Signal Recovery using Gowers' Norms and Gabor Transform

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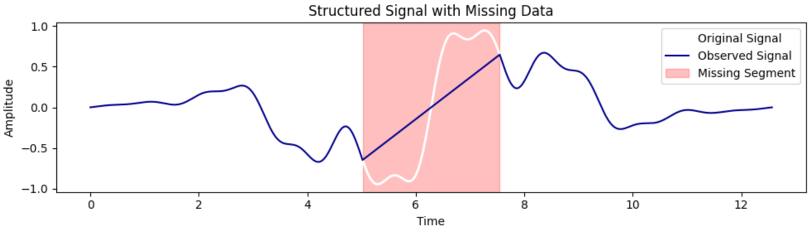
Introduction

Can You Recover the Original Signal?

- You receive only part of a signal/frequency - the rest is missing.



(Illustrative example of a corrupted signal)

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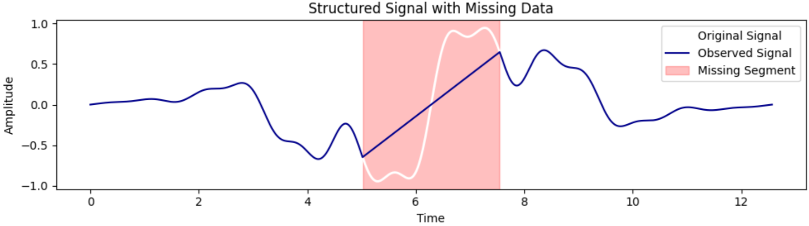

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- Is it possible to reconstruct the full message?

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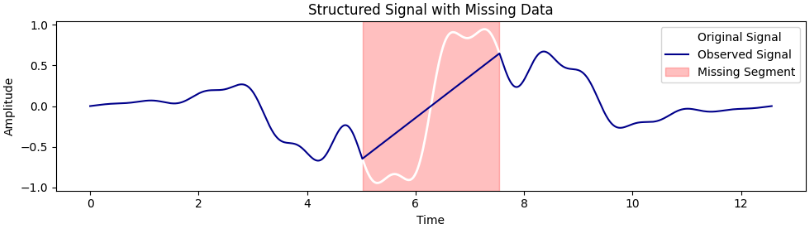

(Illustrative example of a corrupted signal)

- Is it possible to reconstruct the full message?
- Sufficient conditions for reconstruction? What if you know the signal/frequency is "structured"?

Fourier Analysis and Additive Combinatorics

We'll use tools from **Fourier Analysis** and **Additive Combinatorics** to find out.

Fourier Transform

Definition (Discrete Fourier Transform)

For a function $f : \mathbb{Z}_N^d \to \mathbb{C}$, the **normalized DFT** is:

$$\widehat{f}(k) := \frac{1}{\sqrt{N^d}} \sum_{n \in \mathbb{Z}_N^d} f(n)\chi(-kn),$$

where $\chi(x) = e^{-2\pi i k \cdot x/N}$. Then, the inverse transform formula follows:

$$f(n) = \frac{1}{\sqrt{N^d}} \sum_{k \in \mathbb{Z}_N^d} \widehat{f}(k)\chi(kn).$$

Fourier Transform Notation

- We will call an arbitrary function $f : \mathbb{Z}_N^d \to \mathbb{C}$ a **signal**.
- We will call an arbitrary function's fourier transform $\widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ a **frequency**.

Classical Uncertainty Principle

Theorem (Classical Uncertainty Principle)

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ *be a nonzero function with support* $\operatorname{supp}(f) \subseteq \mathbb{Z}_N^d$. *Let* $\widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ *denote the discrete Fourier transform of* f*, with support* $\operatorname{supp}(\widehat{f}) \subseteq \mathbb{Z}_N^d$. *Then the following inequality holds:*

$$|\operatorname{supp}(f)| \cdot |\operatorname{supp}(\widehat{f})| \geq N^d.$$

- This is a discrete version of Heisenberg's Uncertainty Principle!

Discrete L_p-norm

Definition (L_p Norm)

For a function $f : \mathbb{Z}_N \to \mathbb{C}$, the L_p norm is defined as:

$$\|f\|_{L_p(\mathbb{Z}_N)} := \begin{cases} \left(\frac{1}{N}\sum_{n=0}^{N-1} |f(n)|^p\right)^{1/p} & \text{if } 1 \le p < \infty \\ \max_{0 \le n < N} |f(n)| & \text{if } p = \infty \end{cases}$$

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Theorem (Holder's Inequality)

For a function $f, g : \mathbb{Z}_N^d \to \mathbb{C}$ *and* $\frac{1}{p} + \frac{1}{q} = 1$,

$$\|fg\|_{L^1} \leq \|f\|_{L^p} \cdot \|g\|_{L^q}$$

Unique Recovery Principle

Theorem (Classical Recovery Condition)

Let $f : \mathbb{Z}_N \to \mathbb{C}$ supported in $E \subset \mathbb{Z}_N$. Suppose that $\widehat{f}$ is transmitted but the frequencies $\{\widehat{f}(m)\}_{m \in S}$ are unobserved, where $S \subset \mathbb{Z}_N$, with

$$|E| \cdot |S| < \frac{N}{2}. \tag{1}$$

Then f can be recovered exactly and uniquely. Moreover,

$$f = \arg\min_{g} \|g\|_{L^1(\mathbb{Z}_N)} \tag{2}$$

with the constraint $\widehat{f}(m) = \widehat{g}(m), \; m \notin S$.

Gower's Norms

Gowers' Norms

Theorem (Classical Uncertainty Principle)

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$$|\operatorname{supp}(f)| \cdot |\operatorname{supp}(\widehat{f})| \geq N^d.$$

Can we quantify the "structure" of a set more precisely than just size?

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Definition

The **additive energy** of a set $A \subset \mathbb{Z}^d$ is defined as:

$$\Lambda(A) := \left|\left\{(x_1, x_2, x_3, x_4) \in A^4 : x_1 + x_2 = x_3 + x_4\right\}\right|,$$

where $|\cdot|$ denotes the cardinality of the set.

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Definition (Gowers U^2-norm)

For a function $f : \mathbb{Z}_N^d \to \mathbb{C}$, the Gowers U^2 norm is defined as:

$$\|f\|_{U^2}^4 := \mathbb{E}_{x,h_1,h_2}\left[f(x)\overline{f(x+h_1)f(x+h_2)}f(x+h_1+h_2)\right].$$

Additive Uncertainty Principle

Theorem (Additive Uncertainty Principle - Iosevich-Mayeli '25 [AII+25])

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$ be a nonzero signal with support in E, and let $\widehat{f}$ denote its Fourier transform with support in Σ. Then for any $\alpha \in [0,1]$,

(i) $N^d \leq (|E| \cdot \Lambda^{\frac{1}{3}}(\Sigma))^{1-\alpha} \cdot (\Lambda^{\frac{1}{3}}(E) \cdot |\Sigma|)^{\alpha}$.

To prove part (i), it is sufficient to establish the inequality

$$N^d \leq |E| \cdot \Lambda^{\frac{1}{3}}(\Sigma).$$

The inequality $N^d \leq |\Sigma| \cdot \Lambda^{\frac{1}{3}}(E)$ follows by reversing the roles of E and Σ, and the general case follows from these two by writing $N^d = N^{d(1-\alpha)} \cdot N^{d\alpha}$, $0 \leq \alpha \leq 1$. [AII+25]

Additive Uncertainty Principle: Improved

We were able to improve the additive uncertainty principle:

Theorem (SMALL 2025)

Let $f : \mathbb{Z}_N^d \to \mathbb{C}$. Suppose f is supported on $E \subset \mathbb{Z}_N^d$ and $\widehat{f}$ is supported on $\Sigma \subset \mathbb{Z}_N^d$. Then we have the uncertainty principle:

(i) $$N^d \leq |\Sigma| \left(\Lambda_2(E) - |E|^2 \left(1 - \sqrt{\frac{N^d}{|E||\Sigma|}} \sqrt{\frac{\Lambda_2(E)}{|E|^3}} \right) \right)^{1/3}$$

(ii) $$N^d \leq |\Sigma| \left(\frac{\sqrt{B_\Sigma |E| (\Lambda_2(E) - |E|^2)}}{|\Sigma|} + |E|^2 \sqrt{\frac{N^d}{|E|||\Sigma|}} \sqrt{\frac{\Lambda_2(E)}{|E|^3}} \right)^{1/3},$$

where $B_\Sigma = |\Sigma - \Sigma||(\Sigma + \Sigma) - (\Sigma + \Sigma)|$

Additive Uncertainty Principle: Improved

Define

$$1_{x,y,a} := 1_E(x)\, 1_E(y)\, 1_E(x+a)\, 1_E(y+a).$$

We begin by applying the Cauchy-Schwarz inequality to the following sum:

$$\sum_{x,y,a\in\mathbb{Z}_N^d} |f(x)f(y)f(x+a)f(y+a)| \cdot 1_{x,y,a}$$

$$\leq \left(\sum_{x,y,a\in\mathbb{Z}_N^d} |f(x)f(x+a)|^2 \cdot 1_{x,y,a}\right)^{1/2} \left(\sum_{x,y,a\in\mathbb{Z}_N^d} |f(y)f(y+a)|^2 \cdot 1_{x,y,a}\right)^{1/2}$$

$$= \sum_{x,y,a\in\mathbb{Z}_N^d} |f(x)f(x+a)|^2 \cdot 1_{x,y,a}$$

$$= (*)$$

Additive Uncertainty Principle: Improved

$$
\begin{aligned}
(*) = N^{-2d} \sum_{m_1,\dots,m_4} & \widehat{f}(m_1)\overline{\widehat{f}(m_2)}\widehat{f}(m_3)\overline{\widehat{f}(m_4)} \\
& \times \sum_{x,y,a} \chi(x\cdot(m_1 - m_2 + m_3 - m_4))\chi(a\cdot(m_3 - m_4)) \cdot 1_{x,y,a} \\
\le N^{-2d} \sum_{m_1,\dots,m_4} & |\widehat{f}(m_1)\widehat{f}(m_2)\widehat{f}(m_3)\widehat{f}(m_4)| \\
& \times \left| \sum_{\substack{x,y,a \\ a=0}} \chi(x\cdot(m_1 - m_2 + m_3 - m_4))\chi(a\cdot(m_3 - m_4)) 1_{x,y,a} \right| \\
+ N^{-2d} \sum_{m_1,\dots,m_4} & |\widehat{f}(m_1)\widehat{f}(m_2)\widehat{f}(m_3)\widehat{f}(m_4)| \\
& \times \left| \sum_{\substack{x,y,a \\ a\neq 0}} \chi(x\cdot(m_1 - m_2 + m_3 - m_4))\chi(a\cdot(m_3 - m_4)) 1_{x,y,a} \right| \\
=: S_1 + S_2 &
\end{aligned}
$$

Additive Uncertainty Principle: Improved

By applying Cauchy-Schwarz and Hölder's inequalities as well as exploiting the properties of $\chi(x)$, we get the following inequalities

$$S_1 \leq N^{-2d}|E|^2|\Sigma|^3\sqrt{\frac{N^d}{|E||\Sigma|}}\sqrt{\frac{\Lambda_2(E)}{|E|^3}}\left(\sum_m |\widehat{f}(m)|^4\right)$$

$$S_2 \leq N^{-2d}(\Lambda_2(E) - |E|^2)|\Sigma|^3\left(\sum_{m\in\Sigma} |\widehat{f}(m)|^4\right) \quad \text{or}$$

$$\leq N^{-2d}\frac{\sqrt{B_\Sigma|E|(\Lambda_2(E) - |E|^2)}}{|\Sigma|}|\Sigma|^3\left(\sum_{m\in\Sigma} |\widehat{f}(m)|^4\right),$$

where $B_\Sigma = |\Sigma - \Sigma||(\Sigma + \Sigma) - (\Sigma + \Sigma)|$.

Additive Uncertainty Principle: Improved

We are getting the statement of our new theorem by estimating the original $N^{3d} \cdot ||f||_{U_2}^4$ from below:

$$\sum_{x,y,a \in \mathbb{Z}_N^d} |f(x)f(y)f(x+a)f(y+a)| \cdot 1_{x,y,a} \geq N^d \sum_m |\widehat{f}(m)|^4$$

Gabor Transform

What happens if we send a signal multiple times?

Theorem (Burstein et al '25 [BIMN25])

Let $f : \mathbb{Z}_N \to \mathbb{C}$, and suppose that f is transmitted but the values $\{f(x)\}_{x \in M}$ are unobserved. Suppose that

$$\left\| \widehat{f} \right\|_{L^1(S^c)} \leq \frac{\varepsilon}{N} \left\| \widehat{f} \right\|_{L^1(\mathbb{Z}_N)} . \tag{3}$$

Also suppose that $2|M||S| < N$. Let $g = \text{argmin}_u \left\| \widehat{u} \right\|_{L^1(\mathbb{Z}_N)}$ with the constraint $f(x) = u(x)$ for $x \notin M$. Let $h = f - g$. Then,

$$\frac{1}{|M|} \sum_{x \in M} |h(x)| \leq \frac{2\varepsilon}{N - 2|M||S|} \sum_{x \in \mathbb{Z}_N} |f(x)|. \tag{4}$$

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$$\|\widehat{f}\|_{L^1(S^c)} \leq \frac{\varepsilon}{N} \|\widehat{f}\|_{L^1(\mathbb{Z}_N)}. \tag{5}$$

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$$\frac{1}{|M|} \sum_{x \in M} |h(x)| \leq \frac{2\varepsilon}{N - 2|M||S|} \sum_{x \in \mathbb{Z}_N} |f(x)|. \tag{6}$$

- Now: use the idea of L^1-concentration to consider a signal sent multiple times with some noise that makes each iteration of it slightly different.

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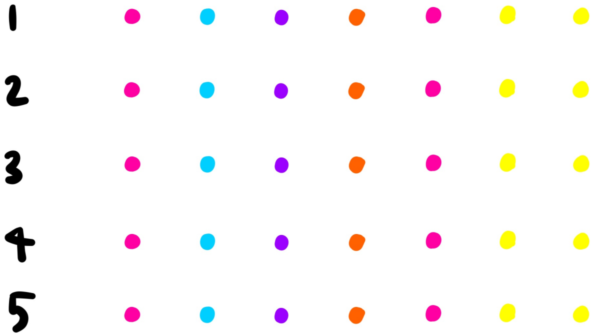

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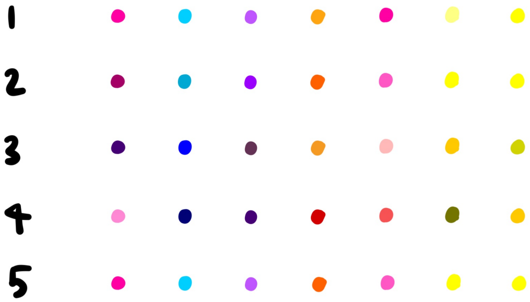

What happens if we send a signal multiple times?

Definition (Gabor Transform)

Given a function $f : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$, we define its Gabor Transform $Gf : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$ by

$$Gf(m, a) := N^{-1/2} \sum_{t \in \mathbb{Z}_N} f(t, a) e^{-2\pi i m \cdot t},$$

i.e., $Gf(m, a) := \widehat{f(—, a)}(m)$.
We thus have the inverse Gabor transform given by

$$f(t, a) = N^{-1/2} \sum_{m \in \mathbb{Z}_N} Gf(m, a) e^{2\pi i t \cdot m}$$

What happens if we send a signal multiple times?

Theorem (SMALL 2025; set-up)

Suppose $f : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$, *we transmit* f, *the values in* $M \times \mathbb{Z}_T$ *are unobserved,* Gf *has support in* $S \times \mathbb{Z}_T$, $2|M||S| < N$, *and for all* $t \notin M$,

$$\sum_{c,b,\in A^c} |f(t,c) - f(t,b)| \leq \frac{\varepsilon}{T^2} \sum_{c,b\in\mathbb{Z}_T} |f(t,c) - f(t,b)| \qquad (7)$$

where $A \subseteq \mathbb{Z}_T$ *and* $0 < \varepsilon < 1$.

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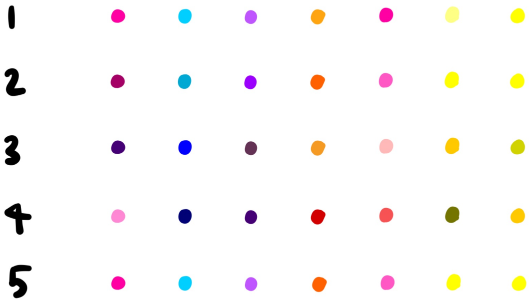

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$$\sum_{c,b,\in A^c} |f(t,c) - f(t,b)| \leq \frac{\varepsilon}{T^2} \sum_{c,b\in\mathbb{Z}_T} |f(t,c) - f(t,b)| \tag{8}$$

where $A \subseteq \mathbb{Z}_T$ *and* $0 < \varepsilon < 1$.

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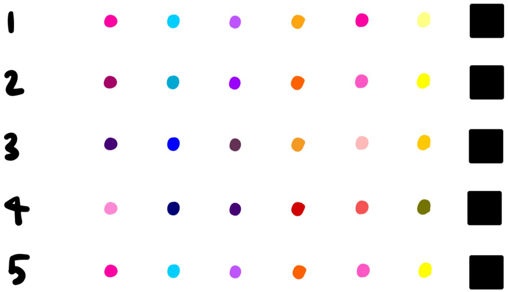

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$$\sum_{c,b,\in A^c} |f(t,c) - f(t,b)| \leq \frac{\varepsilon}{T^2} \sum_{c,b \in \mathbb{Z}_T} |f(t,c) - f(t,b)| \qquad (9)$$

where $A \subseteq \mathbb{Z}_T$ *and* $0 < \varepsilon < 1$.

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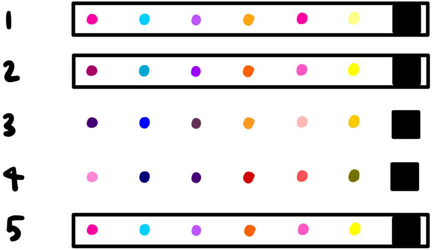

$$\sum_{c,b,\in A^c} |f(t,c) - f(t,b)| \leq \frac{\varepsilon}{T^2} \sum_{c,b\in\mathbb{Z}_T} |f(t,c) - f(t,b)| \qquad (10)$$

What happens if we send a signal multiple times?

Theorem (SMALL 2025; continued)

Define $g := \mathrm{argmin}_u \left\|\widehat{u}\right\|_{L^1(\mathbb{Z}_N)}$, *with the constraint* $u(t) = f(t,a)$ *for* $t \notin M$, *where* $a \in A^c$ *minimizes*

$$\sum_{t\in M^c}\sum_{b\in A^c}|f(t,a)-f(t,b)| \tag{11}$$

and let

$$h(t) := \left(\frac{1}{|A^c|}\sum_{b\in A^c} f(t,b)\right) - g(t). \tag{12}$$

Then,

$$\frac{1}{|M|}\|h\|_{L^1(M)} \leq \frac{3\varepsilon|S|}{|A^c|^2 T^2(N-2|M||S|)}\sum_{t\in M^c}\sum_{b,c\in\mathbb{Z}_t}|f(t,b)-f(t,c)|. \tag{13}$$

Future Works

Closing

Future Works

- Not requiring exact equality in L^1-minimization to prevent overfitting.
- Dealing more specifically with general missing value sets.
- Other ways of partitioning $\mathbb{Z}_N \times \mathbb{Z}_T$.

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SMALL 2025: Additive Energy Uncertainty Principle

Assume, $|\Sigma| \leq N^{d/3}$ and $|\Sigma| \leq (|E| - 1)^{1/4}$, then if we compare

$$\frac{\sqrt{B_\Sigma |E|(\Lambda_2(E) - |E|^2)}}{|\Sigma|} \quad \text{and} \quad \Lambda_2(E) - |E|^2$$

It is the same as comparing

$$B_\Sigma |E| \quad \text{and } |\Sigma|^2(\Lambda_2(E) - |E|^2)$$

We know that $B_\Sigma = |\Sigma - \Sigma||(\Sigma + \Sigma) - (\Sigma + \Sigma)| \leq |\Sigma|^6$. We also know that $\Lambda_2(E) - |E|^2 \geq |E|^2 - |E|$, so

$$B_\Sigma |E| \leq |\Sigma|^6 |E| \leq |\Sigma|^2 |E||\Sigma|^4 \leq |\Sigma|^2(\Lambda_2(E) - |E|^2)$$

Hence, the second inequality is stronger than the first one.