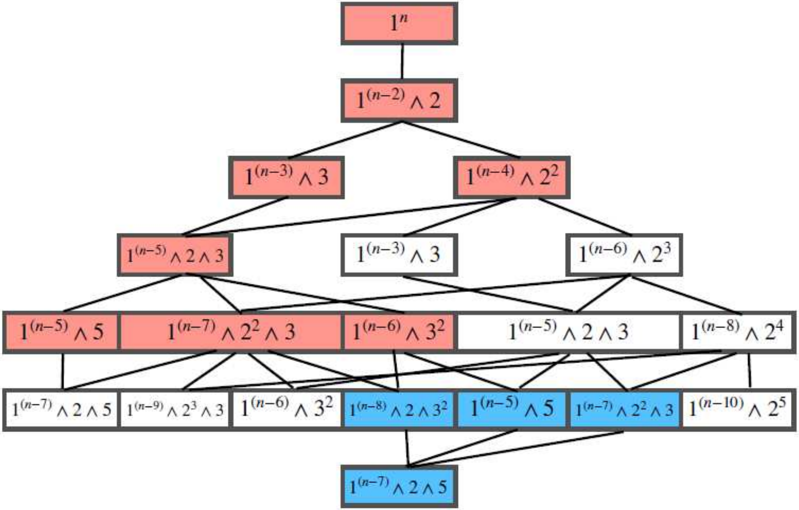

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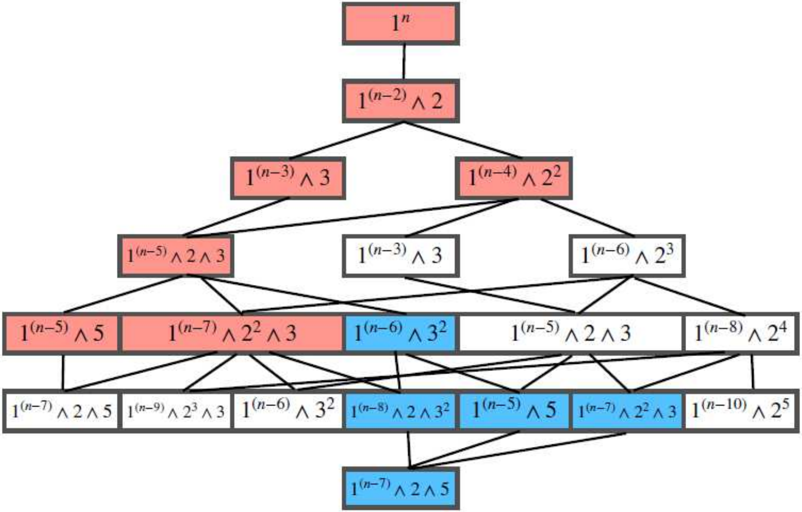

Future Work

- What if $p \geq 3$ people play the Fibonacci game?
- Does the number of moves in random games converge to a Gaussian?
- How long do games take?
- Define k-nacci numbers by $S_{i+1} = S_i + S_{i-1} + \cdots + S_{i-k}$; game terminates but who has the winning strategy?

Rectangle Game

Rectangle Game

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

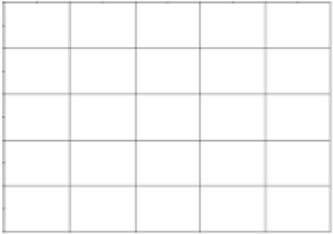

Figure: Winning strategy? Function of board dimension?

Rectangle Game

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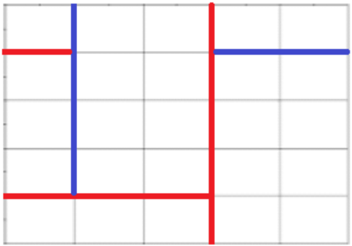

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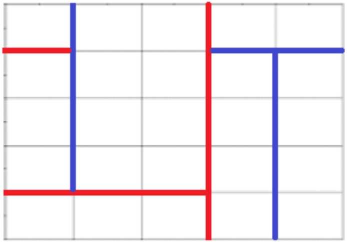

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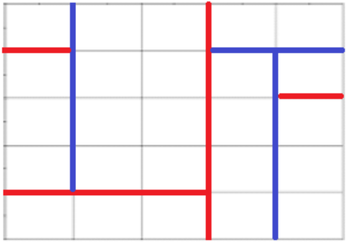

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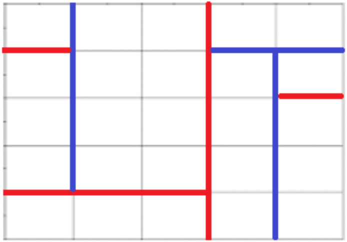
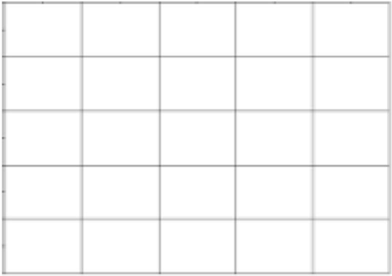


Gather data! Try various sized boards, strategies.

Rectangle Game: Data

RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?



Length	Width	Winner
2	2	--1--
2	3	--1--
3	3	--2--
2	4	--1--
3	4	--1--
4	4	--1--
3	5	--2--

Figure: Do you see a pattern?

Mono-variant

A mono-variant is a quantity that moves on only one direction (either non-decreasing or non-increasing).

Idea: Associate a mono-variant to the rectangle game....

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

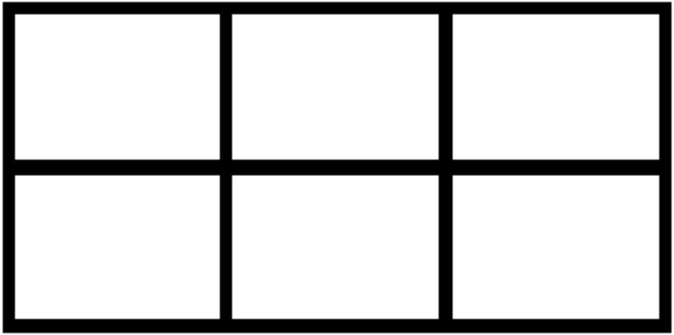


Figure: Move: 0; Pieces: 1.

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

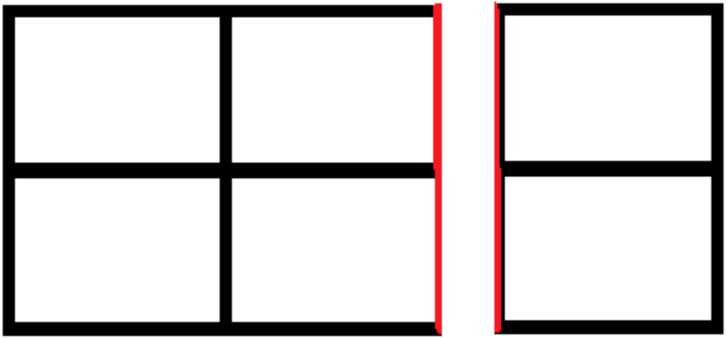


Figure: Move: 1; Pieces: 2.

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

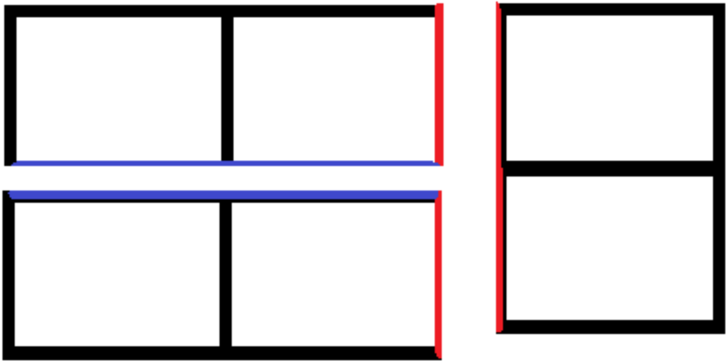


Figure: Move: 2; Pieces: 3.

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

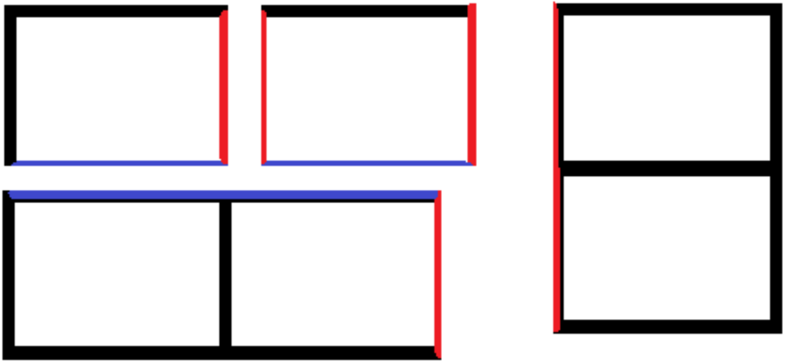


Figure: Move: 3; Pieces: 4.

Rectangle Game: Solution

Every time move, increase number of pieces by 1!

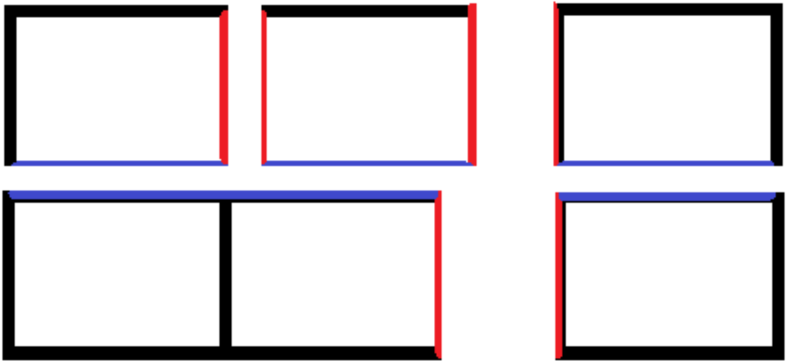


Figure: Move: 4; Pieces: 5.

Rectangle Game: Solution

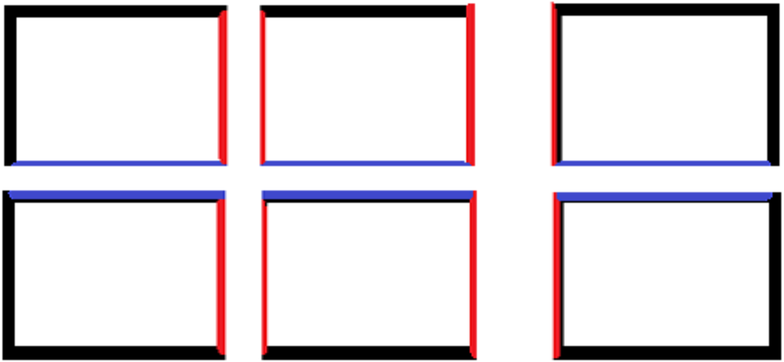


Figure: Move: 5; Pieces: 6. Player 1 Wins.

Rectangle Game: Solution (Continued)

Mono-variant is the number of pieces.

If board is $m \times n$, game ends with mn pieces.

Thus takes $mn - 1$ moves.

If mn even then Player 1 wins else Player 2 wins.