Extending Pythagoras

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Goals of the Talk

- Often multiple proofs: Say a proof rather than the proof.
- Different proofs highlight different aspects.
- Too often rote algebra: Explore! Generalize! Conjecture!
- General: How to find / check proofs: special cases, ‘smell’ test.
- Specific: Pythagorean Theorem.
Pythagorean Theorem
Theorem (Pythagorean Theorem)

*Right triangle with sides $a$, $b$ and hypotenuse $c$, then $a^2 + b^2 = c^2$."

Most students know the statement, but the proof?

Why are proofs important? Can help see big picture.
Geometric Proofs of Pythagoras

**Figure**: Euclid’s Proposition 47, Book I. Why these auxiliary lines? Why are there equalities?
**Geometric Proofs of Pythagoras**

*Figure:* Euclid’s Proposition 47, Book I. Why these auxiliary lines? Why are there equalities?
Figure: A nice matching proof, but how to find these slicings!
Geometric Proofs of Pythagoras

Figure: Four triangles proof: I
Geometric Proofs of Pythagoras

Figure: Four triangles proof: II
Geometric Proofs of Pythagoras

**Figure:** President James Garfield’s (Williams 1856) Proof.
Geometric Proofs of Pythagoras

Lots of different proofs.

Difficulty: how to find these combinations?

At the end of the day, do you know *why* it’s true?
Dimensional Analysis
Possible Pythagorean Theorems....

- $c^2 = a^{17} + b^{17}$.
- $c^2 = a^2 + 17b^2$.
- $c^2 = a^2 - b^2$.
- $c^2 = a^2 + ab + b^2$.
- $c^2 = a^2 + 17ab + b^2$. 
Possible Pythagorean Theorems....

- $c^2 = a^3 + b^3$. No: wrong dimensions.
- $c^2 = a^2 + 2b^2$. No: asymmetric in $a, b$.
- $c^2 = a^2 - b^2$. No: can be negative.
- $c^2 = a^2 + ab + b^2$. Maybe: passes all tests.
- $c^2 = a^2 + 110ab + b^2$. No: violates $a + b > c$. 
Dimensional Analysis Proof of the Pythagorean Theorem

◊ Area is a function of hypotenuse $c$ and angle $x$. 
Dimensional Analysis Proof of the Pythagorean Theorem

- Area is a function of hypotenuse $c$ and angle $x$.
- $\text{Area}(c, x) = f(x)c^2$ for some function $f$ (similar triangles).
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- Must draw an auxiliary line, but where? Need right angles!
Diamond Area is a function of hypotenuse $c$ and angle $x$.

Diamond $\text{Area}(c, x) = f(x)c^2$ for some function $f$ (CPCTC).

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- $f(x)a^2 + f(x)b^2 = f(x)c^2$
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- $\text{Area}(c, x) = f(x)c^2$ for some function $f$ (CPCTC).

- Must draw an auxiliary line, but where? Need right angles!

- $f(x)a^2 + f(x)b^2 = f(x)c^2 \implies a^2 + b^2 = c^2$. 
Dimensional Analysis and the Pendulum

Period: Need combination of quantities to get seconds.

\[ T = f(x) \sqrt{\frac{L}{g}}. \]
Dimensional Analysis and the Pendulum

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Guessing Pythagoras:
Finding the Functional Form

Idea is to try and guess the correct functional form for Pythagoras.

Guess will have some free parameters, determine by special cases.

Natural guesses: linear, quadratic, ....
Linear Attempt

Guess linear relation: $c = \alpha a + \beta b$: what are $\alpha, \beta$?

Consider special cases:
Linear Attempt

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Consider special cases:

- \( a \to 0 \) have very thin triangle so \( b \to c \) and thus \( \beta = 1 \).
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Linear Attempt

Guess linear relation: $c = \alpha a + \beta b$: what are $\alpha, \beta$?

Consider special cases:

- $a \rightarrow 0$ have very thin triangle so $b \rightarrow c$ and thus $\beta = 1$.
- $b \rightarrow 0$ have very thin triangle so $a \rightarrow c$ and thus $\alpha = 1$.

Question: Does $c = a + b$ make sense?
Linear Attempt: Analyzing $c = a + b$ (so $a = b = 1$ implies $c = 2$)

So, *if* linear, *must be* $c = a + b$. Using:
- Area rectangle $x$ by $y$ is $xy$.
- Area right triangle of sides $x$ by $y$ is $\frac{1}{2}xy$. 

Figure: Four triangles and a square, assuming $c = a + b$ and $a = b = 1$. Calculate area of big square two ways: Four triangles, each area $\frac{1}{2} \cdot 1$: total is 2. Square of sides 2: area is $2 \cdot 2 = 4$. Contradiction! Cannot be linear!
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Calculate area of big square two ways:
- Four triangles, each area $\frac{1}{2}1 \cdot 1$: total is 2.
- Square of sides 2: area is $2 \cdot 2 = 4$.

Contradiction! Cannot be linear!
Quadratic Attempt:

Guess quadratic: $c^2 = \alpha a^2 + \gamma ab + \beta b^2$: what are $\alpha, \beta, \gamma$?
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Figure: Four triangles and a square: $a = b = 1$. 
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Figure: Four triangles and a square: $a = b = 1$.

Equating areas: $c^2 = 4 \left( \frac{1}{2} \cdot 1 \cdot 1 \right)$, so $c^2 = 2$ or $c = \sqrt{2}$. 
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Guess quadratic: \( c^2 = \alpha a^2 + \gamma ab + \beta b^2 \): what are \( \alpha, \beta, \gamma \)?

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Figure: Four triangles and a square: \( a = b = 1 \).

Equating areas: \( c^2 = 4 \left( \frac{1}{2} \cdot 1 \cdot 1 \right) \), so \( c^2 = 2 \) or \( c = \sqrt{2} \).
Thus \( 2 = 1 + \gamma 1 \cdot 1 + 1 \), so \( \gamma = 0 \) and \( c^2 = a^2 + b^2 \).
Warnings:

*Not a proof:* just shows that if quadratic, must be $c^2 = a^2 + b^2$.

In lowest terms:

$$\frac{16}{64} = \frac{1}{4}, \frac{19}{95} = \frac{1}{5}, \frac{49}{98} = \frac{1}{2}.$$
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Extending Pythagoras: The Sphere
Pythagoras on a Sphere

What should the Pythagorean Theorem be on a sphere?
Spherical Coordinates

Spherical Coordinates: \( \rho \in [0, R], \phi \in [0, \pi], \theta \in [0, 2\pi) \).

- \( x = \rho \sin(\phi) \cos(\theta) \).
- \( y = \rho \sin(\phi) \sin(\theta) \).
- \( z = \rho \cos(\phi) \).

Note \( z = \rho \cos(\phi) \), then \((x, y)\) from circle of radius \( r = \rho \sin(\phi) \) and angle \( \theta \).
What could the Pythagorean Formula be on a sphere?
Special Cases

What could the Pythagorean Formula be on a sphere?

- If \(a, b, c\) small relative to radius \(R\) should reduce to planar Pythagoras.
- Can have equilateral right triangle with \(a = b = c\).
- Only depends on ratios \(a/R, b/R, c/R\).
Special Cases

What could the Pythagorean Formula be on a sphere?

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- Can have equilateral right triangle with $a = b = c$.
- Only depends on ratios $a/R, b/R, c/R$.

Maybe a relation involving cosines of $a/R, b/R, c/R$ as arc length is related to angle!
cos(u) = 1 - u^2/2! + u^4/4! - \cdots \approx 1 - u^2/2 \ (u \text{ small}).

Ingredients (will consider } R \text{ large relative to } a, b, c:

- \cos(a/R) \approx 1 - \frac{1}{2} \frac{a^2}{R^2}.
- \cos(b/R) \approx 1 - \frac{1}{2} \frac{b^2}{R^2}.
- \cos(c/R) \approx 1 - \frac{1}{2} \frac{c^2}{R^2}.

Algebra: \cos(c/R) \approx \cos(a/R) \cos(b/R):

\[ 1 - \frac{1}{2} \frac{c^2}{R^2} \approx \left(1 - \frac{1}{2} \frac{a^2}{R^2}\right) \left(1 - \frac{1}{2} \frac{b^2}{R^2}\right) = 1 - \frac{a^2 + b^2}{2R^2} + \frac{a^2 b^2}{4R^4} \]

\[ c^2 \approx a^2 + b^2 - \frac{2a^2 b^2}{R^2}. \]
Needed Input: Dot Product $\vec{v} \cdot \vec{w}$

$$(v_1, \ldots, v_n) \cdot (w_1, \ldots, w_n) = v_1 w_1 + \cdots + v_n w_n.$$
Proof of Dot Product Formula (Plane)

Use the Law of Cosines: \( c^2 = a^2 + b^2 - 2ab \cos \theta_{ab}. \)
Proof of Dot Product Formula (Plane)

Use the Law of Cosines: \( c^2 = a^2 + b^2 - 2ab \cos \theta_{ab} \).

\[
(v_1 - w_1)^2 + (v_2 - w_2)^2 = (v_1^2 + v_2^2) + (w_1^2 + w_2^2) - 2|\vec{v}| |\vec{w}| \cos \theta_{vw}.
\]
Proof of Dot Product Formula (Plane)

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\]

After some algebra:
\[
v_1 w_1 + v_2 w_2 = |\vec{v}| |\vec{w}| \cos \theta_{vw},
\]
completing the proof.
Spherical Proof

Three points: $P_0$, $P_A$, $P_B$:

- $P_0 : (R, 0, 0)$.
- $P_A : (R, \phi_A, \theta_A)$: $|\overrightarrow{P_A P_0}| = \frac{\phi_A}{2\pi} R = \frac{a}{R}$.
- $P_B : (R, \phi_B, \theta_B)$: $|\overrightarrow{P_B P_0}| = \frac{\phi_B}{2\pi} R = \frac{b}{R}$.

Length $\overrightarrow{P_B P_A}$ is $\frac{\phi_{AB}}{2\pi} R$, where $\phi_{AB}$ angle between $\overrightarrow{P_A P_0}$ and $\overrightarrow{P_B P_0}$.

Proof follows from dot product:

$$\overrightarrow{P} \cdot \overrightarrow{Q} = |\overrightarrow{P}| |\overrightarrow{Q}| \cos(\phi_{PQ}).$$
Spherical Proof: Continued

Cartesian Coordinates for Dot Product:
Remember right triangle: can take $\theta_A = 0$, $\theta_B = \pi/2$.

- $\vec{P_A P_0} : (R \sin \phi_A, 0, R \cos \phi_A)$, length is $R$.
- $\vec{P_B P_0} : (0, R \sin \phi_B, R \cos \phi_B)$, length is $R$.

$\vec{P_B} \cdot \vec{P_A} = 0 + 0 + R^2 \cos \phi_A \cos \phi_B$.

Dot product now gives

$$\cos(\phi_{AB}) = \frac{\vec{P_B} \cdot \vec{P_A}}{|\vec{P_B P_0}| \ |\vec{P_A P_0}|} = \frac{\vec{P_B} \cdot \vec{P_A}}{R^2}.$$  

Substituting yields

$$\cos(\phi_{AB}) = \frac{R^2 \cos \phi_A \cos \phi_B}{R^2} = \cos \phi_A \cos \phi_B,$$

proving spherical Pythagoras!
Next Step: Generalize

Keep going! Generalize further!

What’s the next natural candidate?
Next Step: Generalize

Keep going! Generalize further!

What’s the next natural candidate? **Hyperbolic!**

Guess:
Next Step: Generalize

Keep going! Generalize further!

What’s the next natural candidate? **Hyperbolic!**

Guess: \( \cosh(c) = \cosh(a) \cosh(b) \), where \( \cosh \) is the hyperbolic cosine!

\[
\cos(x) = \frac{1}{2} \left( e^{ix} + e^{-ix} \right), \quad \cosh(x) = \frac{1}{2} \left( e^x + e^{-x} \right).
\]
Next Step: Generalize

Keep going! Generalize further!

What’s the next natural candidate? Hyperbolic!

Guess: $\cosh(c) = \cosh(a) \cosh(b)$, where $\cosh$ is the hyperbolic cosine!

$$\cos(x) = \frac{1}{2} \left( e^{ix} + e^{-ix} \right), \quad \cosh(x) = \frac{1}{2} \left( e^x + e^{-x} \right).$$

Fun identities:
- $\cosh^2(x) - \sinh^2(x) = 1.$
- $\sinh(x + y) = \sinh(x) \cosh(y) + \cosh(x) \sinh(y).$
- $\cosh(x + y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y).$
Conclusion
Conclusion

◊ Math is not complete – explore and conjecture!

◊ Different proofs highlight different aspects.

◊ Get a sense of what to try / what might work.
Feeling Equations
Sabermetrics is the art of applying mathematics and statistics to baseball.

Danger: not all students like sports (Red Sox aren’t making life easier!).

Lessons: not just for baseball; try to find the right statistics that others miss, competitive advantage (business, politics).
Estimating Winning Percentages

Assume team A wins $p$ percent of their games, and team B wins $q$ percent of their games. Which formula do you think does a good job of predicting the probability that team A beats team B? Why?

\[
\frac{p + pq}{p + q + 2pq}, \quad \frac{p + pq}{p + q - 2pq}, \quad \frac{p - pq}{p + q + 2pq}, \quad \frac{p - pq}{p + q - 2pq}
\]
Estimating Winning Percentages

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\]

How can we test these candidates?

Can you think of answers for special choices of \( p \) and \( q \)?
Estimating Winning Percentages

\[ \frac{p + pq}{p + q + 2pq}, \quad \frac{p + pq}{p + q - 2pq}, \quad \frac{p - pq}{p + q + 2pq}, \quad \frac{p - pq}{p + q - 2pq} \]

Homework: explore the following:

⋄ \( p = 1, q < 1 \) (do not want the battle of the undefeated).

⋄ \( p = 0, q > 0 \) (do not want the Toilet Bowl).

⋄ \( p = q \).

⋄ \( p > q \) (can do \( q < 1/2 \) and \( q > 1/2 \)).

⋄ Anything else where you ‘know’ the answer?
Estimating Winning Percentages

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Estimating Winning Percentages

\[
\frac{p - pq}{p + q - 2pq} = \frac{p(1 - q)}{p(1 - q) + (1 - p)q}
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- \( p > q \) (can do \( q < 1/2 \) and \( q > 1/2 \)).
- Anything else where you ‘know’ the answer?
Estimating Winning Percentages: ‘Proof’

Start

A has a good game with probability p

B has a good game with probability q

Figure: First see how A does, then B.
Estimating Winning Percentages: ‘Proof’

**Figure:** Two possibilities: A has a good day, or A doesn’t.
Estimating Winning Percentages: ‘Proof’

Figure: B has a good day, or doesn’t.
Estimating Winning Percentages: ‘Proof’

**Figure:** Two paths terminate, two start again.
Estimating Winning Percentages: ‘Proof’

Figure: Probability $A$ beats $B$. 

Probability $A$ wins is 

$$
\frac{p (1-q)}{p (1-q) + (1-p) q} = \frac{p - pq}{p + q - 2 pq}
$$
Lessons

Special cases can give clues.

Algebra can suggest answers.

Better formula: Bill James’ Pythagorean Won-Loss formula.
Numerical Observation: Pythagorean Won-Loss Formula

**Parameters**
- $RS_{obs}$: average number of runs scored per game;
- $RA_{obs}$: average number of runs allowed per game;
- $\gamma$: some parameter, constant for a sport.

**James’ Won-Loss Formula (NUMERICAL Observation)**

\[
\text{Won} - \text{Loss Percentage} = \frac{RS_{obs} \gamma}{RS_{obs} \gamma + RA_{obs} \gamma}
\]

$\gamma$ originally taken as 2, numerical studies show best $\gamma$ is about 1.82. Used by ESPN, MLB.

Other Gems
Sums of Integers

\[ S_n := 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{1}{2} n^2. \]
Sums of Integers

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Proof 1: Induction.
Proof 2: Grouping:
\[ 2S_n = (1 + n) + (2 + (n - 1)) + \cdots + (n + 1). \]
Sums of Integers

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Instead of determining sum useful to get sense of size.
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Have \( \frac{n^2}{2} \leq S_n \leq n \); thus \( S_n \) is between \( n^2/4 \) and \( n^2 \), have the correct order of magnitude of \( n \).
Sums of Integers

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Can improve: divide and conquer again: lather, rinse, repeat....

\[ \frac{n^2}{4} + \frac{n 2n}{4} + \frac{n 3n}{4} \leq S_n, \quad \text{so} \quad \frac{6}{16} n^2 \leq S_n. \]
Geometric Irrationality Proofs:
http://arxiv.org/abs/0909.4913

Figure: Geometric proof of the irrationality of \( \sqrt{2} \).
Geometric Irrationality Proofs:
http://arxiv.org/abs/0909.4913

Figure: Geometric proof of the irrationality of $\sqrt{3}$
Geometric Irrationality Proofs:
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Figure: Geometric proof of the irrationality of $\sqrt{5}$. 
Geometric Irrationality Proofs:
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**Figure:** Geometric proof of the irrationality of \( \sqrt{5} \): the kites, triangles and the small pentagons.
Geometric Irrationality Proofs:
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**Figure:** Geometric proof of the irrationality of $\sqrt{6}$. 
The Cookie Problem

The number of ways of dividing $C$ identical cookies among $P$ distinct people is $\binom{C+P-1}{P-1}$.
Preliminaries: The Cookie Problem

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Proof: Consider $C + P - 1$ cookies in a line. 
Cookie Monster eats $P - 1$ cookies: $\binom{C+P-1}{P-1}$ ways to do. Divides the cookies into $P$ sets.
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Example: 8 cookies and 5 people ($C = 8$, $P = 5$):
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Preliminaries: The Cookie Problem

The Cookie Problem

The number of ways of dividing \( C \) identical cookies among \( P \) distinct people is \( \binom{C+P-1}{P-1} \). Solved \( x_1 + \cdots + x_P = C, \ x_i \geq 0 \).

Proof: Consider \( C + P - 1 \) cookies in a line. **Cookie Monster** eats \( P - 1 \) cookies: \( \binom{C+P-1}{P-1} \) ways to do. Divides the cookies into \( P \) sets.

Example: 8 cookies and 5 people \((C = 8, \ P = 5)\):