Why I love Monovariants: From Zombies to Conway’s Soldiers via the Golden Mean

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http://www.williams.edu/Mathematics/sjmiller/public_html

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Zombies
General Advice: What are your tools and how can they be used?

Law of the Hammer:

- Abraham Kaplan: I call it the law of the instrument, and it may be formulated as follows: Give a small boy a hammer, and he will find that everything he encounters needs pounding.

- Abraham Maslow: I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail.

- Bernard Baruch: If all you have is a hammer, everything looks like a nail.
Zombie Infection: Rules

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.
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*Initial Configuration*
Zombie Infection: Rules

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![Initial Configuration](image1)

![One moment later](image2)

![Two moments later](image3)
**Zombie Infection: Rules**

- If share walls with 2 or more infected, become infected.
- Once infected, always infected.

*Initial Configuration*  
*One moment later*

*Two moments later*  
*Three moments later*
Zombie Infection: Conquering The World

Easiest initial state that ensures all eventually infected is...?
Zombie Infection: Conquering The World

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Next simplest initial state ensuring all eventually infected...?
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Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?
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![Zombie grid](image)
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?
Zombie Infection: Conquering The World

Fewest number of initial infections needed to get all...?
Zombie Infection: Can $n - 1$ infect all on an $n \times n$ board?
Zombie Infection: Can $n - 1$ infect all on an $n \times n$ board?

Perimeter of infection unchanged.
Zombie Infection: Can \( n - 1 \) infect all on an \( n \times n \) board?

Perimeter of infection unchanged.
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Perimeter of infection decreases by 2.
Zombie Infection: Can $n - 1$ infect all on an $n \times n$ board?

Perimeter of infection decreases by 4.
Zombie Infection: $n - 1$ cannot infect all

- If $n - 1$ infected, maximum perimeter is $4(n - 1) = 4n - 4$. 
Zombie Infection: \( n - 1 \) cannot infect all

- If \( n - 1 \) infected, maximum perimeter is \( 4(n - 1) = 4n - 4 \).
  
- Mono-variant: As time passes, perimeter of infection never increases.
Zombie Infection: \( n - 1 \) cannot infect all

- If \( n - 1 \) infected, maximum perimeter is \( 4(n - 1) = 4n - 4 \).

- **Mono-variant**: As time passes, perimeter of infection never increases.

- Perimeter of \( n \times n \) square is \( 4n \), so at least 1 square safe!
Figure: John Horton Conway: Image from The Guardian.
So many items to discuss....


- **Monster group:** [https://en.wikipedia.org/wiki/Monster_group](https://en.wikipedia.org/wiki/Monster_group)

- **Audioactive decay:** [https://en.wikipedia.org/wiki/Look-and-say_sequence](https://en.wikipedia.org/wiki/Look-and-say_sequence)

- **15 and 290 theorems:** [https://en.wikipedia.org/wiki/15_and_290_theorems](https://en.wikipedia.org/wiki/15_and_290_theorems)
Conway’s Soldiers

Problem: Infinite checkerboard, pieces at all \((x, y)\) with \(y \leq 0\). Using horizontal / vertical jumps (jumped piece gone forever), how high can you move a piece?

Figure: Left: A subset of the initial configuration. Right: moving a soldier / checker up 1. Can you do 2?
Conway’s Soldiers

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**Figure:** Left: A subset of the initial configuration. Right: moving a soldier / checker up 2. Can you do 3? 4? 5? Any height?
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**Figure:** Left: A subset of the initial configuration. Right: Also moving a soldier / checker up 2. Can you do 3? 4? 5? Any height?
Conway’s Soldiers: The Monovariant: I

Problem: Infinite checkerboard, pieces at all \((x, y)\) with \(y \leq 0\). Using horizontal / vertical jumps (jumped piece gone forever), how high can you move a piece?

Figure: Conway’s monovariant: What is it?
Choose a target point $T$; for us it is a point of height 5 above the checkers: $T = (0, 5)$.

Fix $x$ (to be determined later) and attach $x^{i+j}$ to a point that is $i$ units horizontally and $j$ units vertically from $T$. 
Conway’s Soldiers: The Monovariant: III

Choose a target point \( T \); for us it is a point of height 5 above the checkers: \( T = (0, 5) \).

Fix \( x \) (to be determined later) and attach \( x^{i+j} \) to a point that is \( i \) units horizontally and \( j \) units vertically from \( T \).

What is the value of the initial board?

- Zeroth row: \( \ldots, x^7, x^6, x^5, x^6, x^7, \ldots \): sum is

\[
x^5 + 2 \sum_{k=6}^{\infty} x^k = x^5 + \frac{2x^6}{1-x} = \frac{(1+x)x^5}{1-x}.
\]
Choose a target point $T$; for us it is a point of height 5 above the checkers: $T = (0, 5)$.

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What is the value of the initial board?

- **Zeroth row**: $\ldots, x^7, x^6, x^5, x^6, x^7, \ldots$: sum is

  $$x^5 + 2 \sum_{k=6}^{\infty} x^k = x^5 + \frac{2x^6}{1-x} = \frac{(1+x)x^5}{1-x}.$$

- **Each row is $x$ times previous**: Thus initial board value is

  $$\sum_{n=0}^{\infty} \frac{x^n}{1-x} = \frac{(1+x)x^5}{(1-x)^2}.$$
Conway’s Soldiers: The Monovariant: IV

Two moves: lose 2 pieces and add a piece further from $T$, or lose 2 pieces and add a piece closer to target.

First type of move clearly decreases value of board.
Conway’s Soldiers: The Monovariant: IV

Two moves: lose 2 pieces and add a piece further from $T$, or lose 2 pieces and add a piece closer to target.

Figure: Moving from pieces on $x^6$ and $x^5$ to a piece on $x^4$. Change is $x^4 - x^5 - x^6$, want this to be zero.
Conway’s Soldiers: The Monovariant: IV

Two moves: lose 2 pieces and add a piece further from $T$, or lose 2 pieces and add a piece closer to target.

Second type replaces $x^{n+2}$ and $x^{n+1}$ with an $x^n$: change is $x^n - x^{n+1} - x^{n+2}$. Choose $x$ so that this change is zero.

Thus $1 - x - x^2 = 0$ or $x = (-1 \pm \sqrt{5})/2$. Take positive root, $(-1 + \sqrt{5})/2 = 1 - \varphi$ ($\varphi$ the golden mean).

Monovariant: sum of the values of squares with checkers.
Conway’s Soldiers: The Monovariant: V

Choose a target point $T$; for us it is a point of height 5 above the checkers: $T = (0, 5)$.

- Initial board value is

$$\frac{(1 + x)x^5}{(1 - x)^2} : \text{when } x = \frac{\sqrt{5} - 1}{2} \text{ get } 3 - \sqrt{5} \approx 0.763932.$$

- If have a checker at $(0, 4)$ contributes

$$x = \frac{\sqrt{5} - 1}{2} \approx 0.618034; \text{ as less than } 0.763932 \text{ possible (and can be done).}$$

- If have a checker at the target $T = (0, 5)$ that board’s value is at least 1. As moves never increase board value: IMPOSSIBLE!
Conway’s Soldiers: Generalizations

How could you generalize Conway’s game?
Conway’s Soldiers: Generalizations

How could you generalize Conway’s game?

- What if we had a different lattice, say a hexagon?
- What if we tried to do in higher dimensions?
- What if we have some new moves (diagonal jump, jumping over two pieces)? Similar to hyper-radicals....
- What if you choose if you want to remove the jumped piece?

Goal is to get in the habit of asking questions!
Zeckendorf Games
with Cordwell, Epstein, Flynt, Hlavacek, Huynh, Peterson, Vu
Introduction: Summand Minimality

Fibonacci: \( F_1 = 1, F_2 = 2, F_3 = 3, F_4 = 5, F_{n+2} = F_{n+1} + F_n \).

Zeckendorf’s Theorem

Every positive integer can be written uniquely as a sum of one or more non-consecutive Fibonacci numbers.

Example:

\[ 2020 = 1597 + 377 + 34 + 8 + 3 + 1 = F_{16} + F_{13} + F_8 + F_5 + F_3 + F_1. \]
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Conversely, we can construct the Fibonacci sequence using this property:

$1, 2$
Introduction: Summand Minimality

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Zeckendorf’s Theorem
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Example:

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Conversely, we can construct the Fibonacci sequence using this property:

$1, 2, 3, 5, 8, 13 \ldots$
Summand Minimality

Example

- $18 = 13 + 5 = F_6 + F_4$, legal decomposition, two summands.
- $18 = 13 + 3 + 2 = F_6 + F_3 + F_2$, non-legal decomposition, three summands.

Theorem

The Zeckendorf decomposition is **summand minimal**.

Overall Question

What other recurrences are summand minimal?
Definition

A positive linear recurrence sequence (PLRS) is a sequence given by a recurrence \( \{a_n\} \) with

\[
a_n := c_1 a_{n-1} + \cdots + c_t a_{n-t}
\]

and each \( c_i \geq 0 \) and \( c_1, c_t > 0 \). We use **ideal initial conditions** \( a_{-(n-1)} = 0, \ldots, a_{-1} = 0, a_0 = 1 \) and call \( (c_1, \ldots, c_t) \) the **signature of the sequence**.

Theorem (Cordwell, Hlavacek, Huynh, M., Peterson, Vu)

For a PLRS with signature \( (c_1, c_2, \ldots, c_t) \), the Generalized Zeckendorf Decompositions are summand minimal if and only if

\[
c_1 \geq c_2 \geq \cdots \geq c_t.
\]
Proof for Fibonacci Case

Idea of proof:

\[ D = b_1 F_1 + \cdots + b_n F_n \] decomposition of \( N \), set

\[ \text{Ind}(D) = b_1 \cdot 1 + \cdots + b_n \cdot n. \]
Proof for Fibonacci Case

Idea of proof:

- \( D = b_1 F_1 + \cdots + b_n F_n \) decomposition of \( N \), set \( \text{Ind}(D) = b_1 \cdot 1 + \cdots + b_n \cdot n. \)

- Move to \( D' \) by
  - \( 2F_k = F_{k+1} + F_{k-2} \) (and \( 2F_2 = F_3 + F_1 \)).
  - \( F_k + F_{k+1} = F_{k+2} \) (and \( F_1 + F_1 = F_2 \)).

- Monovariant: Note \( \text{Ind}(D') \leq \text{Ind}(D) \).
  - \( 2F_k = F_{k+1} + F_{k-2} \): \( 2k \) vs \( 2k - 1 \).
  - \( F_k + F_{k+1} = F_{k+2} \): \( 2k + 1 \) vs \( k + 2 \).

- If not at Zeckendorf decomposition can continue, if at Zeckendorf cannot. Better: \( \text{Ind}'(D) = b_1 \sqrt{1} + \cdots + b_n \sqrt{n} \).
Rules

- Two player game, alternate turns, last to move wins.
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- Bins $F_1, F_2, F_3, \ldots$, start with $N$ pieces in $F_1$ and others empty.
Rules

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- Bins $F_1, F_2, F_3, \ldots$, start with $N$ pieces in $F_1$ and others empty.

- A turn is one of the following moves:
  - If have two pieces on $F_k$ can remove and put one piece at $F_{k+1}$ and one at $F_{k-2}$
    (if $k = 1$ then $2F_1$ becomes $1F_2$)
  - If pieces at $F_k$ and $F_{k+1}$ remove and add one at $F_{k+2}$. 

Questions:
- Does the game end? How long?
- For each $N$ who has the winning strategy?
- What is the winning strategy?
Rules

- Two player game, alternate turns, last to move wins.

- Bins $F_1$, $F_2$, $F_3$, ..., start with $N$ pieces in $F_1$ and others empty.

- A turn is one of the following moves:
  - If have two pieces on $F_k$ can remove and put one piece at $F_{k+1}$ and one at $F_{k-2}$
    (if $k = 1$ then $2F_1$ becomes $1F_2$)
  - If pieces at $F_k$ and $F_{k+1}$ remove and add one at $F_{k+2}$.

Questions:

- Does the game end? How long?
- For each $N$ who has the winning strategy?
- What is the winning strategy?
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{cccccc}
10 & 0 & 0 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccccc}
\end{array}
\]

Next move: Player 1: $F_1 + F_1 = F_2$
Sample Game

Start with 10 pieces at $F_1$, rest empty.

<table>
<thead>
<tr>
<th></th>
<th>8</th>
<th>1</th>
<th>0</th>
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<tbody>
<tr>
<td>$F_1 = 1$</td>
<td>$F_2 = 2$</td>
<td>$F_3 = 3$</td>
<td>$F_4 = 5$</td>
<td>$F_5 = 8$</td>
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</tbody>
</table>

Next move: Player 2: $F_1 + F_1 = F_2$
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{cccccc}
6 & 2 & 0 & 0 & 0 \\
\end{array}
\]

Next move: Player 1: $2F_2 = F_3 + F_1$
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{cccccc}
7 & 0 & 1 & 0 & 0 \\
\text{[}F_1 = 1\text{]} & \text{[}F_2 = 2\text{]} & \text{[}F_3 = 3\text{]} & \text{[}F_4 = 5\text{]} & \text{[}F_5 = 8\text{]} \\
\end{array}
\]

Next move: Player 2: $F_1 + F_1 = F_2$
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{cccccc}
5 & 1 & 1 & 0 & 0 \\
\end{array}
\]

Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{cccccc}
5 & 0 & 0 & 1 & 0 \\
\{F_1 = 1\} & \{F_2 = 2\} & \{F_3 = 3\} & \{F_4 = 5\} & \{F_5 = 8\} \\
\end{array}
\]

Next move: Player 2: $F_1 + F_1 = F_2$. 
Sample Game

Start with 10 pieces at $F_1$, rest empty.

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<thead>
<tr>
<th></th>
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<th>1</th>
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<td>$F_1$</td>
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<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
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</tbody>
</table>

Next move: Player 1: $F_1 + F_1 = F_2$. 
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[ \begin{array}{cccccc}
1 & 2 & 0 & 1 & 0 \\
\end{array} \]

Next move: Player 2: $F_1 + F_2 = F_3$. 
Sample Game

Start with 10 pieces at $F_1$, rest empty.

\[
\begin{array}{cccccc}
0 & 1 & 1 & 1 & 0 \\
\end{array}
\]

Next move: Player 1: $F_3 + F_4 = F_5$. 
Sample Game

Start with 10 pieces at $F_1$, rest empty.

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<tr>
<th></th>
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<td>$[F_1 = 1]$</td>
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No moves left, Player One wins.
## Sample Game

Player One won in 9 moves.

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\[ F_1 = 1 \] \[ F_2 = 2 \] \[ F_3 = 3 \] \[ F_4 = 5 \] \[ F_5 = 8 \]
## Sample Game

Player Two won in 10 moves.

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Theorem

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: \((\sqrt{k} + \sqrt{k}) - \sqrt{k + 2} < 0\).
- Splitting: \(2\sqrt{k} - (\sqrt{k + 1} + \sqrt{k + 1}) < 0\).
- Adding 1’s: \(2\sqrt{1} - \sqrt{2} < 0\).
- Splitting 2’s: \(2\sqrt{2} - (\sqrt{3} + \sqrt{1}) < 0\).
Games Lengths: I

**Upper bound:** At most \( n \log_\phi (n\sqrt{5} + 1/2) \) moves.

**Fastest game:** \( n - Z(n) \) moves (\( Z(n) \) is the number of summands in \( n \)'s Zeckendorf decomposition).

From always moving on the largest summand possible (deterministic).
Figure: Frequency graph of the number of moves in 9,999 simulations of the Zeckendorf Game with random moves when $n = 60$ vs a Gaussian. Natural conjecture....
Winning Strategy

Theorem

*Player Two Has a Winning Strategy*

Idea is to show if not, Player Two could steal Player One’s strategy.

Non-constructive!

Will highlight idea with a simpler game.
Winning Strategy: Intuition from Dot Game

Two players, alternate. Turn is choosing a dot at \((i, j)\) and coloring every dot \((m, n)\) with \(i \leq m\) and \(j \leq n\).

Once all dots colored game ends; whomever goes last loses.

Prove Player 1 has a winning strategy!
Winning Strategy: Intuition from Dot Game

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Proof Player 1 has a winning strategy. If have, play; if not, steal.
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Sketch of Proof for Player Two’s Winning Strategy
Sketch of Proof for Player Two’s Winning Strategy
Sketch of Proof for Player Two’s Winning Strategy

\[
\begin{align*}
1^n & \\
1^{(n-2)} \land 2 & \\
1^{(n-3)} \land 3 & \quad 1^{(n-4)} \land 2^2 \\
1^{(n-5)} \land 2 \land 3 & \quad 1^{(n-3)} \land 3 & \quad 1^{(n-6)} \land 2^3 \\
1^{(n-5)} \land 5 & \quad 1^{(n-7)} \land 2^2 \land 3 & \quad 1^{(n-6)} \land 3^2 & \quad 1^{(n-5)} \land 2 \land 3 & \quad 1^{(n-8)} \land 2^4 \\
1^{(n-7)} \land 2 \land 5 & \quad 1^{(n-9)} \land 2^3 \land 3 & \quad 1^{(n-6)} \land 3^2 & \quad 1^{(n-8)} \land 2 \land 3^2 & \quad 1^{(n-5)} \land 5 & \quad 1^{(n-7)} \land 2^2 \land 3 & \quad 1^{(n-10)} \land 2^5 \\
1^{(n-7)} \land 2 \land 5
\end{align*}
\]
Sketch of Proof for Player Two’s Winning Strategy

Diagram of the game state transitions and winning conditions.
Sketch of Proof for Player Two’s Winning Strategy
Sketch of Proof for Player Two’s Winning Strategy
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Sketch of Proof for Player Two’s Winning Strategy
Future Work

- What if $p \geq 3$ people play the Fibonacci game?

- Does the number of moves in random games converge to a Gaussian?

- How long do games take?

- Define $k$-nacci numbers by $S_{i+1} = S_i + S_{i-1} + \cdots + S_{i-k}$; game terminates but who has the winning strategy?
Rectangle Game
**RECTANGLE GAME:** Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

**Figure:** Winning strategy? Function of board dimension?
RECTANGLE GAME: Consider an M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?
**RECTANGLE GAME:** Consider $M \times N$ board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?
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Gather data! Try various sized boards, strategies.
RECTANGLE GAME: Consider M x N board, take turns, each turn can break any piece along one horizontal or along one vertical, last one to break a piece wins. Does someone have a winning strategy?

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Figure: Do you see a pattern?
A **mono-variant** is a quantity that moves on only one direction (either non-decreasing or non-increasing).

Idea: Associate a mono-variant to the rectangle game....
Rectangle Game: Solution

Every time move, increase number of pieces by 1!

Figure: Move: 0; Pieces: 1.
Rectangle Game: Solution

Every time move, increase number of pieces by 1!

Figure: Move: 1; Pieces: 2.
Rectangle Game: Solution

Every time move, increase number of pieces by 1!

Figure: Move: 2; Pieces: 3.
Rectangle Game: Solution

Every time move, increase number of pieces by 1!

Figure: Move: 3; Pieces: 4.
Rectangle Game: Solution

Every time move, increase number of pieces by 1!

**Figure:** Move: 4; Pieces: 5.
Rectangle Game: Solution

Figure: Move: 5; Pieces: 6. Player 1 Wins.
Mono-variant is the number of pieces.

If board is $m \times n$, game ends with $mn$ pieces.

Thus takes $mn – 1$ moves.

If $mn$ even then Player 1 wins else Player 2 wins.