One-Level density for holomorphic cusp forms of arbitrary level

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Introduction

Why study zeros of L-functions?

- Infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Distribution of zeros

- $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.
2. Develop an explicit formula relating what we want to study to something we understand.
3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p\left(s,f\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f)L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

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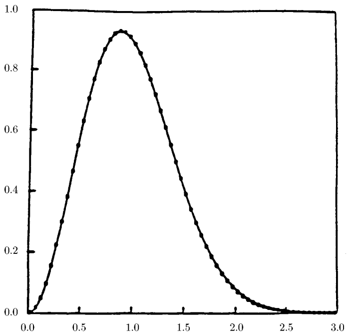
Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

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&= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \text{Good}(s).
\end{aligned}
$$

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$$= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p\log\left(1-p^{-s}\right)$$

$$= \sum_p\frac{\log p\ \cdot\ p^{-s}}{1-p^{-s}} = \sum_p\frac{\log p}{p^s} + \mathrm{Good}(s).$$

Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p \int\left(\frac{x}{p}\right)^s\frac{ds}{s}.$$

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Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)}\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma\log p} e^{-it\log p} ds.$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Cuspidal Newforms

Cuspidal Newforms: Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0) + \frac{1}{2}\phi(0) - \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Measures of Spacings: *n*-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\dots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i \neq j_k\right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
2. 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).
4. n-level correlations for the classical compact groups (Katz-Sarnak).
5. Insensitive to any finite set of zeros.

Measures of Spacings: n-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) \quad = \quad \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f\gamma_f^{(j_1)}\right)\cdots\phi_n\left(L_f\gamma_f^{(j_n)}\right)$$

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1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
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Katz-Sarnak Conjecture

For a ‘nice’ family of L-functions, the n-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1} \sum_{f \in \mathcal{F}_N} \log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(\frac{\log C_f}{2\pi} \gamma_f^{(j_i)} \right)$$
$$\to \int \cdots \int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x) dx.$$

Katz-Sarnak Conjecture

As $C_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Correspondences

Similarities between L-Functions and Nuclei:

Zeros	$\longleftrightarrow$	Energy Levels
Schwartz test function	$\longrightarrow$	Neutron
Support of test function	$\longleftrightarrow$	Neutron Energy.

Cuspidal Newforms
Iwaniec-Luo-Sarnak, Hughes-Miller

Results from Iwaniec-Luo-Sarnak

- Orthogonal: Iwaniec-Luo-Sarnak: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign) in $(-2, 2)$.

- Symplectic: Iwaniec-Luo-Sarnak: 1-level density for $\text{sym}^2(f)$, f holomorphic cuspidal newform.

Will review Orthogonal case.

Modular Form Preliminaries

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{matrix} ad - bc = 1 \\ c \equiv 0(N) \end{matrix} \right\}$$

f is a weight k holomorphic cuspform of level N if

$$\forall \gamma \in \Gamma_0(N), \; f(\gamma z) = (cz + d)^k f(z).$$

- Fourier Expansion: $f(z) = \sum_{n=1}^{\infty} a_f(n) e^{2\pi i z}$, $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$.
- Petersson Norm: $\langle f, g \rangle = \int_{\Gamma_0(N)\backslash \mathbf{H}} f(z)\overline{g(z)} y^{k-2} dxdy$.
- Normalized coefficients:

$$\psi_f(n) = \sqrt{\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}} \frac{1}{||f||} a_f(n).$$

Modular Form Preliminaries: Petersson Formula

$B_k(N)$ an orthonormal basis for weight k level N. Define

$$\Delta_{k,N}(m,n) \; = \sum_{f \in B_k(N)} \psi_f(m)\overline{\psi}_f(n).$$

Petersson Formula

$$\Delta_{k,N}(m,n) \;=\; 2\pi i^k \sum_{c\equiv 0(N)} \frac{S(m,n,c)}{c} J_{k-1}\left(4\pi\frac{\sqrt{mn}}{c}\right) + \; \delta(m,n).$$

Modular Form Preliminaries: Explicit Formula

Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0) + \frac{1}{2}\phi(0) - \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Modular Form Preliminaries: Fourier Coefficient Review

$$\begin{aligned}\lambda_f(n) &= a_f(n)n^{\frac{k-1}{2}}\\ \lambda_f(m)\lambda_f(n) &= \sum_{\substack{d\mid(m,n)\\(d,M)=1}} \lambda_f\left(\frac{mn}{d}\right).\end{aligned}$$

For a newform of level N, $\lambda_f(N)$ is trivially related to the sign of the form:

$$\epsilon_f = i^k \mu(N)\lambda_f(N)\sqrt{N}.$$

Above allows us to split into even and odd families: $1 \pm \epsilon_f$.

Key Kloosterman-Bessel integral from ILS

Ramanujan sum:

$$R(n,q) \;=\; \sum_{a \bmod q}^{*} e(an/q) \;=\; \sum_{d|(n,q)} \mu(q/d)d,$$

where $*$ restricts the sum to a relatively prime to q.

Theorem (ILS)

Let Ψ be an even Schwartz function with $\mathrm{supp}(\widehat{\Psi}) \subset (-2,2)$. Then

$$\sum_{m\le N^{\epsilon}} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y)\widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}y}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty}\Psi(x)\frac{\sin 2\pi x}{2\pi x}\mathrm{d}x - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k\log\log kN}{\log kN}\right),$$

where $R = k^2N$ and φ is Euler's totient function.

Limited Support ($\sigma < 1$): Sketch of proof

- Estimate Kloosterman-Bessel terms trivially.
  $\diamond$ Kloosterman sum: $d\overline{d} \equiv 1 \bmod q$, $\tau(q)$ is the number of divisors of q,

$$S(m,n;q) \;=\; \sum_{d \bmod q}^{*} e\left(\frac{md}{q} + \frac{n\overline{d}}{q}\right)$$

$$|S(m,n;q)| \;\leq\; (m,n,q)\sqrt{\min\left\{\frac{q}{(m,q)}, \frac{q}{(n,q)}\right\}}\;\tau(q).$$

  $\diamond$ Bessel function: integer $k \geq 2$,
  $J_{k-1}(x) \ll \min\left(x, x^{k-1}, x^{-1/2}\right)$.

- Use Fourier Coefficients to split by sign: N fixed:
  $\pm \sum_f \lambda_f(N) * (\cdots)$.

Increasing Support ($\sigma < 2$): Sketch of the proof

- Using Dirichlet Characters, handle Kloosterman terms.
- Have terms like

$$\int_0^\infty J_{k-1}\left(4\pi \frac{\sqrt{m^2 yN}}{c}\right) \widehat{\phi}\left(\frac{\log y}{\log R}\right) \frac{dy}{\sqrt{y}}$$

with arithmetic factors to sum outside.

- Works for support up to $(-2, 2)$.

2-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2N}}{c}\right)\frac{dx_1dx_2}{\sqrt{x_1x_2}}$$

2-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2N}}{c}\right)\frac{dx_1dx_2}{\sqrt{x_1x_2}}$$

Change of variables and Jacobian:

$$\begin{array}{rclcrcl} u_2 & = & x_1x_2 & & x_2 & = & \frac{u_2}{u_1} \\ u_1 & = & x_1 & & x_1 & = & u_1 \end{array}$$

$$\left|\frac{\partial x}{\partial u}\right| = \left|\begin{array}{cc} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1}\end{array}\right| = \frac{1}{u_1}.$$

Left with

$$\int\int\widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log\left(\frac{u_2}{u_1}\right)}{\log R}\right)\frac{1}{\sqrt{u_2}}J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\frac{du_1du_2}{u_1}$$

2-Level Density (cont)

Changing variables, u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right)dw_1.$$

Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) \;=\; \int_{w_1=-\infty}^{\infty} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right)dw_1.$$

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\Psi_2\left(\frac{\log u_2}{\log R}\right)\frac{du_2}{\sqrt{u_2}}$$

General N

with Owen Barrett, Paula Burkhardt, Jonathan DeWitt and Robert Dorward

Notation

Holomorphic cusp newform f, renormalized Fourier coefficients

$$\Psi_f(n) := \left(\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\right)^{1/2} ||f||^{-1}\lambda_f(n),$$

where $\|f\|^2 = \langle f, f\rangle$ and $\langle \cdot, \cdot, \rangle$ denotes the Petersson inner product.

Define

$$\Delta_{k,N}(m,n) := \sum_{g\in\mathcal{B}_k(N)} \overline{\Psi_g(m)}\Psi_g(n),$$

where $\mathcal{B}_k(N)$ is an orthonormal basis for the space of cusp forms of weight k and level N.

Generalized Averaging Formula

Using the orthonormal basis $\mathcal{B}_k(N)$ of Milićević and Blomer, we prove the following (unconditional) formula.

Generalized Averaging Formula

Suppose that $(n, N) = 1$. Then

$$\sum_{f \in H_k^\star(N)} \lambda_f(n) = \frac{k-1}{12} \sum_{LM=N} \mu(L) M \prod_{p^2|M} \left(\frac{p^2}{p^2-1}\right)^{-1} \sum_{(m,M)=1} m^{-1} \Delta_{k,M}(m^2, n).$$

Main Result: Consequence of averaging formula:

1-Level Density [BBDDM]

Fix any $\phi \in \mathcal{S}(\mathbb{R})$ with supp $\widehat{\phi} \subset (-2,2)$. Then, assuming GRH for $L(s,f)$ and $L(s, \text{sym}^2 f)$ for $f \in H_k^*(N)$ and for all Dirichlet L-functions,

$$\lim_{N\to\infty} \frac{1}{\left|H_k^\star(N)\right|} \sum_{f \in H_k^\star(N)} D_1(f;\phi) \;=\; \int_{-\infty}^{\infty} \phi(x) W_1(\mathrm{O})(x)\, dx$$

where $W_1(\mathrm{O})(x) = 1 + \frac{1}{2}\delta_0(x)$; thus the 1-level density for the family $H_k^\star(N)$ agrees only with orthogonal symmetry.

More generally, under the same assumptions the Density Conjecture holds for the family $H_k^\star(N)$ for any test function $\phi(x)$ whose Fourier transform is supported inside $(-u,u)$ with $u < 2\log(kN)/\log(k^2N)$.

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$$\lim_{N\to\infty} \frac{1}{|H_k^\star(N)|} \sum_{f\in H_k^\star(N)} D_1(f;\phi) = \int_{-\infty}^{\infty} \phi(x) W_1(\mathrm{O})(x)\, dx$$

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Can't split by sign: don't have $\epsilon_f = i^k \mu(N)\lambda_f(N)N^{1/2}$ for general N (if $p^2|N$, then the level doesn't determine the local representation and so doesn't determine the root number).

Proof of Averaging Formula: ξ_d

For $f \in H_k^\star(M)$ consider the following arithmetic functions from [BM]:

$$r_f(c) := \sum_{b|c} \frac{\mu(b)\lambda_f(b)^2}{b\sigma_{-1}^{\text{twisted}}(b)^2}, \ \alpha(c) := \sum_{b|c} \frac{\chi_{0;M}(b)\mu(b)}{b^2}, \ \beta(c) := \sum_{b|c} \frac{\chi_{0;M}(b)\mu^2(b)}{b},$$

where $\mu_f(c)$ is the multiplicative function given implicitly by

$$L(f,s)^{-1} = \sum_c \frac{\mu_f(c)}{c^s},$$

or explicitly on prime powers by

$$\mu_f(p^j) = \begin{cases} -\lambda_f(p) & j = 1 \\ \chi_{0;M}(p) & j = 2 \\ 0 & j > 2 \end{cases}$$

and

$$\sigma_{-1}^{\text{twisted}}(b) = \sum_{r|b} \frac{\chi_{0;M}(r)}{r}.$$

Proof of Averaging Formula: ξ_d (cont)

For $\ell \mid d$ define

$$\xi'_d(\ell) := \frac{\mu(d/\ell)\lambda_f(d/\ell)}{r_f(d)^{1/2}(d/\ell)^{1/2}\beta(d/\ell)}, \quad \xi''_d(\ell) := \frac{\mu_f(d/\ell)}{(d/\ell)^{1/2}(r_f(d)\alpha(d))^{1/2}}.$$

Write $d = d_1 d_2$ where d_1 is square-free, d_2 is square-full, and $(d_1, d_2) = 1$. Thus $p \mid\mid d$ implies $p \mid d_1$ and $p^2 \mid d$ implies $p^2 \mid d_2$. Then for $\ell \mid d$ define

$$\xi_d(\ell) := \xi'_{d_1}((d_1, \ell))\xi''_{d_2}((d_2, \ell)).$$

Blomer and Milićević prove if

$$f_d(z) := \sum_{\ell \mid d} \xi_d(\ell) f \mid_\ell (z),$$

where $N = LM$ and $f \in H_k^\star(M)$ is Petersson-normalized with respect to the Petersson norm on level $S_k(N)$, then $\{f_d : d \mid L\}$ is an orthonormal basis of $S_k(L; f)$.

Proof of Averaging Formula: ξ_d (cont)

Orthonormal basis for $S_k(N)$ and sum:

$$\mathcal{B}_k(N) = \bigcup_{LM=N} \bigcup_{f \in H_k^\star(M)} \bigcup_{d|L} f_d'.$$

$$\begin{aligned}
\Delta_{k,N}(m,n) &= \sum_{g \in \mathcal{B}_k(N)} \overline{\left(\frac{\Gamma(k-1)}{(4\pi m)^{k-1}}\right)^{1/2} ||g||^{-1} a_g(m)} \left(\frac{\Gamma(k-1)}{(4\pi n)^{k-1}}\right)^{1/2} ||g||^{-1} a_g(n) \\
&= (4\pi)^{1-k} (mn)^{\frac{1-k}{2}} \Gamma(k-1) \sum_{LM=N} \sum_{f \in H_k^\star(M)} \sum_{f_d' : d|L} ||f_d'||^{-2} \overline{a_{f_d'}(m)} a_{f_d'}(n) \\
&= (4\pi)^{1-k} (mn)^{\frac{1-k}{2}} \Gamma(k-1) \sum_{LM=N} \sum_{f \in H_k^\star(M)} \sum_{d|L} \frac{1}{||f||^2} \overline{\left(\sum_{\ell|(d,m)} \xi_d(\ell) \ell^{k/2} \lambda_f(\tfrac{m}{\ell}) (m/\ell)^{(k-1)/2}\right)} \\
&\qquad \times \left(\sum_{\ell|(d,n)} \xi_d(\ell) \ell^{k/2} a_f(\tfrac{n}{\ell}) (n/\ell)^{(k-1)/2}\right) \\
&= \frac{12}{(k-1)\nu(N)} \sum_{LM=N} \frac{M}{\varphi(M)} \sum_{f \in H_k^\star(M)} \frac{1}{Z(1,f)} \\
&\qquad \times \sum_{d|L} \left(\sum_{\ell|(d,m)} \xi_d(\ell) \ell^{1/2} \lambda_f(\tfrac{m}{\ell})\right) \left(\sum_{\ell|(d,n)} \xi_d(\ell) \ell^{1/2} \lambda_f(\tfrac{n}{\ell})\right).
\end{aligned}$$

Proof of Averaging Formula: ξ_d (cont)

Specialize to $(n, N) = 1$ and $(m, N) = 1$. Then $d|L|N$ and $(m, N) = (n, N) = 1$, $\ell|(d, m)$ implies $\ell = 1$ (and similarly for $\ell|(d, n)$). Find

$$\Delta_{k,N}(m, n) \;=\; \frac{12}{(k-1)\nu(N)} \sum_{LM=N} \frac{M}{\varphi(M)} \sum_{f \in H_k^\star(M)} \frac{\lambda_f(m)\lambda_f(n)}{Z(1, f)} \sum_{d|L} \xi_d(1)^2.$$

Task to understand

$$\sum_{d|L} \xi_d(1)^2$$

Proof of Averaging Formula: ξ_d (cont)

Specialize to $(n, N) = 1$ and $(m, N) = 1$. Then $d|L|N$ and $(m, N) = (n, N) = 1$, $\ell|(d, m)$ implies $\ell = 1$ (and similarly for $\ell|(d, n)$). Find

$$\Delta_{k,N}(m, n) \;=\; \frac{12}{(k-1)\nu(N)} \sum_{LM=N} \frac{M}{\varphi(M)} \sum_{f \in H_k^\star(M)} \frac{\lambda_f(m)\lambda_f(n)}{Z(1, f)} \sum_{d|L} \xi_d(1)^2.$$

Task to understand

$$\sum_{d|L} \xi_d(1)^2 \;=\; \sum_{d|L} \xi_d(1)^2 \;=\; \prod_{\substack{p|L \\ p\nmid M}} \rho_f(p)^{-1} \prod_{\substack{p^2|N \\ p^2\nmid M}} \frac{p^2}{p^2-1}.$$

Sample of the algebra....

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Proof. Using the definition of $\xi_d(\ell)$, writing $d = d_1 d_2$, where d_1 is squarefree and d_2 is squarefull and $(d_1, d_2) = 1$, we have

$$\sum_{d|L} \xi_d(1)^2 = \sum_{d|L} \frac{\lambda_f(d_1)^2 \mu_f(d_2)^2}{r_f(d) d \beta(d_1)^2 \alpha(d_2)}. \tag{3.10}$$

Recall that $\mu_f(1) = 1, \mu_f(p^2) = \chi_{0;M}(p)$, and $\mu_f(p^{e_p}) = 0$ for all $e_p > 2$. As all functions in the sum above are multiplicative, we can factor as follows:

$$\sum_{d|L} \xi_d(1)^2 = \prod_{p^{e_p} \| L} \left(1 + \frac{\lambda_f(p)^2}{r_f(p) p \beta(p)^2} + \frac{\chi_{0;M}(p)(1 - \mu(p^{e_p})^2)}{r_f(p^2) p^2 \alpha(p^2)}\right). \tag{3.11}$$

We now break into cases when $(p, M) = 1$ and $(p, M) \neq 1$ to remove the $\chi_{0;M}(p)$:

$$\sum_{d|L} \xi_d(1)^2 = \prod_{\substack{p^{e_p} \| L \\ (p,M)=1}} \left(1 + \frac{\lambda_f(p)^2}{r_f(p) p \beta(p)^2} + \frac{1 - \mu(p^{e_p})^2}{r_f(p^2) p^2 \alpha(p^2)}\right) \prod_{\substack{p^{e_p} \| L \\ (p,M) \neq 1}} \left(1 + \frac{\lambda_f(p)^2}{r_f(p) p \beta(p)^2}\right). \tag{3.12}$$

We can now simplify many of the terms as follows. If $(p, M) = 1$, then

$$\begin{aligned} \beta(p)^2 &= (1 + 1/p)^2 \\ \alpha(p^2) &= (1 - 1/p^2) \\ r_f(p) &= \rho_f(p). \end{aligned} \tag{3.13}$$

If $(p, M) \neq 1$, we have

$$\begin{aligned} \beta(p) &= \alpha(p^2) = 1 \\ r_f(p) &= 1 - \frac{\lambda_f(p)^2}{p}. \end{aligned} \tag{3.14}$$

In addition, note that $r_f(p) = r_f(p^2)$. Thus we can write the right hand side of (3.12) as

$$\prod_{\substack{p^{e_p} \| L \\ (p,M)=1}} \left(1 + \frac{\lambda_f(p)^2}{\rho_f(p) p \left(1 + \frac{1}{p}\right)^2} + \frac{1 - \mu(p^{e_p})^2}{\rho_f(p) p^2 \left(1 - \frac{1}{p^2}\right)}\right) \prod_{\substack{p^{e_p} \| L \\ (p,M) \neq 1}} \left(1 + \frac{\lambda_f(p)^2}{p \left(1 - \frac{\lambda_f(p)^2}{p}\right)}\right). \tag{3.15}$$

Recall that because $f \in H_k^*(M)$,

$$\lambda_f(p)^2 = \begin{cases} \frac{1}{p} & \text{if } p \| M \\ 0 & \text{if } p^2 \mid M. \end{cases} \tag{3.16}$$

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Therefore we can rewrite the second product in (3.15), and obtain

$$\sum_{d|L} \xi_d(1)^2 = \prod_{\substack{p^{e_p} \| L \\ (p,M)=1}} \left(1 + \frac{\lambda_f(p)^2}{\rho_f(p) p \left(1 + \frac{1}{p}\right)^2} + \frac{1 - \mu(p^{e_p})^2}{\rho_f(p) p^2 \left(1 - \frac{1}{p^2}\right)}\right) \prod_{\substack{p | L \\ p \| M}} \left(\frac{p^2}{p^2 - 1}\right). \tag{3.17}$$

We now simplify the first product above as

$$\begin{aligned} &\prod_{\substack{p^{e_p} \| L \\ (p,M)=1}} \left(\frac{\rho_f(p) p \left(1 + \frac{1}{p}\right)^2 (p^2 - 1) + \lambda_f(p)^2 (p^2 - 1) + p \left(1 + \frac{1}{p}\right)^2 (1 - \mu(p^{e_p})^2)}{\rho_f(p) p \left(1 + \frac{1}{p}\right)^2 (p^2 - 1)}\right) \\ &\quad \times \prod_{\substack{p | L \\ p \| M}} \left(\frac{p^2}{p^2 - 1}\right) \\ &= \prod_{\substack{p^{e_p} \| L \\ (p,M)=1}} \left(\frac{\left(p \left(1 + \frac{1}{p}\right)^2 - \lambda_f(p)^2\right) (p^2 - 1) + \lambda_f(p)^2 (p^2 - 1) + p \left(1 + \frac{1}{p}\right)^2 (1 - \mu(p^{e_p})^2)}{\rho_f(p) p \left(1 + \frac{1}{p}\right)^2 (p^2 - 1)}\right) \\ &\quad \times \prod_{\substack{p | L \\ p \| M}} \left(\frac{p^2}{p^2 - 1}\right) \\ &= \prod_{\substack{p^{e_p} \| L \\ (p,M)=1}} \left(\frac{p \left(1 + \frac{1}{p}\right)^2 (p^2 - \mu(p^{e_p})^2)}{\rho_f(p) p \left(1 + \frac{1}{p}\right)^2 (p^2 - 1)}\right) \prod_{\substack{p | L \\ p \| M}} \left(\frac{p^2}{p^2 - 1}\right) \\ &= \prod_{\substack{p | L \\ p \nmid M}} \rho_f(p)^{-1} \prod_{\substack{p^2 | N \\ p^2 \nmid M}} \left(\frac{p^2}{p^2 - 1}\right), \end{aligned} \tag{3.18}$$

which completes the proof. □

Combining Lemma 3.2 with equations (2.35) and (3.8) yields Lemma 3.1.

4. An inversion and a change from weighted to pure sums

We now introduce the arithmetically weighted sums, as defined in [ILS, (2.53)],

$$\Delta_{k,N}^*(m, n) = \sum_{f \in H_k^*(N)} \frac{\lambda_f(n) \lambda_f(m) Z_N(1, f)}{Z(1, f)}. \tag{4.1}$$

This allows us to state one of our main results, which generalizes work of Iwaniec, Luo, and Sarnak [ILS, Proposition 2.8] and Rouymi [R, Proposition 2.3].

Conclusion and References

Recap

- Choose combinatorics to simplify calculations.
- Extending support often related to deep arithmetic questions.
- Technical issues arise, some formulas only clean in special cases.

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