Extending Pythagoras

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Goals of the Talk

- Often multiple proofs: Say a proof rather than the proof.
- Different proofs highlight different aspects.
- Too often rote algebra: Explore! Generalize! Conjecture!
- General: How to find / check proofs: special cases, ‘smell’ test.
- Specific: Pythagorean Theorem.
Pythagorean Theorem
Theorem (Pythagorean Theorem)

Right triangle with sides $a$, $b$ and hypotenuse $c$, then $a^2 + b^2 = c^2$.

Most students know the statement, but the proof?

Why are proofs important? Can help see big picture.
Geometric Proofs of Pythagoras

Figure: Euclid’s Proposition 47, Book I. Why these auxiliary lines? Why are there equalities?
Geometric Proofs of Pythagoras

Figure: Euclid’s Proposition 47, Book I. Why these auxiliary lines? Why are there equalities?
Figure: A nice matching proof, but how to find these slicings!
Geometric Proofs of Pythagoras

Figure: Four triangles proof: I
Geometric Proofs of Pythagoras

Figure: Four triangles proof: II
Geometric Proofs of Pythagoras

Figure: President James Garfield’s (Williams 1856) Proof.
Geometric Proofs of Pythagoras

Lots of different proofs.

Difficulty: how to find these combinations?

At the end of the day, do you know *why* it’s true?
Dimensional Analysis
Possible Pythagorean Theorems....

\[ c^2 = a^3 + b^3. \]
\[ c^2 = a^2 + 2b^2. \]
\[ c^2 = a^2 - b^2. \]
\[ c^2 = a^2 + ab + b^2. \]
\[ c^2 = a^2 + 110ab + b^2. \]
Possible Pythagorean Theorems....

⋄ $c^2 = a^3 + b^3$. No: wrong dimensions.

⋄ $c^2 = a^2 + 2b^2$. No: asymmetric in $a, b$.

⋄ $c^2 = a^2 - b^2$. No: can be negative.

⋄ $c^2 = a^2 + ab + b^2$. Maybe: passes all tests.

⋄ $c^2 = a^2 + 110ab + b^2$. No: violates $a + b > c$. 
Dimensional Analysis Proof of the Pythagorean Theorem

- Area is a function of hypotenuse $c$ and angle $x$. 
Dimensional Analysis Proof of the Pythagorean Theorem

◊ Area is a function of hypotenuse $c$ and angle $x$.

◊ $\text{Area}(c, x) = f(x)c^2$ for some function $f$ (similar triangles).
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- $f(x)a^2 + f(x)b^2 = f(x)c^2$
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Area$(c, x) = f(x)c^2$ for some function $f$ (CPCTC).

Must draw an auxiliary line, but where? Need right angles!

$f(x)a^2 + f(x)b^2 = f(x)c^2 \Rightarrow a^2 + b^2 = c^2$. 
Dimensional Analysis and the Pendulum

Length: \( L: \) meters
Acceleration: \( g: \) meters/sec\(^2\)
Mass: \( m: \) kilograms
Period: \( T: \) seconds
Angle: \( x: \) radians
Dimensional Analysis and the Pendulum

Period: Need combination of quantities to get seconds.
Dimensional Analysis and the Pendulum

Period: Need combination of quantities to get seconds.

\[ T = f(x) \sqrt{L/g}. \]
Conclusion
Conclusion

- Math is not complete – explore and conjecture!
- Different proofs highlight different aspects.
- Get a sense of what to try / what might work.
Feeling Equations
Sabermetrics is the art of applying mathematics and statistics to baseball.

Danger: not all students like sports (Red Sox aren’t making life easier!).

Lessons: not just for baseball; try to find the right statistics that others miss, competitive advantage (business, politics).
Assume team A wins \( p \) percent of their games, and team B wins \( q \) percent of their games. Which formula do you think does a good job of predicting the probability that team A beats team B? Why?

\[
\frac{p + pq}{p + q + 2pq}, \quad \frac{p + pq}{p + q - 2pq}, \quad \frac{p - pq}{p + q + 2pq}, \quad \frac{p - pq}{p + q - 2pq}
\]
Estimating Winning Percentages

\[
\frac{p + pq}{p + q + 2pq}, \quad \frac{p + pq}{p + q - 2pq}, \quad \frac{p - pq}{p + q + 2pq}, \quad \frac{p - pq}{p + q - 2pq}
\]

How can we test these candidates?

Can you think of answers for special choices of \(p\) and \(q\)?
Estimating Winning Percentages

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\]

Homework: explore the following:

- \(p = 1, q < 1\) (do not want the battle of the undefeated).
- \(p = 0, q > 0\) (do not want the Toilet Bowl).
- \(p = q\).
- \(p > q\) (can do \(q < 1/2\) and \(q > 1/2\)).
- Anything else where you ‘know’ the answer?
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Estimating Winning Percentages

\[
\frac{p - pq}{p + q - 2pq} = \frac{p(1 - q)}{p(1 - q) + (1 - p)q}
\]

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◇ Anything else where you ‘know’ the answer?
Estimating Winning Percentages: ‘Proof’

Start

\[ A \text{ has a good game with probability } p \]

\[ B \text{ has a good game with probability } q \]

**Figure:** First see how \( A \) does, then \( B \).
Estimating Winning Percentages: ‘Proof’

**Figure:** Two possibilities: A has a good day, or A doesn’t.
Estimating Winning Percentages: ‘Proof’

Figure: $B$ has a good day, or doesn’t.
Estimating Winning Percentages: ‘Proof’

Figure: Two paths terminate, two start again.
Estimating Winning Percentages: ‘Proof’

Probability A wins is \( \frac{p(1-q)}{p(1-q) + (1-p)q} \) = \( \frac{p - pq}{p + q - 2pq} \)

**Figure:** Probability A beats B.
Lessons

Special cases can give clues.

Algebra can suggest answers.

Better formula: Bill James’ Pythagorean Won-Loss formula.
Numerical Observation: Pythagorean Won-Loss Formula

Parameters

- \( RS_{\text{obs}} \): average number of runs scored per game;
- \( RA_{\text{obs}} \): average number of runs allowed per game;
- \( \gamma \): some parameter, constant for a sport.

James’ Won-Loss Formula (NUMERICAL Observation)

\[
\text{Won – Loss Percentage} = \frac{RS_{\text{obs}} \gamma}{RS_{\text{obs}} \gamma + RA_{\text{obs}} \gamma}
\]

\( \gamma \) originally taken as 2, numerical studies show best \( \gamma \) is about 1.82. Used by ESPN, MLB. See http://arxiv.org/abs/math/0509698 for a ‘derivation’.
Other Gems
Sums of Integers

\[ S_n := 1 + 2 + \cdots + n = \frac{n(n + 1)}{2} \approx \frac{1}{2} n^2. \]
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Proof 1: Induction.

Proof 2: Grouping:

\[ 2S_n = (1 + n) + (2 + (n - 1)) + \cdots + (n + 1). \]
Sums of Integers

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Have \[ \frac{n \cdot n}{2} \leq S_n \leq n; \] thus \( S_n \) is between \( n^2 / 4 \) and \( n^2 \), have the correct order of magnitude of \( n \).
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Can improve: divide and conquer again: lather, rinse, repeat....

\[ \frac{n}{4} \cdot \frac{n}{4} + \frac{n}{4} \cdot \frac{2n}{4} + \frac{n}{4} \cdot \frac{3n}{4} \leq S_n, \quad \text{so} \quad \frac{6}{16}n^2 \leq S_n. \]
Geometric Irrationality Proofs:
http://arxiv.org/abs/0909.4913

Figure: Geometric proof of the irrationality of $\sqrt{2}$. 
Geometric Irrationality Proofs:
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Figure: Geometric proof of the irrationality of $\sqrt{3}$
Geometric Irrationality Proofs:

http://arxiv.org/abs/0909.4913

Figure: Geometric proof of the irrationality of $\sqrt{5}$.
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**Figure:** Geometric proof of the irrationality of $\sqrt{5}$: the kites, triangles and the small pentagons.
Geometric Irrationality Proofs:
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Figure: Geometric proof of the irrationality of $\sqrt{6}$. 
### Preliminaries: The Cookie Problem

#### The Cookie Problem

The number of ways of dividing $C$ identical cookies among $P$ distinct people is \[^{C+P-1}_{P-1}\].
Preliminaries: The Cookie Problem

The Cookie Problem

The number of ways of dividing $C$ identical cookies among $P$ distinct people is $\binom{C+P-1}{P-1}$.

Proof: Consider $C + P - 1$ cookies in a line. **Cookie Monster** eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do. Divides the cookies into $P$ sets.
Preliminaries: The Cookie Problem

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Cookie Monster eats $P - 1$ cookies: \( \binom{C+P-1}{P-1} \) ways to do. 
Divides the cookies into $P$ sets.
Example: 8 cookies and 5 people ($C = 8$, $P = 5$):
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Preliminaries: The Cookie Problem

The Cookie Problem

The number of ways of dividing \( C \) identical cookies among \( P \) distinct people is \( \binom{C+P-1}{P-1} \). Solved \( x_1 + \cdots + x_P = C \), \( x_i \geq 0 \).

**Proof**: Consider \( C + P - 1 \) cookies in a line. **Cookie Monster** eats \( P - 1 \) cookies: \( \binom{C+P-1}{P-1} \) ways to do. Divides the cookies into \( P \) sets.

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