Spectral Properties of Randomly Thinned d-Regular Graphs

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Motivation

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- Motivation: Can number theory operations (convolving L-functions) be modeled in classical RMT?
- Hadamard Product (studying d-regular graphs, ‘cousin’ for checkerboard matrices: Virginia Tech).

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- Eigenvalues encode global structure (expansion, randomness, mixing).
- For many random ensembles, eigenvalue distributions have universal limits.
- Regular graphs have known limiting law (McKay).
- What happens if we randomly delete edges?

The McKay Ensemble

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- Aside from trivial eigenvalue d, remaining eigenvalues admit a limiting distribution as $n \to \infty$.

The McKay Law

McKay: excluding trivial eigenvalue d, the empirical spectral distribution converges to

$$f(x) \ := \ \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)} \quad \text{for } |x| \leq 2\sqrt{d-1}.$$

- Regular-graph analogue of semicircle law. Can create a new random matrix ensemble, similar to the ensemble of checkerboard matrices.

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- Question: What is the limiting eigenvalue distribution of H?

Empirical Spectral Distribution (ESD)

Let $H \in \mathbb{R}^{n\times n}$ be symmetric, eigenvalues $\lambda_1, \ldots, \lambda_n$:

$$\mu_H \;:=\; \frac{1}{n}\sum_{i=1}^{n}\delta_{\lambda_i}.$$

- μ_H encodes the global eigenvalue distribution of H.

Moment Method

Define the k-th moment of the ESD by

$$m_k(H) \;:=\; \int x^k \, d\mu_H(x) \;:=\; \frac{1}{n}\sum_{i=1}^{n} \lambda_i^k \;:=\; \frac{1}{n}\,\mathrm{Tr}\Big(H^k\Big).$$

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- Moments often determine the limiting distribution.
- From Graph Theory: $\operatorname{Tr}(A^k)$ counts closed walks of length k in the graph of A.

Low Moments: Already Different from McKay

$$m_1(H) = \frac{1}{n}\mathbb{E}[\mathrm{Tr}(H)] = 0 \qquad \text{(no self-loops).}$$
$$m_2(H) = \frac{1}{n}\mathbb{E}[\mathrm{Tr}(H^2)] = dp.$$

Compare with McKay:

$$m_2^{\mathrm{McKay}} = d.$$

Conclusion

For any fixed $p \in (0,1)$, the Hadamard-thinned ensemble is not McKay.

Difference From McKay (Formal Statement)

Theorem (Difference From McKay)

For any fixed $p \in (0,1)$, the empirical spectral distribution μ_H of the Hadamard-thinned ensemble does not converge to the McKay distribution or a scaled McKay distribution.

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- Reason: second moments differ ($dp \neq d$), so limiting laws cannot agree.
- Fourth moments $pd + 2p^2 d(d-1)$ versus $2d^2 - d$; if rescale so second moments equal, scaled fourth differ.

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- Thinning produces a subgraph, can compare closed-walk counts.

Subgraph Argument (Squeeze)

Obtain H by deleting edges of A; any closed walk in H is a closed walk in A. Thus, for odd k,

$$0 \le \mathrm{Tr}\Big(H^k\Big) \le \mathrm{Tr}\Big(A^k\Big).$$

Taking expectations, dividing by n, and letting $n \to \infty$ gives

$$0 \le \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[\mathrm{Tr}\Big(H^k\Big)] \le \lim_{n\to\infty} \frac{1}{n}\mathbb{E}[\mathrm{Tr}\Big(A^k\Big)] = 0.$$

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Corollary (Asymptotic Symmetry)

The limiting spectral measure of the Hadamard ensemble is symmetric about 0 for all $p \in (0,1)$.

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- A walk using r distinct edges survives with probability p^r.
- Asymptotically, dominant contribution from tree-like closed walks (as in McKay's analysis).

Even Moments of the Hadamard Ensemble

Let $A_{k,r}$ denote the number of closed walks of length $2k$ using exactly r distinct edges (tree-like walks dominate asymptotically). Then for $k \geq 1$,

$$m_{2k}(H) \; = \; \sum_{r=1}^{k} A_{k,r} \, p^r \, d(d-1)^{\, r-1}.$$

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- Setting $p = 1$ recovers the McKay even moments.
- Yields explicit two-parameter family of limiting measures $\mu_{d,p}$.

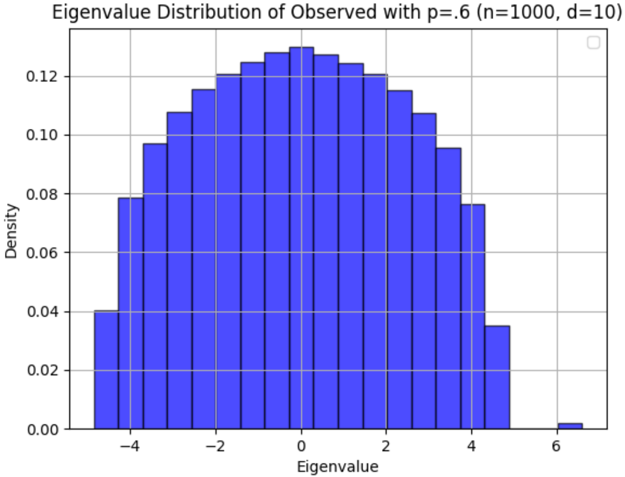
Convergence of the ESD

Theorem (ESD Converges to $\mu_{d,p}$)

As $n \to \infty$, the empirical spectral distribution of the Hadamard ensemble converges (in distribution) to a limiting measure $\mu_{d,p}$ whose moments are given by the above formulas.

- Moment convergence + determinacy $\Rightarrow$ weak convergence.

Shape of the ESD



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- Odd moments vanish $\Rightarrow$ asymptotic symmetry about 0.
- Even moments are an explicit polynomial in p and d.
- This defines a new family of limiting distributions $\mu_{d,p}$.

References

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- T. Hasegawa and S. Saito, "A note on the moments of the Kesten distribution," *Discrete Mathematics*, vol. 344, no. 10, p. 112524, 2021. doi:10.1016/j.disc.2021.112524.

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Thank You.