Conway Checkers: Monovariant methods and Fibonacci jumping

Joint with the Conway Checkers group (SMALL 2024)
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⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

To move, jump a checker over another checker into an unoccupied square, either vertically or horizontally, then the jumped-over checker is removed.

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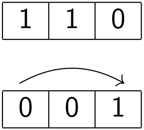

The Pagoda function

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Fix a target square T (on row n) and assign each square values of powers of x according to the Taxicab metric from T:

x^4	x^3	x^2	x^1	x^0	x^1	x^2	x^3	x^4
x^5	x^4	x^3	x^2	x^1	x^2	x^3	x^4	x^5
x^6	x^5	x^4	x^3	x^2	x^3	x^4	x^5	x^6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x^{n+4}	x^{n+3}	x^{n+2}	x^{n+1}	x^n	x^{n+1}	x^{n+2}	x^{n+3}	x^{n+4}
x^{n+5}	x^{n+4}	x^{n+3}	x^{n+2}	x^{n+1}	x^{n+2}	x^{n+3}	x^{n+4}	x^{n+5}
x^{n+6}	x^{n+5}	x^{n+4}	x^{n+3}	x^{n+2}	x^{n+3}	x^{n+4}	x^{n+5}	x^{n+6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

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Setting this to 0 and taking the reasonable value of x gives

$$x = \frac{\sqrt{5}-1}{2} = \frac{1}{\varphi} =: \alpha.$$

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It is reasonably easy to check that rows 1 through 4 can be reached.

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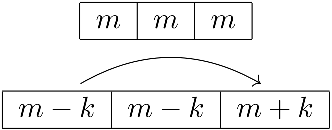

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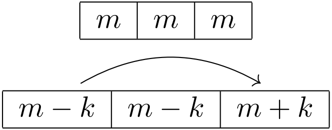

for any $0 < k \leq m$.

All other rules remain the same. We call this the m-game.

Fibonacci jumping

Here and throughout we define the Fibonacci numbers

$$F(k) \ := \ \frac{1}{\sqrt{5}}(\varphi^k - (-\alpha)^k)$$

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Suppose it is possible to reach the state

$\vdots$
0
$F(k)$
$F(k-1)$

somewhere on the top row of checkers.

Fibonacci jumping

From this state, consider the following sequence of moves:

0	0	...	0	1
$\vdots$	$\vdots$	...	$F(2)$	0
0	0	...	$F(1)$	0
0	0	...	0	0
0	0	...	0	0
0	$F(k-1)$	...	0	0
$F(k)$	$F(k-2)$	...	0	0
$F(k-1)$	0	...	0	0

where the final 1 is on the $k-1^{\text{th}}$ row.

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$F(k)$	$F(k-2)$	...	0	0
$F(k-1)$	0	...	0	0

where the final 1 is on the $k-1^{\text{th}}$ row.
Hence if $m = F(k) + \varepsilon$ for some $0 \leq \varepsilon < F(k-1)$ it is possible to reach at least the $k-1^{\text{th}}$ row.

Upper bound on rows attainable

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It is theoretically possible to reach the n^{th} row in finite moves if $E_0(n) > 1$. Solving this for n gives that the maximum row attainable satisfies

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$$n_m \; < \; \log_{\varphi}(m) + 5.$$

Since n_m is always an integer,

$$n_m \; \leq \; \left\lfloor \log_{\varphi}(m) + 5 \right\rfloor .$$

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We want to find a strong lower bound for the number of rows attainable.

Constructive proof

Consider the following table (for $m \neq 1$):

move 1	target	amount needed
$F(n+1)$	$F(n+1)$	0
$2m - F(n+1)$	$F(n)$	$F(n+2) - 2m$
$2m - F(n+1)$	$F(n+2) - 2m$	$F(n+3) - 4m$
m	$F(n+4) - 6m$	$F(n+4) - 7m$
m	$F(n+5) - 11m$	$F(n+5) - 12m$
m	$F(n+6) - 19m$	$F(n+6) - 20m$
$\vdots$	$\vdots$	$\vdots$

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$\vdots$	$\vdots$	$\vdots$

The target column gives the checkers required in the column, and the amount needed gives the amount needed to be added to that square.

Constructive proof

It can be seen that the general term for the amount needed to be added (in the -k^{th} row) is

$$F(n+k+1) - a_k m$$

with $a_{k+2} = a_{k+1} + a_k + 1$.

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This can be solved to give

$$a_k = F(k+3) - 1.$$

Condition on attainability

Using this method, the n^{th} row is attainable if, for some k,

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A sufficient condition for this is

$$\lim_{k\to\infty} \frac{m a_k}{F(n+k+1)} \;>\; 1.$$

To find this limit, we use Binet's formula.

Finding the limit

Noting that $|\alpha| < 1$ and substituting the a_k found earlier, we have

$$\lim_{k\to\infty}\left(\frac{mF(k+3)}{F(n+k+1)} - \frac{m}{F(n+k+1)}\right)$$

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$$= \lim_{k\to\infty}\frac{m\varphi^{k+3} - \sqrt{5}m}{\varphi^{n+k+1}} = \frac{m}{\varphi^{n-2}}.$$

Setting this to be greater than 1 and solving gives a bound on n_m:

$$n_m \;\geq\; \left\lfloor \log_\varphi(m) + 2 \right\rfloor .$$

The result

Theorem

In the one column Conway Checkers m-game, for $m > 1$, the maximum row theoretically obtainable in finite moves is

$$n_m = \lfloor \log_{\varphi}(m) + 2 \rfloor$$

and it is always possible to achieve this row.
If $m = 1$, it is only possible to reach row 1.

Back to the whole board

We now wish to apply this result to the whole board.

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The following state can be relatively easily obtained from the starting board:

0	0	0	0	0
0	0	m	0	0
0	0	m	0	0
0	0	m	0	0
0	0	m	0	0
0	0	m	0	0
0	0	m	m	m
m	m	m	m	m

Hence it is possible to raise a column up by 2 squares in finite time.

Statements

We now apply our previous result to this raised column.

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Theorem

In the Conway Checkers m-game the maximum row attainable n_m satisfies

$$\left\lfloor \log_\varphi(m) + 4 \right\rfloor \ \leq \ n_m \ \leq \ \left\lfloor \log_\varphi(m) + 5 \right\rfloor .$$

In particular, it is always possible to reach within 1 row of the theoretical maximum.
Note this is true for $m = 1$ by earlier arguments.

Conjecture

In the m-game, for $m > 1$, n_m satisfies

$$n_m \ = \ \left\lfloor \log_\varphi(m) + 5 \right\rfloor .$$

Other results

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For m sufficiently large, we have an algorithm to obtain the bounds $\left\lfloor \log_\varphi(m) + 4.67 \right\rfloor \leq n_m \leq \left\lfloor \log_\varphi(m) + 5 \right\rfloor$.

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For m sufficiently large, we have an algorithm to obtain the bounds $\left\lfloor \log_\varphi(m) + 4.67 \right\rfloor \leq n_m \leq \left\lfloor \log_\varphi(m) + 5 \right\rfloor$.

For many choices of m, this gives a constructive method to reach the theoretical maximum row.

Other results

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m	m	m	0

$m-k$	$m-k$	$m-k$	k

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With these rules, the optimal jumping method uses the Tribonacci numbers in the same way we earlier used the Fibonacci numbers.

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By similar methods to earlier, it is always possible to reach within 1 of the upper bound.

Predictably, if you jump over n squares at a time, then the $(n+1)$-nacci numbers are the optimal jumping method.

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