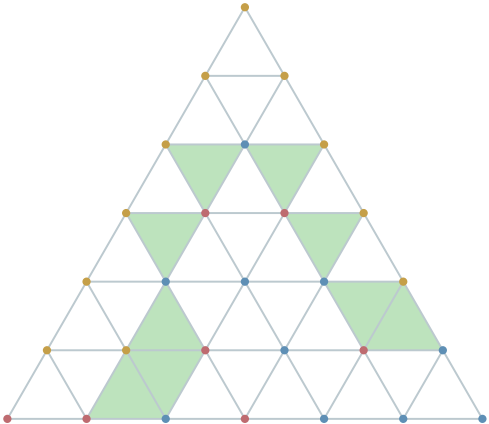
Sperner Scoring Game

Scores, Strategies,
and Complexity

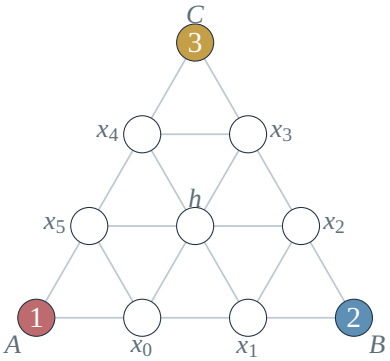
Simon Harnik
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Advised by Steven J. Miller
Williams College
sjm1@williams.edu



Let’s Play a Coloring Game

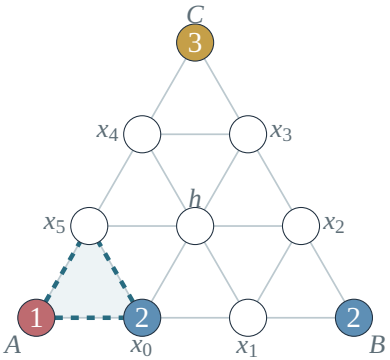
Guided audience participation



Don’t let a fully-colored triangle form!

Let’s Play a Coloring Game

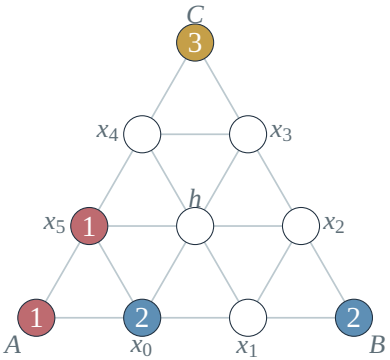
Guided audience participation



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Let’s Play a Coloring Game

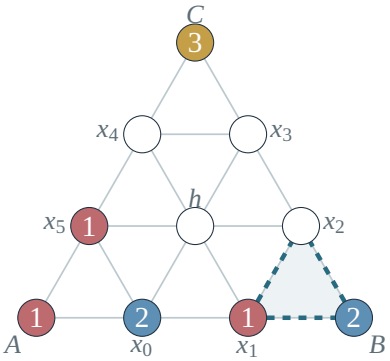
Guided audience participation



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Let’s Play a Coloring Game

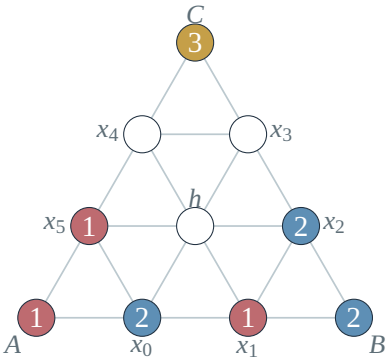
Guided audience participation



Don’t let a fully-colored triangle form!

Let’s Play a Coloring Game

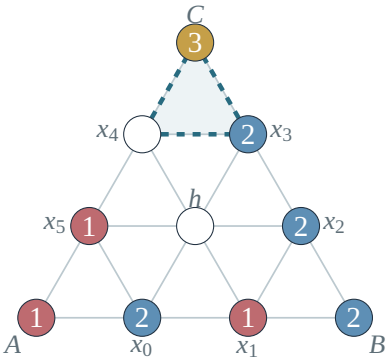
Guided audience participation



Don’t let a fully-colored triangle form!

Let’s Play a Coloring Game

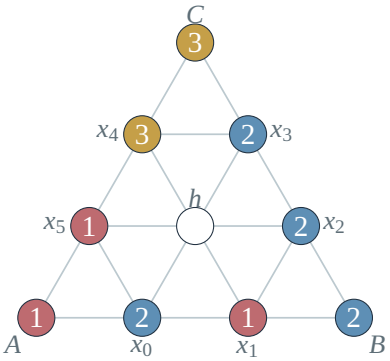
Guided audience participation



Don’t let a fully-colored triangle form!

Let’s Play a Coloring Game

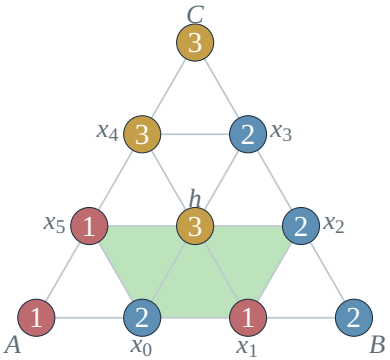
Guided audience participation



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Let’s Play a Coloring Game

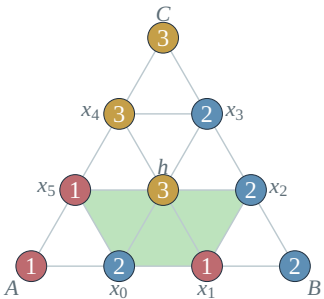
Guided audience participation



Don’t let a fully-colored triangle form!

Sperner guarantees a fully labeled triangle

Existence, parity, and applications



Every completed legal Sperner labeling contains an **odd** number of fully labeled triangles.

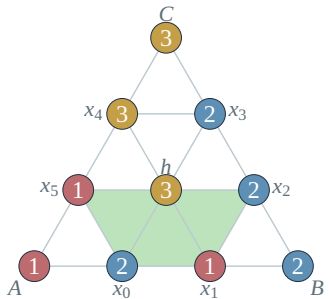
Count incidences with 1–2 edges.

Interior edges count twice. The 1–2 side contributes an odd boundary count.

Sperner (1928); Su (1999); Daskalakis and Farina (2026)

Sperner guarantees a fully labeled triangle

Existence, parity, and applications



The Sperner Scoring Game

FC maximizes the number of fully labeled triangles.

NFC minimizes that count.

Alternate turns: any blank vertex and any legal label.

If we cannot prevent fully labeled triangles entirely, how many can each player force?

Sperner (1928); Su (1999); Daskalakis and Farina (2026)

Sperner guarantees a fully labeled triangle

Existence, parity, and applications

Brouwer fixed-point theorem

A continuous map of a simplex into itself has a fixed point.

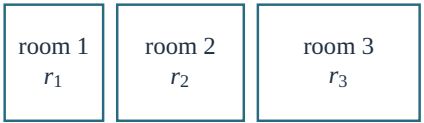
$$f : \Delta^d \longrightarrow \Delta^d, \qquad \exists x \in \Delta^d : \; f(x) = x.$$

Label finer meshes using coordinate displacement.

Sperner gives shrinking fully labeled cells; continuity gives a fixed point at a limit.

Sperner guarantees a fully labeled triangle

Existence, parity, and applications



three different bedrooms
one total rent: $r_1 + r_2 + r_3 = R$

Rental harmony — Su (1999)

Choose room prices and assign each roommate a distinct favorite room.

Hypotheses: an acceptable room at every price vector, closed preference sets, and preference for free rooms.

A Sperner-type rent-price labeling gives existence.

Optimal play on a line

Initial path: opposite fixed endpoints; all internal vertices blank



FC: opposite label



NFC: matching label

FC plays opposite to a colored neighbor, securing a differing segment.

NFC matches a colored neighbor, securing a matching segment.

V: FC's optimal score in differing segments; t: FC's turns.

$$t \leq V \leq t + 1.$$

Optimal play on a line

Initial path: opposite fixed endpoints; all internal vertices blank



FC: opposite label



NFC: matching label

FC plays opposite to a colored neighbor, securing a differing segment.

NFC matches a colored neighbor, securing a matching segment.

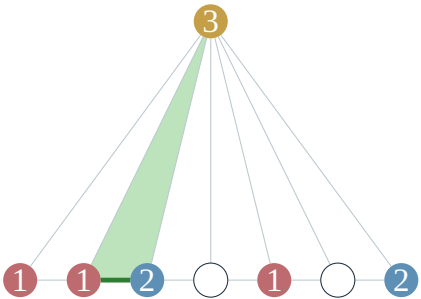
V: FC's optimal score in differing segments; t: FC's turns.

$$t \leq V \leq t + 1.$$

Opposite endpoints force oddness, so $\boxed{V = 2\left\lfloor \frac{t}{2} \right\rfloor + 1}$.

Fixed-apex fans are equivalent to 1D paths

Exact move-and-score correspondence

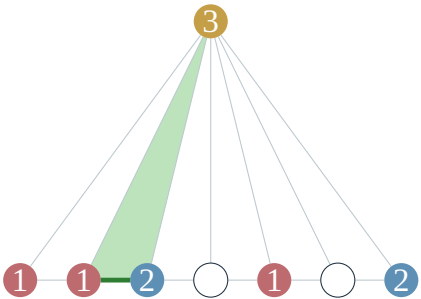


The apex is already fixed to 3.

Base vertices keep labels $\{1, 2\}$ and their freely chosen move order.

Fixed-apex fans are equivalent to 1D paths

Exact move-and-score correspondence



The apex is already fixed to 3.

Base vertices keep labels $\{1, 2\}$ and their freely chosen move order.

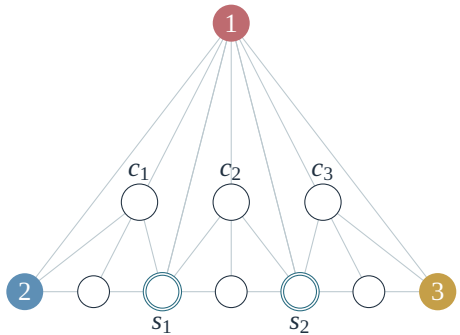
Different base labels
$\Longleftrightarrow$ fully labeled triangle.

Same legal moves. Same score.

The correspondence also holds for partial base positions.

Fans with shared vertices

Shared separators



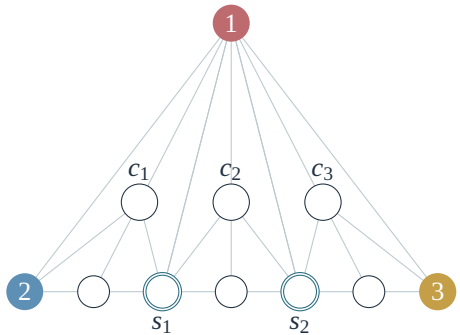
Center labels: 1, 2, and 3.

Base and shared separators: labels 2, 3.

A separator move affects both neighboring sectors.

Fans with shared vertices

Shared separators



Center labels: 1, 2, and 3.

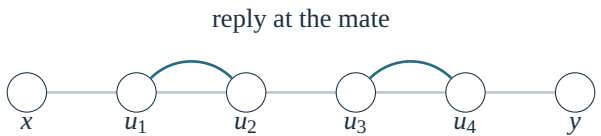
Base and shared separators: labels 2, 3.

A separator move affects both neighboring sectors.

A center labeled 1 selects its base transitions; centers 2 or 3 contribute only separator terms.

Four blank vertices can be compressed

Contextual replacement



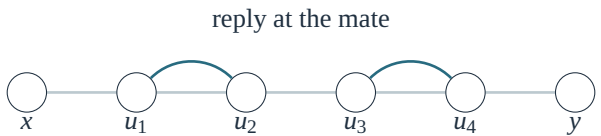
$$T = \mathrm{tr}(x, u_1, u_2, u_3, u_4, y) \quad \rightsquigarrow \quad 2 + \mathbf{1}\{x \neq y\}.$$

Blank binary path block; no exterior dependencies.

Independent fixed-length exterior; payoff monotone in T.

Four blank vertices can be compressed

Contextual replacement



$$T = \mathrm{tr}(x, u_1, u_2, u_3, u_4, y) \quad \rightsquigarrow \quad 2 + \mathbf{1}\{x \neq y\}.$$

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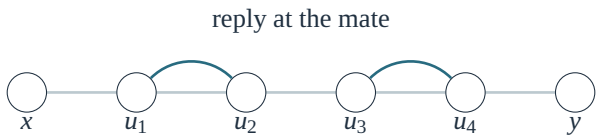
Independent fixed-length exterior; payoff monotone in T.

FC replies opposite; NFC replies matching.

Two moves restore the exterior turn.

Four blank vertices can be compressed

Contextual replacement



$$T = \mathrm{tr}(x, u_1, u_2, u_3, u_4, y) \quad \rightsquigarrow \quad 2 + \mathbf{1}\{x \neq y\}.$$

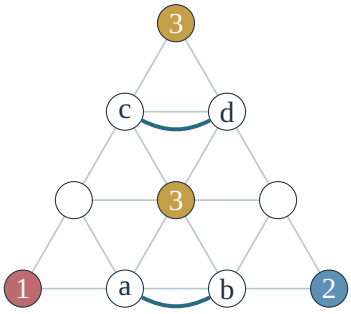
Original k-sector fans retain at most $5k - 1$ moves.

The +2 applies only when the center selects this path.

Paired replies preserve value for either next role. A game replacement, not a smaller triangulation.

Optimal play on the opening board

Corner-only K_3; either next role



blue arcs: response pairs

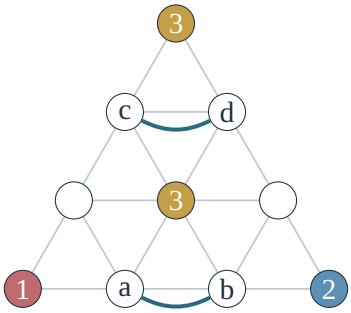
Fixing the center leaves two disjoint boundary threats.

NFC answers at the paired vertex to block each threat.

Shown center 3: block $(a, b) = (2, 1)$ and $(c, d) = (1, 2)$.

Optimal play on the opening board

Corner-only K_3; either next role



blue arcs: response pairs

Let q count the two unblocked threats.

$$X = 1 + 2q.$$

NFC can force $q = 0$; Sperner gives $X \geq 1$.

$$\boxed{V_F(K_3) = V_N(K_3) = 1}.$$

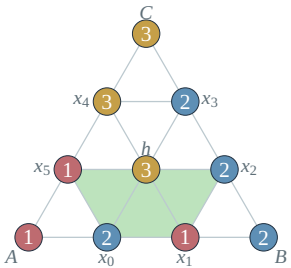
V_F: FC next; V_N: NFC next.

If FC starts on a side, NFC chooses a center label that makes that first move irrelevant to both threats.

Sperner (1928)

Random play on K_3

Independent uniform legal labeling



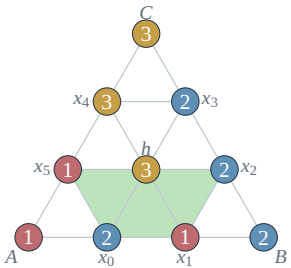
Corners fixed. Sides fair binary.
Interior fair ternary. All independent.

X = number of fully labeled triangles.

x	1	3	5
$\Pr(X = x)$	$9/16$	$6/16$	$1/16$

Random play on K_3

Independent uniform legal labeling



Corners fixed. Sides fair binary.
Interior fair ternary. All independent.

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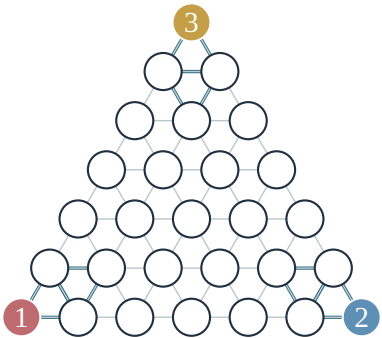
x	1	3	5
$\Pr(X = x)$	$9/16$	$6/16$	$1/16$

$$\mathbb{E}X = 2, \qquad \operatorname{Var} X = \frac{3}{2}.$$

The two boundary threats each occur with probability $1/4$; $X = 1 + 2Y$, $Y \sim \operatorname{Bin}(2, 1/4)$.

Random play on K_m

Independent uniform legal labeling



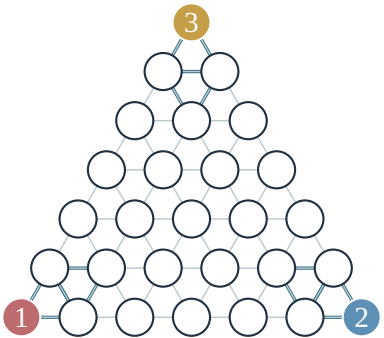
m subdivisions per side; m^2 triangles.

Number of triangles	Fully labeled probability
6 exceptional	$1/4$
$3(m-2)$	$1/6$
$m(m-3)$	$2/9$

$$\mathbb{E}X = \frac{4m^2 - 3m + 9}{18} \quad (m \geq 3).$$

Random play on K_m

Independent uniform legal labeling



m subdivisions per side; m^2 triangles.

Number of triangles	Fully labeled probability
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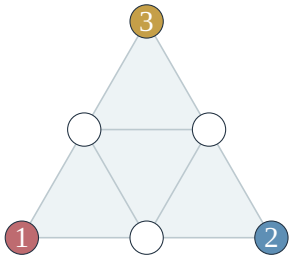
$$\mathbb{E}X = \frac{4m^2 - 3m + 9}{18} \quad (m \geq 3).$$

$$\frac{\mathbb{E}X}{m^2} \longrightarrow \frac{2}{9}.$$

Limit of $\mathbb{E}[X]/m^2$, not game value.

Random play on K_m

Independent uniform legal labeling



K_2: four triangles.
Corners fixed; side labels independent and fair.

Score invariance: every legal completion has the same count.

Corner triangles: one successful assignment out of four. Central triangle: two out of eight.

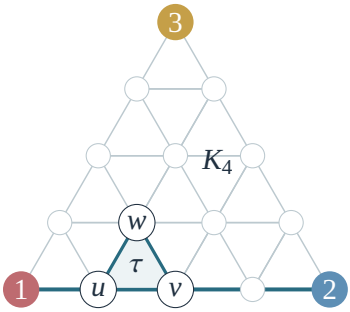
$$\mathbb{E}[X] = 4 \cdot \frac{1}{4} = 1, \qquad X \geq 1 \text{ by Sperner.}$$

Therefore $X = 1$ in every completion.

K_1 is the single fully labeled triangle.

Random play: expectation and variance

Any legal partial 2D position; independent uniform legal labeling



u, v: on outer side 1–2
w: interior

$A(v)$: legal labels at v; a prescribed label gives a singleton.

u, v lie on side 1–2; w is interior.

$$I_\tau = \mathbf{1}_{\{\tau \text{ fully labeled}\}} \in \{0, 1\}.$$

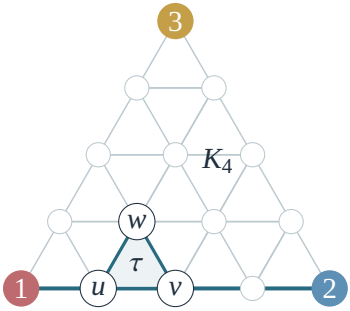
The indicator is 1 when the event holds and 0 otherwise.

$$X = \sum_\tau I_\tau.$$

X: number of fully labeled triangles.

Random play: expectation and variance

Any legal partial 2D position; independent uniform legal labeling



u, v: on outer side 1–2
w: interior

$$\tau = \{u, v, w\};\ (v_1, v_2, v_3) = (u, v, w).$$

$$(M_\tau)_{ij} = \mathbf{1}_{\{j \in A(v_i)\}}.$$

$$M_\tau = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Rows: (u, v, w). Columns: $(1, 2, 3)$.

$A(u) = A(v) = \{1, 2\}$; $A(w) = \{1, 2, 3\}$.

Random play: expectation and variance

Any legal partial 2D position; independent uniform legal labeling

$$M_\tau = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Rows (u, v, w); columns $(1, 2, 3)$.

$$p_\tau = \Pr(I_\tau = 1) = \frac{\operatorname{per}(M_\tau)}{|A(u)|\,|A(v)|\,|A(w)|}.$$

Permanent: sum the six products choosing one entry from each row and column, without alternating signs.

$$p_\tau = \frac{2}{2 \cdot 2 \cdot 3} = \frac{1}{6}.$$

The permanent counts fully labeled legal assignments for any triangle.

Random play: expectation and variance

Any legal partial 2D position; independent uniform legal labeling

X = number of fully labeled triangles; I_τ is its 0–1 triangle indicator.

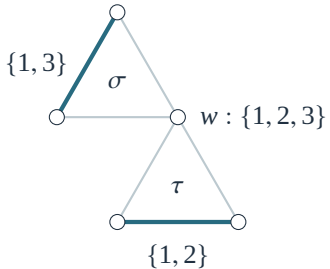
$$X = \sum_\tau I_\tau, \qquad \mathbb{E}[I_\tau] = p_\tau.$$

$$\boxed{\mathbb{E}[X] = \sum_\tau p_\tau}.$$

Linearity needs no independence among the triangle indicators.

Random play: expectation and variance

Any legal partial 2D position; independent uniform legal labeling



one shared vertex
joint probability 0

$$p_{\tau\sigma} = \Pr(I_\tau = 1 \text{ and } I_\sigma = 1).$$

Count legal assignments on the union of both triangles' vertices.

$$p_\tau = p_\sigma = \frac{1}{6}, \quad p_{\tau\sigma} = 0.$$

$$\mathrm{Cov}(I_\tau, I_\sigma) = -\frac{1}{36}.$$

One shared vertex suffices. Disjoint vertex sets give independent indicators.

Random play: expectation and variance

Any legal partial 2D position; independent uniform legal labeling

$$\mathbb{E}[X] = \sum_{\tau} p_\tau,$$

$$\mathrm{Var}(X) = \sum_{\tau} p_\tau(1 - p_\tau) + 2 \sum_{\substack{\tau<\sigma \\ \tau\cap\sigma\neq\varnothing}} (p_{\tau\sigma} - p_\tau p_\sigma).$$

$\tau < \sigma$: count each unordered pair of distinct triangles once.

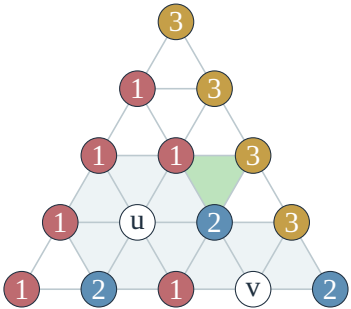
$p_\tau = \Pr(I_\tau = 1)$; $p_{\tau\sigma} = \Pr(I_\tau = 1 \text{ and } I_\sigma = 1)$.

Include pairs sharing one vertex as well as a full edge.

Independent vertices: largest swing first

Exact strategy; supplied legal partial positions

Critical: at most one blank per triangle. $\lambda(v)$: the label at v.



The hollow vertices u, v share no triangle.

Each unresolved triangle depends on just one remaining label.

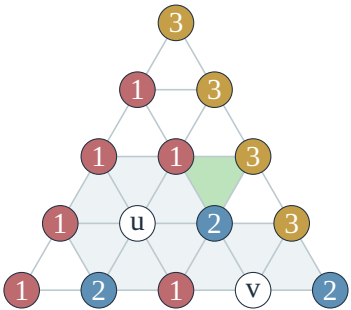
One already completed triangle contributes the constant 1.

$$X = 1 + 4\mathbf{1}_{\{\lambda(u)=3\}} + 2\mathbf{1}_{\{\lambda(v)=1\}}.$$

Independent vertices: largest swing first

Exact strategy; supplied legal partial positions

Critical: at most one blank per triangle. $\lambda(v)$: the label at v.



Swing: largest minus smallest local contribution.

u: contributions $0, 0, 4$; swing 4.

v: contributions $2, 0$; swing 2.

Completion maximum: 7.

Independent vertices: largest swing first

Exact strategy; supplied legal partial positions

Critical: at most one blank per triangle. $\lambda(v)$: the label at v.

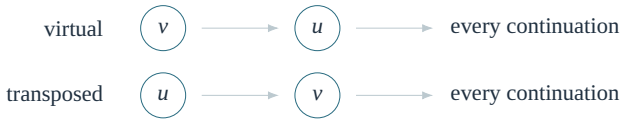
FC next: $V_F = 5$.

FC takes u with label 3; NFC takes v with label 2.

NFC next: $V_N = 3$.

NFC takes u with label 1 or 2; FC takes v with label 1.

Both roles take a largest remaining swing, using their own best label.

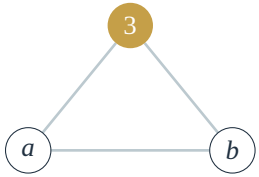


Transpose the two choices throughout every continuation.

Taking a largest remaining swing cannot worsen the mover's guaranteed score.

A shared triangle couples two choices

Group terms by their uncolored vertices



two blank vertices
one fixed label

With one blank per triangle, contributions separate.

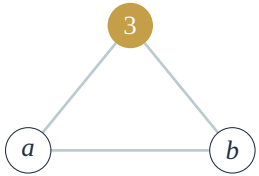
Blank vertices *a* and *b* share a triangle.

Legal partial mesh ⟶ triangle indicators ⟶ payoff interactions.

Physical edges show possible interactions; terms may cancel. Three-vertex terms give a hypergraph.

A shared triangle couples two choices

Group terms by their uncolored vertices



two blank vertices
one fixed label

Triangle indicator:

$$I_\tau = \mathbf{1}_{\{\lambda(a)=1,\lambda(b)=2\}} + \mathbf{1}_{\{\lambda(a)=2,\lambda(b)=1\}}.$$

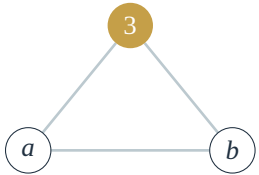
The score effect at a depends on b.

Legal partial mesh ⟶ triangle indicators ⟶ payoff interactions.

Physical edges show possible interactions; terms may cancel. Three-vertex terms give a hypergraph.

A shared triangle couples two choices

Group terms by their uncolored vertices



two blank vertices
one fixed label

$\lambda(a)\backslash\lambda(b)$	1	2
1	0	1
2	1	0

The payoff does not separate into a and b contributions.

Legal partial mesh ⟶ triangle indicators ⟶ payoff interactions.

Physical edges show possible interactions; terms may cancel. Three-vertex terms give a hypergraph.

Completion vs. guaranteed score

Existential and alternating quantifiers

Common input: explicit triangulation, legal partial labeling, and threshold k.

SSG-COMPLETION

Does any legal completion
reach at least k?

$\exists$ labeling.

Reduction source:

Planar Independent Set.

Each move chooses both a vertex and a legal label. Existence $\neq$ forced outcome.

Completion vs. guaranteed score

Existential and alternating quantifiers

Common input: explicit triangulation, legal partial labeling, and threshold k.

SSG-COMPLETION

Does any legal completion reach at least k?

$$\exists \text{ labeling.}$$

Reduction source:

Planar Independent Set.

SSG-VALUE

Can FC force at least k against NFC?

$$\exists m_1 \; \forall m_2 \; \exists m_3 \cdots : \; X \geq k$$

Next role supplied; FC next shown.

Reduction source: quantified Boolean formulas.

Each move chooses both a vertex and a legal label. Existence $\neq$ forced outcome.

Completion vs. guaranteed score

Existential and alternating quantifiers

Common input: explicit triangulation, legal partial labeling, and threshold k.

SSG-COMPLETION

Does any legal completion reach at least k?

$$\exists \text{ labeling.}$$

Reduction source:

Planar Independent Set.

NP membership:
polynomial-time completion check.

SSG-VALUE

Can FC force at least k against NFC?

$$\exists m_1 \; \forall m_2 \; \exists m_3 \cdots : \; X \geq k$$

Next role supplied; FC next shown.

Reduction source: quantified Boolean formulas.

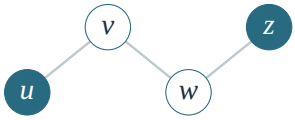
PSPACE membership:
depth-limited minimax, polynomial space.

Each move chooses both a vertex and a legal label. Existence ≠ forced outcome.

Encoding graph selections with legal triangles

SSG-COMPLETION; supplied legal partial positions

Planar Independent Set: select at least q_0 vertices with no selected adjacent pair.



Encode graph vertices as blank SSG vertices.

Label 2: selected.

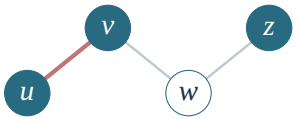
Label 1: not selected.

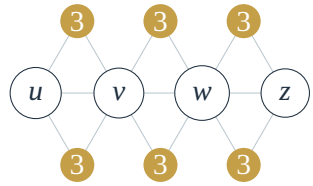
Source problem: Garey, Johnson, and Stockmeyer (1976)

Encoding graph selections with legal triangles

SSG-COMPLETION; supplied legal partial positions

Graph choice ⟶ SSG label ⟶ score



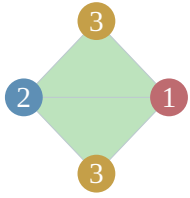


Fixed-3 pins form local patches. All three labels remain legal.

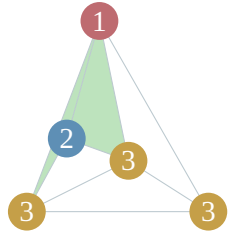
Source problem: Garey, Johnson, and Stockmeyer (1976)

Encoding graph selections with legal triangles

SSG-COMPLETION; supplied legal partial positions



edge disagreement
2 fully labeled triangles



label-1 unary unit
2 fully labeled triangles

Edge units reward unequal labels. Unary units cancel degree-dependent vertex rewards.

For completion maximization, label 3 can be replaced by 1 without reducing the score.

Source problem: Garey, Johnson, and Stockmeyer (1976)

Degree-independent rewards for selected vertices

Completion reduction: degree cancellation

Make the reward independent of a graph vertex's degree.

Incident edge units $+2\deg_G(v)$ per selected v,

$\deg_G(v) - 1$ unary units $-2(\deg_G(v) - 1)$ per selected v.

$$2\deg_G(v) - 2(\deg_G(v) - 1) = 2.$$

Remove isolated source vertices first, adjusting the source target.

Degree-independent rewards for selected vertices

Completion reduction: degree cancellation

S: selected graph vertices; $e_G(S)$: edges with both ends selected.

Fully labeled triangle count:

$$\boxed{X = B + 2\big(|S| - 2e_G(S)\big)}.$$

B: fixed score independent of S, including the outer frame.

Reward: $+2$ per selected vertex.

Penalty: -4 per selected edge.

Fixed subdivisions remove unwanted interactions; fixed filler triangles contribute only to B.

Removing a selected endpoint improves the score



$|S| = 3$, $e_G(S) = 1$; $X = 7$. $|S| = 2$, $e_G(S) = 0$; $X = 9$.

Remove a selected endpoint: lose one selected vertex, eliminate at least one selected edge.

The score increases by at least two.

The same example baseline is $B = 5$.

Removing a selected endpoint improves the score

NP-completeness; supplied legal partial positions

An optimal selection is independent.

Let $\alpha(G)$ be the largest independent-set size.

$$\max X = B + 2\alpha(G), \qquad k = B + 2q_0.$$

$$\exists \text{ legal completion with } X \geq k \iff \alpha(G) \geq q_0.$$

Removing a selected endpoint improves the score

NP-completeness; supplied legal partial positions

SSG-COMPLETION is NP-complete.

Polynomial-size, unweighted Sperner triangulation.

NP membership: check legal labels and count fully labeled triangles.

Hardness is for supplied legal partial positions with fixed interior labels.

Priority exchange and quantified choices

PSPACE proof outline; unrestricted vertex choice

Start with a planar quantified Boolean formula: alternating truth choices, followed by a clause check.



Ownership is 1 when FC takes a logical item and 0 when NFC takes it.

The controller chooses a bit by taking the logical item or its companion.

Logical items: v, w; companions: v', w'; parity move: d.

Source problem: Lichtenstein (1982)

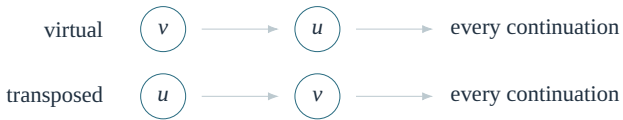
Priority exchange and quantified choices

PSPACE proof outline; unrestricted vertex choice

Let w_u, w_v be the items' priority bonuses. Their gap must cover the remaining verifier's range.

$$w_u - w_v \geq B_R, \qquad B_R = \max R - \min R.$$

u: higher-priority item; v: lower-priority item. R: remaining verifier payoff.



Transpose item names throughout the whole continuation tree, including the opponent's moves.

Equal final owners: unchanged. Different owners: the priority gain covers the change in R.

Source problem: Lichtenstein (1982)

Priority exchange and quantified choices

PSPACE proof outline; unrestricted vertex choice

Optimal priority-ordered strategies exist.

Every unclaimed item remains selectable.

Two-item groups preserve the next role; a singleton changes it.

Quantified choices: assigned controller. Verification choices: FC.

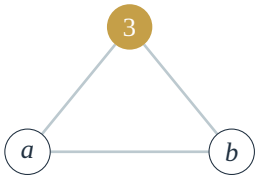
Priority exchange preserves the unrestricted game value.

Source problem: Lichtenstein (1982)

SSG-VALUE is PSPACE-complete

Theorem: supplied legal 2D partial positions

Quantified choices ⟶ verifier payoff ⟶ unweighted Sperner triangulation.



two blank vertices
one fixed label

Fixed-label fans realize unary rewards for vertex ownership.

All three labels stay legal.

The triangulation has polynomial size and coordinate bit length.

Hardness holds with interior blanks, at most two blanks per triangle, and maximum degree 3 in the graph induced by blanks. The full triangulation may have large degree.

Source problem: Lichtenstein (1982)

SSG-VALUE is PSPACE-complete

Theorem: supplied legal 2D partial positions

SSG-VALUE is PSPACE-complete.

The reduction computes a threshold k.

Source formula true FC can force k,

Source formula false FC's value is at most $k - 4$.

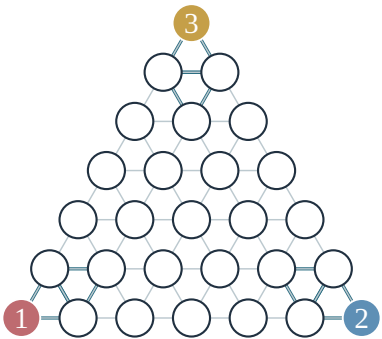
Polynomial-depth minimax uses polynomial space.

Hardness holds with interior blanks, at most two blanks per triangle, and maximum degree 3 in the graph induced by blanks. The full triangulation may have large degree.

Source problem: Lichtenstein (1982)

Optimal play from corner-only boards

Scope and open problems



Supplied legal partial positions

One blank per triangle: sort the swings.

Two blanks already permit hard instances.

Corner-only starting boards

General K_m optimal strategies remain open.

References

External mathematical literature

Sperner, E. (1928). Neuer Beweis für die Invarianz der Dimensionszahl und des Gebietes.

Kannai, Y. (2013). Using oriented volume to prove Sperner's lemma. *Economic Theory Bulletin* 1, 11–19.

Daskalakis, C., and Farina, G. (2026). *Brouwer and Sperner*. MIT 6.7980 lecture notes.

Su, F. E. (1999). Rental Harmony: Sperner's Lemma in Fair Division. *American Mathematical Monthly* 106, 930–942.

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Garey, M. R., Johnson, D. S., and Stockmeyer, L. (1976). Some simplified NP-complete graph problems. *Theoretical Computer Science* 1, 237–267.

Lichtenstein, D. (1982). Planar formulae and their uses. *SIAM Journal on Computing* 11, 329–343.

Tutte, W. T. (1963). How to draw a graph. *Proceedings of the London Mathematical Society* 13, 743–767.

de Fraysseix, H., Pach, J., and Pollack, R. (1990). How to draw a planar graph on a grid. *Combinatorica* 10, 41–51.

Sperner Scoring Game

Definition

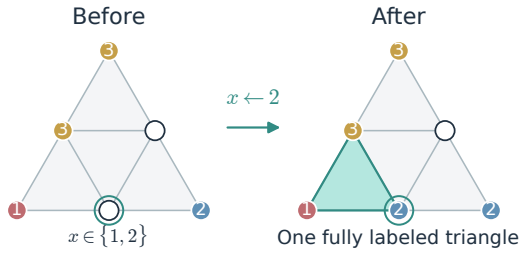
Let K triangulate Δ^d, with corner labels $1, \ldots, d+1$.

- Use labels of the smallest containing face.
- Players alternate coloring blank vertices.
- A fully labeled d-simplex uses all $d+1$ labels.

Fully labeled simplices can already be counted in partial positions. The counts N_{FC} and N_{NFC} refer to the *completed* labeling.

P_{FC} maximizes N_{FC}.
P_{NFC} maximizes $N_{\mathrm{NFC}} = |\mathcal{S}_d(K)| - N_{\mathrm{FC}}$.

In $d = 2$**:** Interior vertices use $\{1, 2, 3\}$; outer sides permit only their two corner labels.



Hollow vertices are uncolored.
The shaded triangle is fully labeled.

Corner-only: only corners fixed. **Partial:** other vertices may also be fixed.

Sperner and Fixed Points

Proof sketch

Let $f : \Delta^d \to \Delta^d$ be continuous. If a mesh vertex is already fixed, the conclusion holds. Otherwise label a vertex by a coordinate i with $x_i > f_i(x)$.

Since the coordinate sums of x and $f(x)$ agree, such a coordinate exists. A coordinate that is zero cannot be chosen, giving the Sperner boundary condition.

For meshes whose maximum cell diameter tends to zero, Sperner supplies fully labeled cells. Compactness gives a subsequence shrinking to x^*. For each coordinate i, nearby vertices satisfy $x_i > f_i(x)$; continuity gives $x_i^* \geq f_i(x^*)$. Equality of the coordinate sums forces equality in every coordinate.

$$f(x^*) = x^*.$$

This is an existence argument; it does not bound the running time of a search algorithm.

Rental Harmony: Hypotheses and Labeling

Conditional application

For n roommates and n rooms, prices satisfy $r_i \geq 0$ and $\sum_i r_i = R$. Assume:

1. At every price vector, each roommate finds a room acceptable.
2. Each roommate prefers a free room to every nonfree room.
3. Preferred-room sets are closed under limits of price vectors.

Then prices and a bijection exist giving each roommate a favorite room; ties are allowed.

Su's owner-labeled price mesh uses a dual Sperner boundary rule, rather than the ordinary boundary rule unchanged. Fully labeled cells, refinement, and closed preferences yield one limiting price vector and assignment.

The secretive-roommate extension is Frick, Houston-Edwards, and Meunier (2019), under its theorem's hypotheses; it uses preferences from only $n - 1$ housemates.

Su (1999), Sections 6–7; Frick, Houston-Edwards, and Meunier (2019)

Notation for Labels, Counts, and Values

Conventions

For a completed two-dimensional labeling λ, write

$$X(\lambda) = N_{\mathrm{FC}}(\lambda) = \sum_{\tau \in \mathcal{S}_2(K)} I_\tau(\lambda), \qquad I_\tau(\lambda) = \mathbf{1}_{\{\tau \text{ fully labeled}\}}.$$

An indicator is 0 or 1. Under the independently uniform legal-label experiment, $p_\tau = \Pr(I_\tau = 1)$ and $p_{\tau\sigma} = \Pr(I_\tau = 1 \text{ and } I_\sigma = 1)$ are probabilities. The same terminal count is a function in strategy arguments and a random variable in that experiment.

$A(v)$ is the legal-label set for an uncolored vertex, or the prescribed singleton for a colored one. For a label triple, $R(a, b, c) = \mathbf{1}_{\{\{a,b,c\}=\{1,2,3\}\}}$ is the same fully labeled test evaluated at those three labels.

V_F, V_N are optimal terminal counts with FC or NFC next. FC maximizes X; NFC minimizes X and receives $|\mathcal{S}_2(K)| - X$. In one dimension X_1 counts fully labeled line segments. Explicitly introduced auxiliary symbols in residue tables and higher-dimensional formulas have local meanings.

Sperner Parity Without Orientation

Parity proof

Let a_τ be the number of triangle edges whose endpoint labels are $1, 2$. A fully labeled triangle has $a_\tau = 1$; a triangle using only labels $1, 2$ has $a_\tau = 0$ or 2; every other triangle has $a_\tau = 0$.

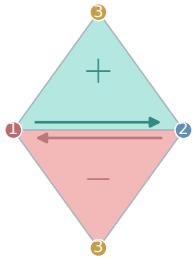
$$\#\{\text{fully labeled triangles}\} \equiv \sum_\tau a_\tau \pmod 2.$$

Every interior 1–2 edge is counted twice. Only the outer side joining corners $1, 2$ can contribute boundary 1–2 edges. Its binary word starts at 1 and ends at 2, so its number of transitions is odd. The total number of fully labeled triangles is therefore odd, and in particular positive.

This proof needs no signed orientation. The next frame gives the signed strengthening used in negative-face arguments.

Signed Sperner Identity

Identity and proof



interior edge cancels

Let N_+ count CCW fully labeled orders $123, 231, 312$, and N_- count $132, 321, 213$. Give a directed edge weight $+1$ for $1 \to 2$, -1 for $2 \to 1$, and 0 otherwise.

The sum around a counterclockwise triangle is $+1$, -1, or 0, according to its fully labeled orientation.

Every interior edge appears twice, oppositely directed. On the outer boundary, only side 1–2 contributes; its binary label word has net transition count one.

$$N_+ - N_- = 1, \qquad \boxed{X = 1 + 2N_-}.$$

This converts a local negative-face strategy into two additional fully labeled triangles, and lets us test whether a board's score can vary.

Sperner (1928); Kannai (2013)

The One-Dimensional Solution

Initial binary path: all internal vertices uncolored



FC: opposite



NFC: matching

FC: play beside a colored vertex, using the opposite label. Each FC turn secures a fully labeled line segment.

NFC: use the matching label. Each NFC turn secures a non-fully-labeled line segment.

If FC has t turns on a path with opposite fixed endpoints,

$$t \le V \le t+1, \qquad X_1 \text{ is odd}, \qquad \boxed{V = 2\left\lfloor \frac{t}{2} \right\rfloor + 1}.$$

The complete frontier guarantee, matching upper bound, and endpoint parity argument follow in Appendix B8.

Initial Binary Path: Matching Bounds

Complete frontier proof

An initial binary path has u uncolored internal vertices between opposite fixed endpoints. There are $u + 1$ line segments. If FC has t turns, NFC has $u - t$ turns.

Until all vertices are colored, a blank adjacent to a colored vertex exists. Coloring it opposite to that neighbor permanently completes a new fully labeled line segment. Thus FC can secure at least t such segments, regardless of the other moves.

NFC instead copies a colored neighbor at each frontier turn, permanently completing at least $u - t$ matching segments. Therefore the number X_1 of fully labeled segments satisfies

$$t \le V \le (u + 1) - (u - t) = t + 1.$$

Every binary word with opposite endpoints has an odd number of transitions. Exactly one integer in $[t, t + 1]$ is odd, proving

$$V = 2\lfloor t/2 \rfloor + 1.$$

Scope: the initial path. The next page gives a partial-position counterexample.

Initial Greedy Play and Partial-Path Scope

Counterexample and theorem summary

The frontier proof makes every favorable-edge greedy strategy optimal from the initial path with opposite fixed endpoints and all internal vertices uncolored. It does not make every such move optimal in every partial position.

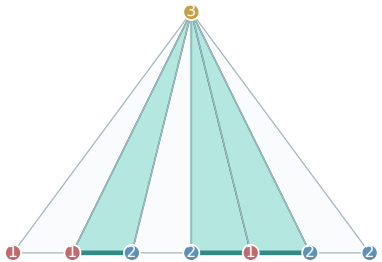
$$(1, *, 0, *, 0), \qquad \text{FC next.}$$

Coloring the left blank 0 immediately secures a differing segment, but NFC can color the right blank 0 and leave score 1. Coloring the right blank 1 instead fixes two differing segments there; either NFC choice on the left leaves one more, so the value is 3.

The full fixed-endpoint partial-path theorem uses interval signatures $M + \varepsilon J + \eta W$, not unrestricted local greediness. Here J is a one-move choice of $-1, +1$, and W is one mandatory zero-score move. The outcomes are $(M + \varepsilon, M - \varepsilon)$, retaining null-move parity.

Single-Side Fan Reduction

Reduction



color change = fully labeled cone

Cone each path edge to a fixed apex labeled 3.

fully labeled triangle $\iff$ base labels differ.

The base vertices retain their legal labels $\{1, 2\}$ and their freely chosen move order.

Because the apex is fixed, coning introduces no new moves; each fully labeled line segment corresponds to exactly one fully labeled triangle.

The path and fan are identical scoring games.

Fixed-Edge Stellar Stacks and Coning

Reduction

The fixed-edge stellar family E_n has a binary-path quotient for arbitrary legal partial positions. Fixed endpoints split a partial path into maximal uncolored intervals.

Coning to a fixed apex carrying a new label preserves the scoring game:

$$\sigma \text{ fully labeled} \quad \Longleftrightarrow \quad \sigma * \{a\} \text{ fully labeled.}$$

New simplex faces have the standard Sperner domains. Repeated fixed-apex cones lift solved games into higher dimensions without adding a playable apex.

This reduction applies to these coned triangulations, rather than to all higher-dimensional boards.

Appendix B10

Selector Cones: Optimal Play

Theorem and proof outline



If $z = i$, only the opposite path P_i scores.

For each current path P_i, put $L_i = V_F(P_i)$ and $R_i = V_N(P_i)$. The binary-path solution gives $L_i \geq R_i$. Set

$$A = \max_i R_i, \quad B = \min_i L_i.$$

Choose the center now. Coloring $z = i$ uses a turn: FC then gets R_i, while NFC gets L_i. Their best center choices therefore give A and B.

If $A < B$, delay instead. A prime denotes the state after one boundary move.

Next role	Optimal move on one path	New extrema and opponent's value
FC	Choose $L_p = B$; attain $R'_p = B$.	$A' = B \leq B'$, hence $V'_N = B$.
NFC	Choose $R_p = A$; attain $L'_p = A$.	$A' \leq A = B'$, hence $V'_F = A$.

The recurrences $L_p = \max R'_p$, $R_p = \min L'_p$ give the matching converse bounds. Induct on blanks:

$$\boxed{V_F = \max(A, B), \qquad V_N = \min(A, B).}$$

Selector Cones: Attaining the Bounds

Extremal induction

For each boundary path P_i, write $L_i = V_F(P_i)$ and $R_i = V_N(P_i)$. Binary paths satisfy, at every partial state,

$$L_i \geq R_i, \qquad L_i = \max_{P_i \to H} R(H), \qquad R_i = \min_{P_i \to H} L(H).$$

Off-path vertices become null waits after the center is fixed. The binary-path wait lemma leaves both values unchanged. Hence choosing center label i has follower value R_i for FC and L_i for NFC.

$$A = \max_i R_i, \qquad B = \min_i L_i, \qquad V_F = \max(A, B), \quad V_N = \min(A, B).$$

Induct on the number of boundary blanks. If $A \geq B$, selecting the center attains A for FC or B for NFC. If $A < B$:

- **FC delays.** Choose p with $L_p = B$ and an optimal path move with $R'_p = B$. Since every other $R_i \leq A < B$ and $L'_p \geq R'_p$, the new bounds satisfy $A' = B \leq B'$. Induction gives the NFC-next value $\min(A', B') = B$.
- **NFC delays.** Choose p with $R_p = A$ and an optimal path move with $L'_p = A$. Since every other $L_i \geq B > A$ and $R'_p \leq L'_p$, we have $A' \leq A = B'$. Induction gives the FC-next value $\max(A', B') = A$.

These paths are nonterminal: a terminal path has $L_p = R_p$, contradicting $A < B$. The zero-blank case is immediate. Appendix B14 proves that no other move improves either bound. If the center is fixed to i, the values are (L_i, R_i). Appendix B12

Selector Cones: Converse Bounds

Converse bounds

Keep $A = \max_i R_i$, $B = \min_i L_i$, and the induction from Appendix B13. Let

$$C = \max(A, B), \qquad D = \min(A, B).$$

FC cannot exceed C.

A center move has value $R_i \le A \le C$.

After any move on path p, the NFC-next value is $\min(A', B')$.

- If another path still attains $L_i = B$, then $B' \le B \le C$.
- Otherwise p was the unique minimizer. Its recurrence gives $R'_p \le L_p = B$, while all unchanged $R_i \le A$. Thus $A' \le C$.

Either way $\min(A', B') \le C$.

NFC cannot go below D.

A center move has value $L_i \ge B \ge D$.

After any move on path p, the FC-next value is $\max(A', B')$.

- If another path still attains $R_i = A$, then $A' \ge A \ge D$.
- Otherwise p was the unique maximizer. Its recurrence gives $L'_p \ge R_p = A$, while all unchanged $L_i \ge B$. Thus $B' \ge D$.

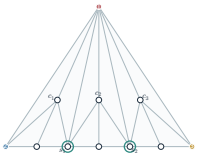
Either way $\max(A', B') \ge D$.

Appendix B13 attains both bounds. Thus $V_F = C$ and $V_N = D$, with every legal center and boundary move included.

Appendix B12

Shared-Sector Fans: Conditional Score Identity

Reduction and proof



D_k: coned sectors with shared, playable separators.

Let T_j count fully labeled line segments along sector j's base; write its separator labels as bits $s_{j-1}, s_j \in \{0, 1\}$. Counting the triangles gives

$$f_j = \begin{cases} T_j, & c_j = 1, \\ s_{j-1} + s_j, & c_j = 2, \\ 2 - s_{j-1} - s_j, & c_j = 3, \end{cases} \qquad X = \sum_j f_j.$$



$\rightsquigarrow \quad 2 + \mathbf{1}_{x \neq y}$

Audited use of compression. Four fresh binary internal vertices occur only through their path transition count. The exterior is an independent fixed-length game and its terminal payoff is nondecreasing in that count after all exterior data are fixed. Paired replies preserve the exterior turn. The endpoint-aware completion lemma supplies the final inequalities when exterior play ends.

The resulting object is a joint scoring game, with shared separators retained. A deleted block adds 2 only in the $c_j = 1$ branch. Original D_k boards reduce to at most $5k - 1$ retained moves; this is not an unconditional triangle equality or an automatic smaller unweighted triangulation.

Four-Blank Replacement: Exact Hypotheses

Contextual theorem

Let $\mathcal{E}$ be an independent finite fixed-length exterior game. Its complete datum ω includes endpoint bits x, y, fixed now or eventually chosen in $\mathcal{E}$. Four fresh binary moves $u_1, \dots, u_4$ occur nowhere else in the rules or payoff, except through

$$T = \mathrm{tr}(x, u_1, u_2, u_3, u_4, y).$$

The terminal payoff is $\Phi(T, \omega)$, nondecreasing in T for every fixed ω. Then deleting those four moves and replacing their input by

$$T^{\#} = 2 + \mathbf{1}\{x \neq y\}$$

preserves both starting-role values, including in every further independent fixed-length additive context. Endpoint moves remain in the exterior game.

This is a contextual scoring-game substitution. It does not assert pointwise equality for arbitrary labels, an unconditional +2 offset in a fan, or realizability of the compressed object by a smaller unweighted triangulation.

Endpoint-Aware Completion: Complete Finite Premise

144-case certificate

Pair (u_1, u_2) and (u_3, u_4). In an FC-compatible state each pair is blank or completed oppositely; in an NFC-compatible state it is blank or completed equally. There are $3^2 = 9$ states of each kind.

$$\Delta_{x,y}(u) = \mathrm{tr}(x, u_1, u_2, u_3, u_4, y) - 2 - \mathbf{1}\{x \neq y\}.$$

For fixed endpoints and next role, use the exact recurrence

$$V_F(s) = \max_{i:s_i=*,\, b\in\{0,1\}} V_N(s[i \leftarrow b]), \quad V_N(s) = \min_{i:s_i=*,\, b\in\{0,1\}} V_F(s[i \leftarrow b]),$$

with terminal value Δ.

Endpoint Completion: Bounds and Verification

144-case certificate

The exhaustive margins are

completed pairs	0	1	2
FC-compatible range	$[0, 0]$	$[0, 2]$	$[0, 2]$
NFC-compatible range	$[0, 0]$	$[-2, 0]$	$[-2, 0]$

Each row includes all four endpoint assignments and both next roles. Thus FC can force $\Delta \geq 0$ from every FC-compatible state, NFC can force $\Delta \leq 0$ from every NFC-compatible state. All $2 \cdot 9 \cdot 4 \cdot 2 = 144$ inequalities are reproduced by `prepare_assets.py`. This finite premise is explicitly computer-assisted.

Contextual Replacement: FC Simulation

Proof for both roles

Absorb any further independent fixed-length context into the exterior game. Simulate an optimal FC strategy in the shortened game.

Copy every exterior move. While exterior play remains, FC never initiates the internal block. On NFC's entry into a fresh pair, immediately color its mate oppositely. Those two physical moves make no simulated exterior move, restoring the same next role. Every completed pair is opposite.

When exterior play ends, both endpoints have known labels. The remaining block is FC-compatible. The endpoint-aware completion lemma applies whichever role moves next and gives

$$T \geq 2 + \mathbf{1}\{x \neq y\} = T^{\#}.$$

The copied exterior history has exactly the same datum ω as the simulated history. Since $\Phi(t, \omega)$ is nondecreasing in t, FC obtains at least the shortened-game value.

No endpoint-oblivious completion policy is needed: FC only initiates the remaining block after exterior play has ended.

Contextual Replacement: NFC Simulation

Matching upper bound

Simulate an optimal NFC strategy in the shortened game, copying exterior moves. NFC never initiates the internal block while exterior play remains. On any FC entry into a fresh pair, immediately color its mate equally. The block remains NFC-compatible, and the next exterior role is unchanged after the two physical moves.

At exterior termination both endpoints are known. The endpoint-aware completion lemma, for either next role, gives

$$T \le 2 + \mathbf{1}\{x \neq y\} = T^{\#}.$$

For the same copied exterior datum ω, monotonicity holds the full payoff to the shortened value. Together with the FC lower bound, this proves equality for either initial role.

The proof includes every legal exterior and internal choice. Four deleted moves are even, paired replies preserve the exterior turn, and no hidden mandatory wait is discarded. The argument remains valid in any further independent fixed-length additive context.

Sector-Fan Residue Cores

Kernel bounds with conditional offsets

In $D_k(n_1, \ldots, n_k)$, retain the k centers and $k-1$ shared separators. For $n_j = 4a_j + r_j$, $0 \le r_j < 4$, the exact reduced sector contribution is

$$f_j^{\#} = \begin{cases} 2a_j + \operatorname{tr}(s_{j-1}, w_j, s_j), & c_j = 1, \\ s_{j-1} + s_j, & c_j = 2, \\ 2 - s_{j-1} - s_j, & c_j = 3, \end{cases} \qquad |w_j| = r_j.$$

The original corner-only position has at most $3k + (k-1) + k = 5k - 1$ retained moves. Replacements compose because every sector score is nondecreasing in its transition count and separators are retained in one joint game.

Partial Sector Fans: Residual Tails and Waits

Kernel bounds with conditional offsets

For arbitrary partial positions, let $u \le k - 1$ separators remain uncolored. Reduce each uncolored endpoint tail modulo four; compress fixed-endpoint middles by their contextual path signatures. At most $6u$ tail moves, k hot J moves, u separators, one retained null wait, and k centers remain:

$$\#\text{moves} \le 2k + 7u + 1 \le 9k - 6 \quad (k \ge 2).$$

State bounds are $2 \cdot 3^{4k-1}4^k$ for original positions and $4 \cdot 3^{k+7u}4^k$ for partial positions. They are exponential in unbounded k, not a general polynomial solver.

Sector-Fan Value Formulas

Value formulas

For original D_2, D_3 positions, put $n_{\max} = \max(n_1, n_2)$ and $n_{\min} = \min(n_1, n_2)$. Then

$$V_F(D_2) = \max\left\{3, 2\left\lfloor \frac{n_{\max}+2}{4} \right\rfloor + 1\right\}, \quad V_N(D_2) = 2\left\lfloor \frac{n_{\min}+1}{4} \right\rfloor + 1.$$

For D_3, write $n_i = 4a_i + r_i$, $(a_1, a_2, a_3) = (x, y, z)$, and $0 \le r_i < 4$. Set

$$\delta_{ij} = \mathbf{1}_{r_i+r_j\ge 2} + \mathbf{1}_{r_i=r_j=3}, \quad P_{ij} = a_i + a_j + \delta_{ij}, \quad U = \max_{r_j \le 1}(a_j + 1),$$

with $\max \varnothing = -\infty$. Using $\vee = \max$, $\wedge = \min$,

$$E_L = \begin{cases} P_{12} \wedge (x+2) & (r_1, r_3) = (2, 3), \\ -\infty & \text{otherwise}, \end{cases} \qquad E_R = \begin{cases} P_{23} \wedge (z+2) & (r_1, r_3) = (3, 2), \\ -\infty & \text{otherwise}. \end{cases}$$

$$\boxed{q_F = U \vee \operatorname{med}(P_{12}, P_{13}, P_{23}) \vee E_L \vee E_R, \quad V_F = 2q_F + 1.}$$

The residue step is proved by complete symbolic certificates. General D_k retains a parameter-exponential fixed-k core. Appendix B15

D_3 NFC-First Residue Table: I

Residue table

Write $n_i = 4a_i + r_i$, $(a_1, a_2, a_3) = (x, y, z)$ and classify residues as $0, L = \{1, 2\}, 3$. Put $X = x + 1$, $Y = y + 1$, $Z = z + 1$, $A = x + y$, $B = x + z$, $C = y + z$.

Class word	q_N ($\vee = \max$, $\wedge = \min$)
000	$1 \vee (y \wedge B \wedge X) \vee (y \wedge B \wedge Z) \vee (A \wedge C \wedge X \wedge Z)$
003	$1 \vee (y \wedge X) \vee (y \wedge Z) \vee (A \wedge X \wedge Z)$
030	$(B \wedge X \wedge Y) \vee (X \wedge Z) \vee (B \wedge Y \wedge Z)$
300	$1 \vee (y \wedge X) \vee (y \wedge Z) \vee (C \wedge X \wedge Z)$
303	$(y \wedge X) \vee (y \wedge Z) \vee (X \wedge Z) \vee (y \wedge (x + 2) \wedge (z + 2))$
00L	$1 \vee (y \wedge B \wedge X) \vee (y \wedge Z) \vee (A \wedge X \wedge Z)$

The optimal terminal count is $V_N = 2q_N + 1$. These are exact symbolic formulas for every $x, y, z \geq 0$.

Residues 1 and 2 give the same NFC-first value; no move-set isomorphism is asserted. Appendix B15

D_3 NFC-First Residue Table: II

Residue table

Continue $X = x + 1$, $Y = y + 1$, $Z = z + 1$, $A = x + y$, $B = x + z$, $C = y + z$, with $\vee = \max$ and $\wedge = \min$.

Class word	q_N
$0L0$	$(B \wedge X \wedge Y) \vee (A \wedge C \wedge X \wedge Z)$ $\vee (B \wedge Y \wedge Z) \vee (X \wedge Y \wedge Z)$
$0L3$	$(X \wedge Y) \vee (A \wedge X \wedge Z) \vee (Y \wedge Z)$
$03L$	$(B \wedge X \wedge Y) \vee (X \wedge Z) \vee (Y \wedge Z)$
$L00$	$1 \vee (y \wedge X) \vee (y \wedge B \wedge Z) \vee (C \wedge X \wedge Z)$
$L30$	$(X \wedge Y) \vee (X \wedge Z) \vee (B \wedge Y \wedge Z)$
$3L0$	$(X \wedge Y) \vee (C \wedge X \wedge Z) \vee (Y \wedge Z)$

The contextual reduction retains both playable endpoints, arbitrary exterior interleaving, and shared-separator timing. Appendix B15

D_3 NFC-First Residue Table: III

Residue table

In addition to the preceding definitions, let $D = \operatorname{med}(y, X, Z)$ and $M = \operatorname{med}(X, Y, Z)$.

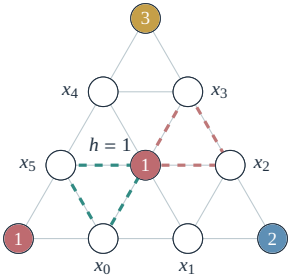
Class word(s)	q_N
$3L3, 333$	$(X \wedge Y) \vee (X \wedge Z) \vee (Y \wedge Z) \vee (Y \wedge (x+2) \wedge (z+2))$
$0LL$	$(B \wedge X \wedge Y) \vee (A \wedge X \wedge Z) \vee (Y \wedge Z)$
$LL0$	$(X \wedge Y) \vee (C \wedge X \wedge Z) \vee (B \wedge Y \wedge Z)$
$L0L, L03, 30L$	D
$033, LLL, LL3, L3L, L33,$ $3LL, 330, 33L$	M

The 17 rows partition all 27 class words and cover all 64 literal residue triples. Every value is $V_N = 2q_N + 1$. Complete symbolic certificates prove the formulas on the full nonnegative quotient orthant. Appendix B15

Optimal Play on K_3

Theorem and proof

K_m: m subdivisions per side, m^2 scored triangles. Boundary labels below are bits $0/1$.



Bits $0/1$: $x_0, x_1 : 1/2$; $x_2, x_3 : 2/3$; $x_4, x_5 : 3/1$.
Hollow = blank; dashed = candidate face.

Let q count CCW fully labeled triples $132, 321, 213$. The nine faces give

h	q	null bits
1	$x_0(1-x_5)+x_2(1-x_3)$	x_1, x_4
2	$x_2(1-x_1)+x_4(1-x_5)$	x_0, x_3
3	$x_0(1-x_1)+x_4(1-x_3)$	x_2, x_5

Block a pair $u(1-v)$. Set $u = 0$ or $v = 1$. If FC sets $u = 1$, reply $v = 1$; if FC sets $v = 0$, reply $u = 0$. Otherwise secure an untouched pair.

Each FC move touches at most one pair; each reply permanently kills a new summand.

Starting roles. NFC opens $h = 1$, or FC opens at h and NFC uses that row: NFC has three boundary turns. After FC opens x_i, NFC chooses $h = (2, 1, 3, 2, 1, 3)_i$ to make x_i null, leaving two NFC turns for two untouched pairs.

Score conversion. The signed Sperner identity gives $X = 1 + 2q$. Since $q \geq 0$ and NFC forces $q = 0$,

$$V_F(K_3) = V_N(K_3) = 1.$$

K_3 Threat Algebra and Responses

Strategy proof

Fixing the center leaves two disjoint threats. In CCW boundary order let $x_0, \dots, x_5$ be the bits at $(1,0), (2,0), (2,1), (1,2), (0,2), (0,1)$. Their $0/1$ labels are $1/2, 1/2, 2/3, 2/3, 3/1, 3/1$.

$\lambda(h)$	$q = N_-$	null bits
1	$x_0(1-x_5) + x_2(1-x_3)$	x_1, x_4
2	$x_2(1-x_1) + x_4(1-x_5)$	x_0, x_3
3	$x_0(1-x_1) + x_4(1-x_3)$	x_2, x_5

For a threat $u(1-v)$, NFC secures it by $u = 0$ or $v = 1$. Answer $u = 1$ by $v = 1$, and answer $v = 0$ by $u = 0$.

NFC opens at $h = 1$ if first. If FC opens at h, use the corresponding threat row. If FC opens at x_i, select h so x_i is null:

i	0	1	2	3	4	5
$\lambda(h)$	2	1	3	2	1	3

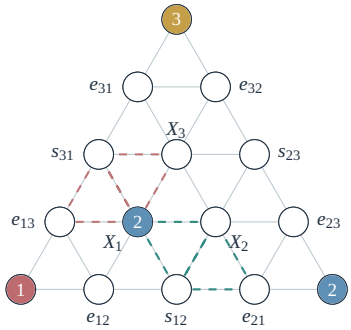
.

NFC then has enough turns to secure both pairs. Thus $q = 0$ and $V_F = V_N = 1$. Appendix B27

A Forced Negative Face on K_4

Lower-bound proof

By the signed Sperner identity, write $q = N_-$ and $X = 1 + 2q$.



Hollow = blank; dashed = candidate face.
$A = \{X_2, s_{12}, e_{21}\}$, $B = \{X_3, s_{31}, e_{13}\}$.

FC first: open $X_1 = 2$. NFC touches at most one of the disjoint wings A, B. In the untouched wing, play the pivot below.

Pivot	Selector's alternatives
$X_2 = 3$	$s_{12} = 1$: $(s_{12}, X_2, X_1) = 132$ $s_{12} = 2$: finish $e_{21} = 1$ to get 213
$X_3 = 1$	$s_{31} = 3$: $(X_1, X_3, s_{31}) = 213$ $s_{31} = 1$: finish $e_{13} = 3$ to get 321

If NFC takes the backup first, choose the direct selector alternative; outside play also leaves it free. Thus $q \geq 1$ by FC's third turn.

NFC first: five opening types $e_{12} = 1$, $e_{12} = 2$, $s_{12} = 1$, $X_1 = 1$, $X_1 = 2$. Three single-switch tactics and two anchor tactics ($e_{21} = 1$, test e_{23}) force $q \geq 1$ by FC's third turn; backup covers all 27 actions.
Hence $V_F(K_4), V_N(K_4) \geq 3$; the first-player construction also gives $V_F(K_m) \geq 3$ for all $m \geq 4$.

K_4 Exact Seven-Bit Decomposition

Payoff identity

Let $X_1 = (1,1)$, $X_2 = (2,1)$, $X_3 = (1,2)$. On cyclic side ij, use ears e_{ij}, e_{ji} and midpoint s_{ij}. For cyclic (i,j,k),

$$\begin{aligned} S_{ij} &= [e_{ij} = j, s_{ij} = i, X_i = k] + [s_{ij} = i, X_i = j, X_j = k] \\ &\quad + [s_{ij} = j, X_i = k, X_j = i] + [s_{ij} = j, e_{ji} = i, X_j = k], \\ G_i &= [e_{ij} = j, e_{ik} = k, X_i = i], \quad C = [(X_1, X_2, X_3) \in \{(1,3,2), (3,2,1), (2,1,3)\}]. \end{aligned}$$

The four events in a sector are mutually exclusive, and the exact face partition gives

$$\boxed{q = C + G_1 + G_2 + G_3 + S_{12} + S_{23} + S_{31}}.$$

Simultaneously permuting corners, labels, and barycentric coordinates preserves q. The 27 opening actions reduce to five symmetry types. The complete opening-conditioned response relations accompany the upper-bound certificate below. Appendix B37

The Double Switch: Every Reply

Tactical proof

In K_m, $m \geq 4$, use $X_1 = (1,1)$, $X_2 = (2,1)$, $X_3 = (1,2)$, $s_{12} = (2,0)$, $e_{21} = (3,0)$, $s_{31} = (0,2)$, $e_{13} = (0,1)$. All required labels below are legal.

FC opens $X_1 = 2$. NFC's reply touches at most one of $A = \{X_2, s_{12}, e_{21}\}$ and $B = \{X_3, s_{31}, e_{13}\}$. Activate an untouched wing.

Pivot	NFC's next move	FC completion or permanent negative face
$X_2 = 3$	$s_{12} = 1$	$(s_{12}, X_2, X_1) = 132$ already scores.
	$s_{12} = 2$	Set $e_{21} = 1$: $(s_{12}, e_{21}, X_2) = 213$.
	$e_{21} = 1$	Set $s_{12} = 2$ for that same face.
	$e_{21} \neq 1$, or outside	Set $s_{12} = 1$ for the direct face.
$X_3 = 1$	$s_{31} = 3$	$(X_1, X_3, s_{31}) = 213$ already scores.
	$s_{31} = 1$	Set $e_{13} = 3$: $(e_{13}, X_1, s_{31}) = 321$.
	$e_{13} = 3$	Set $s_{31} = 1$ for that same face.
	$e_{13} \neq 3$, or outside	Set $s_{31} = 3$ for the direct face.

Each negative face is permanent, proving $X \geq 3$ by FC's third turn. Appendix B29

NFC-First K_4: Three Single-Switch Cases

Tactical proof

Simultaneous permutation of corners, labels, and barycentric coordinates preserves negative-face count: an odd permutation reverses both geometric and label orientation. The 27 legal actions have five orbits of sizes $6, 6, 6, 3, 6$.

Three representatives have these complete two-turn FC replies:

NFC opens	FC plays	After NFC's next move
$e_{12} = 2$	$e_{13} = 3$	If X_1 is blank, set $X_1 = 1$. Label 1 already completes G_1; label 2 is answered by $s_{31} = 1$; label 3 by $s_{12} = 1$.
$s_{12} = 1$	$e_{12} = 2$	If X_1 is blank, set $X_1 = 3$. Label 3 already scores; label 1 is answered by $e_{13} = 3$; label 2 by $X_2 = 3$.
$X_1 = 2$	$e_{13} = 3$	If s_{31} is blank, set it to 1. Label 1 already scores; label 3 is answered by $X_3 = 1$.

An outside move leaves the displayed selector blank. If NFC labels that selector instead, the alternative completion vertex is untouched. Thus every branch creates a negative face, rather than merely identifying a possible terminal assignment. Appendix B29

NFC-First K_4: The Ear-Opening Anchor

Tactical proof

NFC opens $e_{12} = 1$. FC anchors $e_{21} = 1$ and tests the reply at e_{23}.

NFC's reply	FC's continuation
$e_{23} = 3$	Play $X_2 = 2$ and complete G_2.
$e_{23} = 2$	Play $X_2 = 3$, then use the s_{12} switch below.
Any other move	Play $e_{23} = 3$. After the next NFC move: blank X_2 is set to 2; $X_2 = 2$ already scores; $X_2 = 3$ uses the s_{12} switch; $X_2 = 1$ uses the s_{23} switch.

s_{12} **switch:** if blank, set $s_{12} = 2$. Label 2 already gives a negative face with $e_{21} = 1, X_2 = 3$. Label 1 is answered by $X_1 = 2$, giving the other orientation in the side sector.

s_{23} **switch:** if blank, set $s_{23} = 2$. Label 2 already gives a negative face with $e_{23} = 3, X_2 = 1$. Label 3 is answered by $X_3 = 2$.

Ownership check. If X_2 is already occupied, one of the two NFC moves after the anchor occupied it; the other can touch only one vertex of the relevant switch. The required backup X_1 or X_3 is therefore free when the selector is blocked. FC scores by its third move. Appendix B29

NFC-First K_4: The Interior-Opening Anchor

Tactical proof

NFC opens $X_1 = 1$. FC anchors $e_{21} = 1$.

NFC's reply	FC's continuation
$e_{23} = 3$	Play $X_2 = 2$, completing G_2.
$e_{23} = 2$	Play $X_2 = 3$. The distinct vertices $s_{12} = 2$ and $X_3 = 2$ are two immediate negative-face completions. NFC can touch at most one.
Any other move	Play $e_{23} = 3$. If X_2 is blank, set it to 2; label 2 already scores. If $X_2 = 3$, use the two completions above. If $X_2 = 1$, use the s_{23} switch below.

s_{23} **switch:** blank is set to 2; label 2 already scores; label 3 is answered by $X_3 = 2$.

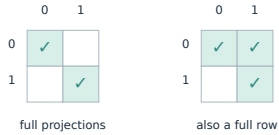
Once X_2 is occupied, only one other NFC move after the anchor is available to touch the two completion vertices. A free completion remains, or NFC itself has already created the negative face.

Together with the preceding four representatives, these cases cover all 27 NFC openings. Therefore $V_N(K_4) \geq 3$, independently of the upper-bound strategy. Appendix B29

When Pair Responses Compose

Composition theorem

The opponent moves next. Partition the blanks into ordered pairs with legal domains A_i, B_i, plus h unrestricted single vertices. Require each pair to finish in $R_i \subseteq A_i \times B_i$.



Full projections: every label at either endpoint has a compatible label at the mate.

Initiable: some row or column is full, so one label secures the relation before the mate is played.

h even	all R_i have full projections,
h odd	all have full projections and are initiable.

Sufficiency. Reply at the mate of an entered pair. Pair off free moves if h is even. With one exposed free vertex, its occupation triggers a securing move in a fresh pair; the unused mate becomes the exposed vertex. The invariant used on K_4 is preserved.

Necessity. A missing projection is defeated by that endpoint label. For odd h, the opponent can exhaust everything outside a noninitiable pair. The responder must open it, allowing a forbidden completion next.

Pair-Response Parity

Composition theorem

The opponent moves next. For each ordered pair (a_i, b_i) require its final labels in $R_i \subseteq A_i \times B_i$, with h other free vertices.

h even	Every relation has both coordinate projections full.
h odd	Full projections and a full row or column in every relation.

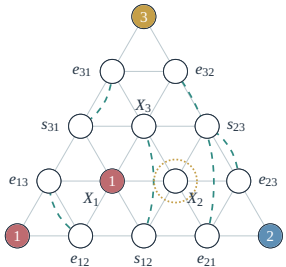
Necessity. A missing projection lets the opponent select a label with no reply. With odd free parity the opponent can consume the free interface, forcing the responder to initiate a pair; that requires a full row or column.

Sufficiency. Answer an entered pair using a matching coordinate. Match the free vertices in pairs when h is even. When h is odd, keep one exposed hole; if it is taken, initiate a pair along a full row or column and expose its unused mate.

These strategies survive arbitrary interleaving. The next pages give the exact K_4 response data. Appendix B35 Appendix B37

Optimal Play on K_4: $V_F = V_N = 3$

Upper-bound proof



NFC opens $X_1 = 1$; X_2 is the exposed blank. Dashed arcs are response pairs, not physical edges.

Enforce $a = \alpha$ or $b = \beta$ in each pair:

(a, b)	(α, β)
(e_{12}, e_{13})	$(1, 1)$
(e_{21}, e_{32})	$(2, 3)$
(e_{23}, s_{23})	$(2, 3)$
(e_{31}, s_{31})	$(3, 1)$
(s_{12}, X_3)	$(1, 3)$

Invariant. If FC enters a pair, set its mate's safe label. If FC takes the exposed vertex, set $a = \alpha$ in a fresh pair and expose b. One exposed blank and the completed safe clauses are preserved.

Score certificate. In the face partition, C, G_i, S_{ij} are the center, corner, and side-strip $0/1$ indicators. Thus $q = C + \sum G_i + \sum S_{ij}$. The clauses kill G_1, S_{31}; the five others are pairwise incompatible.

FC opens	$e_{12} = 1$	$e_{12} = 2$	$s_{12} = 1$	$X_1 = 1$	$X_1 = 2$
reply	$e_{21} = 2$	$X_2 = 2$	$e_{12} = 1$	$X_2 = 2$	$e_{32} = 3$

Every label has a mate reply; each row forces $q \le 1$ (certificates in backup).

K_4 NFC-Second Response Relations: I

Response relations

A relation is imposed on the ordered pair displayed. Each projects onto both full legal domains. Reply once, then answer every entry into a fresh pair at its mate.

FC opening	NFC reply	Five pair relations
$e_{12} = 1$	$e_{21} = 2$	$e_{23} = e_{32}$; $(e_{31}, s_{12}) \in \{(1,2),(3,1)\}$; $(e_{13}, s_{23}) \in \{(1,3),(3,2)\}$; $(s_{31}, X_3) \in \{(1,1),(1,2),(3,3)\}$; $X_1 = X_2$.
$e_{12} = 2$	$X_2 = 2$	$(e_{21}, e_{23}) \in \{(1,2),(2,3)\}$; $(e_{32}, e_{31}) \in \{(2,3),(3,1)\}$; $(e_{13}, s_{23}) \in \{(1,3),(3,2)\}$; $(s_{12}, s_{31}) \in \{(1,3),(2,1)\}$; $X_1 = X_3$.
$s_{12} = 1$	$e_{12} = 1$	$(e_{21}, e_{23}) \in \{(1,2),(2,3)\}$; $(e_{32}, e_{31}) \in \{(2,3),(3,1)\}$; $(e_{13}, s_{23}) \in \{(1,3),(3,2)\}$; $(s_{31}, X_3) \in \{(1,1),(1,2),(3,3)\}$; $X_1 = X_2$.

Substitution into the seven-bit identity leaves at most one nonzero bit, so $q \leq 1$. Appendix B37

K_4 NFC-Second Response Relations: II

Response relations

FC opening	NFC reply	Five pair relations
$X_1 = 1$	$X_2 = 2$	$e_{12} = e_{21}$; $e_{23} = e_{32}$; $e_{31} = s_{31}$; $(e_{13}, X_3) \in \{(1,1),(1,3),(3,2)\}$; $(s_{12}, s_{23}) \in \{(1,2),(2,3)\}$.
$X_1 = 2$	$e_{32} = 3$	$e_{12} = s_{12}$; $(e_{21}, e_{31}) \in \{(1,3),(2,1)\}$; $e_{23} = s_{23}$; $e_{13} = s_{31}$; $X_2 = X_3$.

The possible nonzero-bit supports in these rows are respectively

$$\varnothing, \{G_1\}, \{G_2\}, \{G_3\}, \{S_{23}\}; \qquad \varnothing, \{G_2\}, \{S_{12}\}, \{S_{31}\}.$$

Hence $q \le 1$ after every legal FC opening, and the double switch gives the matching lower bound.

Appendix B37

K_4 NFC-First Safe Clauses

Response strategy

Open $X_1 = 1$, with X_2 as the initial hole.

Pair (a, b)	Safe labels (α, β)	Enforced clause
(e_{12}, e_{13})	$(1, 1)$	$e_{12} = 1$ or $e_{13} = 1$
(e_{21}, e_{32})	$(2, 3)$	$e_{21} = 2$ or $e_{32} = 3$
(e_{23}, s_{23})	$(2, 3)$	$e_{23} = 2$ or $s_{23} = 3$
(e_{31}, s_{31})	$(3, 1)$	$e_{31} = 3$ or $s_{31} = 1$
(s_{12}, X_3)	$(1, 3)$	$s_{12} = 1$ or $X_3 = 3$

If FC enters a pair, NFC assigns the mate's safe label. If FC takes the hole, NFC plays $a = \alpha$ in an untouched pair and exposes b as the new hole.

The clauses kill G_1, S_{31} and make the remaining possible score bits pairwise incompatible, so $q \leq 1$. FC's five-opening tactical proof supplies $q \geq 1$ for this starting role too. Appendix B37 Appendix B35

K_4 Safe Clauses: The Indicator Check

Score certificate

After NFC opens $X_1 = 1$, the five safe clauses in Appendix B37 kill G_1 and S_{31}. The remaining candidates are $S_{12}, G_2, G_3, C, S_{23}$.

If nonzero	Consequences of the safe clauses
S_{12}	Its $s_{12} = 2, e_{21} = 1, X_2 = 3$ force $X_3 = 3, e_{32} = 3$. These kill G_2, G_3, C and every event of S_{23}.
G_2	Its $e_{23} = 3, e_{21} = 1, X_2 = 2$ force $e_{32} = 3, s_{23} = 3$. These kill S_{12}, G_3, C and S_{23}.
G_3	Its $e_{31} = 1, e_{32} = 2, X_3 = 3$ force $e_{21} = 2$. This kills S_{12}, G_2, C; the e_{23}, s_{23} clause kills the only remaining side event.
C	Its $X_2 = 3, X_3 = 2$ force $s_{12} = 1$, killing S_{12}; the fixed interior labels kill G_2, G_3 and all S_{23} events.

If none of these four indicators is one, only S_{23} can remain, and its events are mutually exclusive. Thus every completion satisfying the clauses has $q \leq 1$. This is a terminal-label certificate; the exposed-vertex invariant proves NFC can enforce it against every move order. Appendix B37

Dominant Boundary Ears

Dominance

For every $m \geq 3$, exactly six corner-adjacent boundary vertices have globally FC-preferred labels. Deeper boundary vertices have marginals of both signs.

Fixing all six ears to their FC-preferred labels on K_4 leaves a six-blank follower with both-starter value five.

ear-fixed follower: 5 original K_4 : 3.

The original game includes the timing and ownership of the ears. A preferred-label statement alone does not authorize precoloring them in the game. Appendix B29

Somerville #1: Construction and Scoring

Definition

Start with one triangle K_0. To form K_{r+1}, insert one center in every triangle of K_r and join it to that triangle's three vertices. Retain all old vertices and edges.

$$|\mathcal{S}_2(K_r)| = 3^r, \qquad \#\text{noncorner vertices} = (3^r - 1)/2.$$

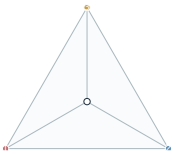
Only the newest-generation triangles are scored. All noncorner vertices, including those from earlier generations, remain playable unless fixed in a supplied partial position.

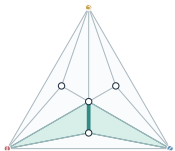
The matching theorem below gives an all-depth guarantee. It is not an exact optimal-strategy formula for arbitrary depth. Exact values at the smallest depths and isolated finite computations are labeled separately.

Somerville #1: A Matching Bound

Theorem and proof

At depth r, star every preceding face. Put $F = 3^{r-1}$ and $I = (F-1)/2$: the old face and interior-vertex counts.


depth 1: 3 final triangles


depth 2: 9 final triangles

In the bipartite graph *old interior vertices versus incident old faces*,

$$3|S| \le \sum_{v \in S} \deg_F(v) \le 3|N(S)|.$$

Every interior link has at least three edges. Hall's theorem therefore matches each old interior vertex to a distinct new face center.

Response strategy. On every matched edge, NFC copies the opponent's label to its mate. Both incident child triangles are non-fully-labeled. Distinct matched centers put these two-triangle blocks in distinct old faces.

Pair the unmatched centers as waiting moves. If NFC starts and moves last, first color one unmatched center; otherwise NFC responds throughout. This accounts for every move outside the matched edges.

$$X \le 3F - 2I = 2 \cdot 3^{r-1} + 1 \quad (r \ge 1), \quad \text{if NFC moves last.}$$

This is an all-depth guarantee, not an exact-value formula. Only the newest 3^r triangles are scored.

Somerville #1 Scoring Convention

Matching bound

At depth r, $|\mathcal{S}_2(K_r)| = 3^r$ and the number of noncorner moves is $n_r = (3^r - 1)/2$.

r	0	1	2	3
(V_F, V_N)	$(1,1)$	$(1,1)$	$(3,3)$	$(7,7)$
evidence	proof	proof	proof	finite computation

Matching theorem. For a one-step face-stellar refinement of a disk with F old faces and I old interior vertices, NFC moving last forces $X \le 3F - 2I$.

Hall's inequality follows from

$$3|S| \le \sum_{v \in S} \deg_F(v) \le 3|N(S)|.$$

Pair every old interior vertex with the center of its matched incident face and copy labels. For Somerville #1 this gives $2 \cdot 3^{r-1} + 1$ when r is even and FC starts, or r is odd and NFC starts.

The bound gives $3, 7, 19$ at depths $1, 2, 3$ and is not an exact all-depth value. Appendix B44

Matched-Edge Strategy Bounds

Strategy bound

Suppose FC is next, the number of blanks is even, and disjoint equal-domain edges pair chosen blanks. NFC copies the mate’s label after FC enters a pair.

Each equality edge blocks its incident triangles from being fully labeled. Maximizing the total number of distinct blocked faces gives the strongest bound in this restricted response family.

Equality is a full-projection relation. With an odd free interface it lacks a full row or column, so the general parity theorem identifies why a bare unmatched singleton cannot always be ignored. Appendix B44 Appendix B35

The Independent Uniform Legal-Label Experiment

Probability model

Fix any legal partial 2D position. At an uncolored vertex v, let $A(v)$ be its legal labels; at a colored vertex, use the prescribed singleton. Sample the remaining labels independently and uniformly from these sets. Thus an unfixed side vertex is fair binary and an unfixed interior vertex fair ternary.

$$I_\tau = \mathbf{1}_{\{\tau \text{ fully labeled}\}}, \qquad X = \sum_{\tau \in \mathcal{S}_2(K)} I_\tau,$$

$$p_\tau = \Pr(I_\tau = 1), \qquad p_{\tau\sigma} = \Pr(I_\tau = 1 \text{ and } I_\sigma = 1).$$

Sequentially selecting a vertex, even adaptively, then using a fresh conditionally uniform legal label gives this same product law. Uniform marginals alone do not give independence.

This is an expected-score experiment. It is separate from strategic optimal play; the limit $\mathbb{E}[X_{K_m}]/m^2 \to 2/9$ concerns random-play density.

The Eligibility Matrix and Local Probability

Finite counting proof

For $\tau = \{u, v, w\}$, order $(v_1, v_2, v_3) = (u, v, w)$ and set

$$(M_\tau)_{ij} = \mathbf{1}_{\{j \in A(v_i)\}}, \qquad i, j \in \{1, 2, 3\}.$$

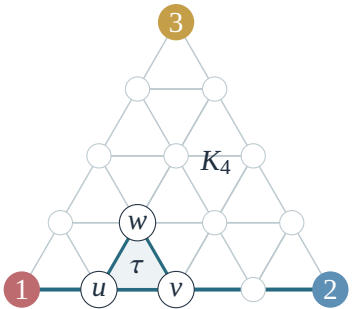
The permanent is

$$\operatorname{per}(M) = \sum_{\pi \in \mathfrak{S}_3} \prod_{i=1}^{3} M_{i,\pi(i)}, \qquad p_\tau = \frac{\operatorname{per}(M_\tau)}{\prod_{v \in \tau} |A(v)|}.$$

Here $\mathfrak{S}_3$ is the set of six permutations of labels $1, 2, 3$. Every fully labeled assignment is exactly one such permutation. Its product is 1 exactly when all three vertex-label choices are legal. There are no alternating signs. The denominator counts all equally likely local legal assignments, including prescribed singleton labels.

The Local Matrix: A Literal K4 Triangle

Worked probability



u, v: on outer side 1–2

w: interior

$$M_\tau = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \qquad p_\tau = \frac{2}{12} = \frac{1}{6}.$$

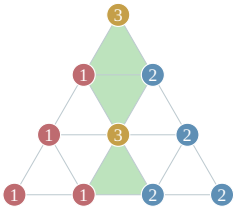
Rows are u, v, w; columns are labels $1, 2, 3$. Of the six label permutations, only $(1, 2, 3)$ and $(2, 1, 3)$ are legal.

This triangle is literally $\{(1, 0), (2, 0), (1, 1)\}$ in K_4. Thus $A(u) = A(v) = \{1, 2\}$ and $A(w) = \{1, 2, 3\}$. There are 12 legal local assignments; exactly 2 are fully labeled.

Uniform Legal Labeling on K_3

Expectation and variance

Labels are independent and uniform on their legal domains: corners fixed, six boundary labels fair binary, center uniform on $\{1, 2, 3\}$. Thus all 192 completions are equally likely.



x	1	3	5
$\Pr(X = x)$	$9/16$	$6/16$	$1/16$

$$\mathbb{E}X = 2, \quad \operatorname{Var} X = 3/2.$$

Conditional on the center, direct face counting gives $X = 1 + 2Y$, where Y counts two disjoint boundary patterns, each with probability $1/4$. Their labels are independent, hence $Y \sim \operatorname{Bin}(2, 1/4)$.

The displayed board is one completed example with score 3; the distribution concerns all independently uniform legal completions, not optimal play.

Joint Probabilities on a Vertex Union

Finite counting proof

For distinct triangles τ, σ, let $U_{\tau\sigma} = \tau \cup \sigma$. Then

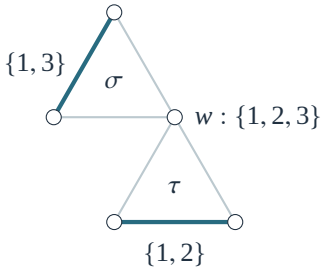
$$p_{\tau\sigma} = \frac{\displaystyle\sum_{a \in \prod_{v \in U_{\tau\sigma}} A(v)} I_\tau(a) I_\sigma(a)}{\displaystyle\prod_{v \in U_{\tau\sigma}} |A(v)|}.$$

The shared vertices are assigned once. Counting the triangles separately would incorrectly treat their shared label as two independent labels.

If $\tau \cap \sigma = \varnothing$, their indicators depend on disjoint independent vertex labels, so $p_{\tau\sigma} = p_\tau p_\sigma$.

One Shared Vertex Gives Nonzero Covariance

Literal example



one shared vertex
joint probability 0

Each triangle has fully labeled probability $1/6$. The lower triangle requires the common interior vertex w to have label 3; the upper one requires 2. They cannot both be fully labeled.

$$p_{\tau\sigma} = 0, \qquad \mathrm{Cov}(I_\tau, I_\sigma) = 0 - \frac{1}{6}\frac{1}{6} = -\frac{1}{36}.$$

There is exactly one shared vertex, and no shared edge.

The literal K_4 triangles are $\tau = \{(1,0),(2,0),(1,1)\}$ and $\sigma = \{(0,1),(0,2),(1,1)\}$. Their common vertex is $w = (1,1)$. Counting assignments on this union confirms the displayed probabilities.

Variance Under Independent Uniform Legal Labeling

Variance

For any legal partial 2D position under independent uniform legal labeling,

$$X = \sum_{\tau} I_\tau, \qquad \mathbb{E}[X] = \sum_{\tau} p_\tau,$$

$$\mathrm{Var}(X) = \sum_{\tau} p_\tau(1 - p_\tau) + 2 \sum_{\substack{\tau < \sigma \\ \tau \cap \sigma \neq \varnothing}} (p_{\tau\sigma} - p_\tau p_\sigma).$$

$\tau < \sigma$ counts each unordered pair of distinct triangles once. Since $I_\tau^2 = I_\tau$, its variance is $p_\tau - p_\tau^2$. Expanding the square of the sum gives one covariance for each ordered distinct pair, hence the factor two for unordered pairs.

Pairs with disjoint vertex sets have zero covariance under the product law and may be omitted. All intersecting pairs remain, including those sharing only one vertex. Independence of the triangle indicators is not required for linearity of expectation.

The Second Moment on K3

Exact moment check

The K_3 distribution is

$$\Pr(X = 1) = 9/16, \quad \Pr(X = 3) = 6/16, \quad \Pr(X = 5) = 1/16.$$

Consequently

$$\mathbb{E}[X] = (9 + 18 + 5)/16 = 2,$$

$$\mathbb{E}[X^2] = (9 + 54 + 25)/16 = 11/2, \qquad \operatorname{Var}(X) = 11/2 - 4 = 3/2.$$

The general local-probability and covariance formulas give these same moments. The K_3 variance is a small-mesh exception to the equilateral variance polynomial, whose range is $m \geq 4$.

Local Fully Labeled Probabilities

Probability

An all-interior d-simplex has $(d+1)^{d+1}$ equiprobable legal labelings, of which $(d+1)!$ are fully labeled:

$$p_d = \frac{(d+1)!}{(d+1)^{d+1}}, \qquad p_2 = \frac{2}{9}.$$

Let $X^{(d)}$ count fully labeled d-simplices; $X^{(2)} = X$. With one 0–1 indicator I_τ per d-simplex, linearity gives

$$\mathbb{E}[X^{(d)}] = \sum_{\tau \in \mathcal{S}_d(K)} p_\tau$$

even though these indicators need not be independent.

For equilateral K_m with $m \geq 3$,

$$\boxed{\mathbb{E}[X] = \frac{4m^2 - 3m + 9}{18}}.$$

Appendix B56 Appendix B57

Equilateral Mean: Boundary Corrections

Face classification

Assume $m \geq 3$.

Face class	Count	Fully labeled probability
Corner faces	3	$1/4$
Inverted faces next to a corner	3	$1/4$
Other outer-edge faces	$3(m-2)$	$1/6$
Remaining faces	$m(m-3)$	$2/9$

$$\mathbb{E}[X] = 6 \cdot \frac{1}{4} + 3(m-2) \cdot \frac{1}{6} + m(m-3) \cdot \frac{2}{9} = \frac{4m^2 - 3m + 9}{18}.$$

A face can have a boundary vertex without an outer boundary edge. The six exceptional faces must be separated from the other classes.

Five Face Classes in the Equilateral Mean

Exact local counts

For $m \geq 3$, use the independent uniform legal-label experiment. Up to permuting corner labels:

Local legal domains / location	Count	Fully labeled probability
$\{1\}, \{1,2\}, \{1,3\}$; corner triangle	3	$1/4$
$\{1,2\}, \{1,3\}, [3]$; inside a corner	3	$1/4$
$\{1,2\}, \{1,2\}, [3]$; along one side	$3(m-2)$	$1/6$
One side vertex and two interior vertices	$3(m-3)$	$2/9$
Three interior vertices	$(m-3)^2$	$2/9$

Fully labeled / legal assignment ratios are $1/4, 3/12, 2/12, 4/18, 6/27$. The five face counts sum to m^2. The interior remainder forms K_{m-3}; the boundary counts follow from the three sides and corners.

$$\mathbb{E}X = \frac{6}{4} + \frac{3(m-2)}{6} + \frac{m(m-3)\,2}{9} = \frac{4m^2 - 3m + 9}{18}.$$

Equilateral Covariances: Shared Edges

Three nonzero edge classes

For unordered pairs and $m \geq 4$, local legal-domain counting gives:

Pair type	Number	Joint probability	Covariance
Corner and immediately inner triangle	3	$1/12$	$1/48$
$p = 1/6$ face with a $p = 2/9$ face	$6(m-3)$	$1/18$	$1/54$
Two $p = 2/9$ faces; shared edge interior endpoints	$3(m-2)(m-3)/2$	$2/27$	$2/81$

For any pair τ, σ, enumerate legal labels on $\tau \cup \sigma$ and divide the number making both fully labeled by $\prod_{v \in \tau \cup \sigma} |A(v)|$. Subtract $p_\tau p_\sigma$.

Shared-Edge Covariances: Counting and Sum

Three nonzero edge classes

When all four vertices are interior, the common-edge labels differ with probability $2/3$ and each third vertex matches the missing label with probability $1/3$, giving $2/27$. Boundary cases require their own legal-domain count.

$$C_m^{(e)} = 3/48 + 6(m-3)/54 + \frac{3(m-2)(m-3)}{2}\frac{2}{81} = \frac{m^2}{27} - \frac{2m}{27} - \frac{7}{144}.$$

There is one corner pair per corner, $2(m-3)$ second-type edges per side, and the stated number of remaining interior-interior edges.

Equilateral Covariances: One Shared Vertex

Four nonzero vertex classes

For unordered pairs and $m \geq 4$:

Local pair condition	Number	Joint probability	Covariance
Two $p = 1/6$ faces near different sides of one corner	3	0	$-1/36$
$p = 1/6$ and $p = 2/9$ faces near one corner	6	$1/36$	$-1/108$
Two $p = 2/9$ faces reaching different sides near one corner	3	$5/108$	$-1/324$
Two $p = 2/9$ faces with boundary vertices on the same side	$3(m-4)$	$1/18$	$1/162$

Shared-Vertex Covariances: Counting and Sum

Four nonzero vertex classes

The first and third configurations occur once per corner, the second twice per corner, and the last $m-4$ times per side. All omitted intersecting types have zero covariance; disjoint vertex sets are independent.

$$C_m^{(v)} = -3/36 - 6/108 - 3/324 + 3(m-4)/162 = \frac{m-12}{54}.$$

For the first class, a shared interior vertex would need label 3 for a face based on side 1–2 and label 2 for a face based on side 1–3. Both faces cannot be fully labeled, explaining the joint probability zero.

Equilateral Variance and Small-Mesh Exceptions

Exact moment formulas

The sum of individual Bernoulli variances is

$$D_m = 6\frac{1}{4}\frac{3}{4} + 3(m-2)\frac{1}{6}\frac{5}{6} + m(m-3)\frac{2}{9}\frac{7}{9}$$
$$= \frac{14m^2}{81} - \frac{11m}{108} + \frac{7}{24}.$$

Adding both covariance sums yields

$$\operatorname{Var} X = D_m + 2(C_m^{(e)} + C_m^{(v)}) = \boxed{\frac{80m^2 - 69m - 81}{324}} \qquad (m \geq 4).$$

Uniform Legal Labeling: Small-Mesh Exceptions

Exact moment formulas

For K_3, $D_3 = 37/24$, $C_3^{(e)} = 1/16$, $C_3^{(v)} = -1/12$, giving variance $3/2$. Equivalently use its exact three-point distribution.

For K_1, K_2, $X = 1$ deterministically and variance is zero. For K_2, the expectation is 1, while Sperner gives $X \geq 1$ in every labeling, so equality holds for every outcome.

The local class counts prove the all-m formulas. The delivered verifier checks their legal-domain probabilities and counts through K_8 as a finite implementation audit; it does not replace the class-count argument.

Somerville #1: Expected Score

Exact face-counting proof

Use the corner-only Somerville #1 construction and independently uniform legal labels. At depth $r \geq 1$, there are 3^r final triangles. Face-stellar splitting introduces only interior vertices; the three fixed corners remain the only boundary vertices.

Each of the three outer edges belongs to exactly one final triangle. These are exactly the three triangles with two fixed corners. Every remaining triangle contains at most one fixed corner.

With two distinct fixed labels, the random third vertex completes the triangle with probability $1/3$. With zero or one fixed corner, the fully labeled probability is $2/9$.

$$\mathbb{E}[X_r] = 3 \cdot \frac{1}{3} + (3^r - 3)\frac{2}{9} = \frac{2}{9}\,3^r + \frac{1}{3}.$$

$$\boxed{\frac{\mathbb{E}[X_r]}{3^r} = \frac{2}{9} + 3^{-(r+1)} \qquad (r \geq 1).}$$

Construction: Appendix B43. Uniform-domain counting: Appendix B48.

Boundary Approximation

Bound

Let M be the number of top-dimensional simplices and B_∂ the number containing a vertex on the outer boundary. Let $X^{(d)}$ count fully labeled d-simplices. All other simplices have fully labeled probability p_d.

$$\left|\frac{\mathbb{E}[X^{(d)}]}{M} - p_d\right| \le \frac{B_\partial}{M}.$$

Thus $B_\partial / M \to 0$ gives the interior limit.

For Somerville #1, final-generation scoring gives

$$\frac{\mathbb{E}[X_r]}{3^r} = \frac{2}{9} + 3^{-(r+1)} \qquad (r \ge 1).$$

Appendix B64

Boundary-Heavy Refinement

Limitation

Full barycentric subdivision has a vanishing boundary-simplex fraction.

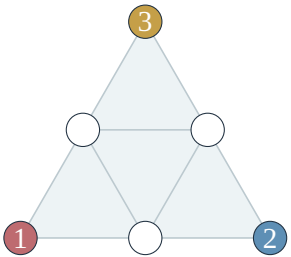
Arbitrary uneven refinements can retain many simplices incident with boundary vertices, even when the maximum simplex diameter tends to zero.

The bound B_∂ / M identifies the needed condition. Vanishing mesh diameter alone does not supply the unconditional limit p_d.

Why K2 Is Score-Invariant

Exact counting and parity

Score invariance means that every legal completion has the same terminal number X of fully labeled triangles. It does not require at most one uncolored vertex per triangle.



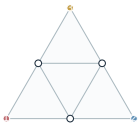
In corner-only K_2, the three side labels are independent fair choices from $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$. Each corner triangle is fully labeled for one of its four assignments. The central triangle is fully labeled for two of eight assignments. Thus each of the four triangles has probability $1/4$ and $\mathbb{E}[X] = 1$.

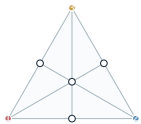
Sperner guarantees $X \geq 1$ in every completion. Every legal completion has positive probability, so a mean of 1 forces $X = 1$ in every completion.

The central triangle initially has three uncolored vertices. This invariant board is therefore not a critical partial position. Conversely, the critical K_4 example has independently variable contributions and is not score-invariant. The following characterization explains the geometry of general corner-only invariant boards.

Score-Invariant Triangulations

Characterization and proof outline





For a nondegenerate straight-line 2D board with **only corners fixed**, every completion has the same score iff

1. Each outer side has at most one noncorner vertex.
2. There is no interior vertex, or one interior vertex whose triangles cone the entire boundary.

The invariant score is $X = 1$.

Local obstruction. A negative-capable triangle is one whose legal domains admit a negative fully labeled triple. That triple extends to a full legal labeling, giving score at least three. Every board also admits a score-one labeling: use label 1 off side 23, label 2 along that side except corner 3, and label 3 there.

Necessity. Two consecutive noncorner vertices on a side admit reversed side labels and an opposite third label. Two interior vertices in a face, or an interior chord in an interior vertex's cyclic link, likewise admit a negative face. Without such a face, that link must be the entire boundary cycle.

Sufficiency. Under the two stated conditions, boundary order, or the selected side of the cone, makes every fully labeled triangle positive. The signed identity then forces exactly one. Full link cases remain in backup; arbitrary partial positions are outside this classification.

Score Invariance: The Local Obstruction

Proof

Assume a nondegenerate straight-line triangular board and only the three corners fixed.

A score-one completion always exists. Label every vertex off side 23 by 1; on that side use 2 except at corner 3. A fully labeled triangle must contain corner 3 and its nearest boundary neighbor on side 23. The unique triangle incident to that boundary edge is therefore the only fully labeled triangle: a simplicial edge cannot skip the neighbor.

A negative-capable face is a variation certificate. Test the three negative CCW label permutations against its legal vertex domains. A successful local assignment extends independently to the rest of the board. Then $N_- \geq 1$, hence $X \geq 3$.

Thus score invariance is equivalent to the absence of a negative-capable face.

If two consecutive noncorner vertices lie on one outer side, reverse their two side labels on the incident face and give its off-side third vertex the opposite corner label. This is negative and legal. Every side consequently has at most one noncorner vertex on an invariant board. Appendix B68

Score Invariance: Why the Only Interior Case Is a Cone

Proof

Two interior vertices in a face allow the two remaining distinct labels in either order, so one orientation is negative. In the absence of negative-capable faces, an interior vertex z therefore has only boundary neighbors.

An interior chord between consecutive neighbors u, v in the cyclic link of z is also impossible. Same-side endpoints cannot form such a straight interior chord. Otherwise, up to renaming, their domains are either

$$\{1\}, \{2,3\}, \{1,2,3\}, \quad \text{or} \quad \{1,2\}, \{2,3\}, \{1,2,3\}.$$

Each permits both orientations of a fully labeled triple. Thus every link edge is an outer boundary edge. A cycle subgraph of the simple boundary cycle must be the whole cycle: the star of z fills the board, leaving no other interior vertex.

Converses. With no interior vertex and at most one noncorner vertex per side, boundary labels have three cyclic blocks in order $1, 2, 3$, so all fully labeled triangles are positive. In the full cone, fixing center label i selects the opposite side's binary word, which changes once. In either case $X = 1$. Appendix B68

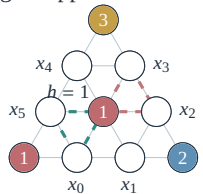
From Local Scores to Payoff Interactions

Representation

For blanks U, the terminal count is $X(\lambda_U) = \sum_\tau \mathbf{1}\{\tau \text{ fully labeled}\}$. Choose a reference legal label at each v and put $\phi_v^a = \mathbf{1}\{\lambda(v) = a\}$ for each other label.

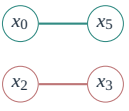
$$X = C + \sum_{1 \le |S| \le 3} \sum_{a_S} c_{S,a_S} \prod_{v \in S} \phi_v^{a_v}.$$

The reference-label indicators form a basis of each local payoff table. Each triangle uses at most three blanks, so no larger support occurs.



The legal K_3 position with $h = 1$ gives

$$X = 1 + 2\big[x_0(1 - x_5) + x_2(1 - x_3)\big].$$



Nonzero pair interactions

Nonzero supports S form the **interaction hypergraph**. A vertex in some support is **active**; otherwise it is **null**, but remains a move.

A **binary quadratic** graph needs binary domains and no cubic coefficients. At most two blanks per face ensures pairwise terms, not binary legal domains.

Deriving the Sparse Payoff Expansion

Representation proof

Let b_v be a reference label in the legal domain at each blank. Write $b[T : a]$ for the complete labeling obtained by replacing precisely the reference labels in T with the specified nonreference labels a_T. Define

$$c_{S,a_S} = \sum_{T \subseteq S} (-1)^{|S|-|T|} X\big(b[T : a]\big).$$

For a completion λ, let $D = \{v : \lambda(v) \neq b_v\}$. Substitution in the proposed expansion gives the coefficient of $X(b[T : \lambda])$ as

$$\sum_{T \subseteq S \subseteq D} (-1)^{|S|-|T|} = \begin{cases} 1 & T = D, \\ 0 & T \neq D. \end{cases}$$

This recovers $X(\lambda)$ and proves uniqueness.

Apply the coefficient formula to one triangle indicator. If some $v \in S$ lies outside that triangle, terms with and without v cancel. Only supports contained in its at most three blanks survive.

Collect contributions from *all* triangles before removing zero coefficients. Cancellation is why payoff-active vertices and interaction edges need not equal the uncolored vertices and physical triangulation edges. Appendix B71

Three Different Graphs and Mandatory Null Moves

Definitions and scope

Physical residual graph: the triangulation's graph induced by its uncolored vertices, retaining physical edges between them.

Collected payoff interaction hypergraph: expand local score indicators in a reference-label basis, sum coefficients across all triangles, delete zero terms, then use each surviving support as a hyperedge. Pair supports give an ordinary graph. An interior vertex may still have three legal labels.

Weak dual: one vertex per triangle, adjacent across a shared physical edge. It is neither of the preceding graphs.

At most two uncolored vertices per triangle gives categorical pairwise terms, not automatically binary variables. A binary quadratic theorem also needs binary domains and vanishing cubic coefficients.

A payoff-null uncolored vertex is still one mandatory move. Generally cancel only pairs of independent null moves and retain one if the number is odd. Arbitrary wait erasure requires a hereditary no-zugzwang hypothesis; the pure-cycle example below shows why.

Preferred Labels and Turn-Order Comparison

Lifting theorem

Write $F = V_F$, $N = V_N$. A game is **no-zugzwang** if $F \geq N$, and **hereditary** if this holds at every follower.

First, a mandatory null move W can be erased from a hereditary game H. Induction on remaining moves gives

$$F(H + W) = \max\{F(H), N(H)\} = F(H), \quad N(H + W) = \min\{N(H), F(H)\} = N(H).$$

Preferred-label lifting

If a legal label a at v satisfies $f(a, x) \geq f(b, x)$ for every other label b and every completion x, and the fixed-label game $H = (v = a)$ is hereditary, then $F(G) \geq N(G)$.

Proof. Replace every terminal payoff by $f(a, x)$ while retaining the move at v. This is exactly $H + W$ on the same action tree. Pointwise comparison survives every max/min, while FC may open $v = a$:

$$\boxed{N(G) \leq N(H + W) = N(H) \leq F(G).}$$

Reverse the payoff comparison for an NFC-preferred label. To infer *heredity*, apply the certificate in every follower; hereditary fixed-label slices provide the needed induction hypothesis.

Null Vertices, Heredity, and Wait Erasure

Induction proof

If a payoff is independent of each null variable separately, it is jointly independent of them: change their labels one at a time. Fixing them legally leaves the same payoff function on the active core and a genuine partial SSG position.

Let W be a one-turn shared null move. For a hereditarily no-zugzwang game H, induct on ordinary moves and then on k. Ordinary followers of $H + kW$ have the values of the corresponding follower of H; the wait follower has the values of H. Thus

$$V_F(H + kW) = \max\{V_F(H), V_N(H)\} = V_F(H),$$
$$V_N(H + kW) = \min\{V_N(H), V_F(H)\} = V_N(H).$$

With no ordinary moves the payoff is constant. The induction applies at every follower.

The two-variable geometric theorem and the three-variable root theorem together make every at-most-three-blank SSG position hereditary. Erasing and restoring the null vertices therefore proves the stronger at-most-three-*active*-vertex result. Appendix B74 Appendix B78

Preferred-Slice Induction

Lifting theorem

1. A label is pointwise preferred by one role at a chosen vertex, against every assignment of the other vertices.
2. Its fixed-label slice and every follower are no-zugzwang.
3. Compare continuations through that slice. Pointwise inequalities survive alternating maxima and minima and lift the next-role inequality.

The argument allows different vertices to have labels preferred by different roles. In particular, all but at most one blank globally dominant vertex is sufficient under the corresponding inductive hypotheses.

Preferred labels do not impose a legal-move rule and need not preserve the exact value of an arbitrary precolored board. Appendix B74

Unary Closure and Face Collars

Geometric constraint

Adding independent unary moves preserves hereditary no-zugzwang, although exact values can change. A legal three-active core plus independent unary vertices remains hereditary.

For a literal triangle whose three vertices are all blank, each edge borders an additional face. Its neighboring fixed collar can create unavoidable extra interaction terms.

Earlier abstract cyclic-face-plus-unary kernels omitted those collar terms. Their abstract values do not by themselves establish a theorem for a genuine Sperner triangulation. Appendix B74

At Most Three Active Vertices

Theorem and proof outline

Theorem. On a nondegenerate straight-line 2D Sperner board, every legal partial position with at most three payoff-active vertices is hereditarily no-zugzwang. Null moves preserve both values.

Two variables: geometry.
Only shared FC indicators R couple labels a, b:

$$M_{ab} = C + u_a + v_b + \sum_{j=1}^{m} R(a, b, c_j), \quad m \le 2.$$

A reversal needs low/high entries in every row and column, with gap at least two. Unary terms cancel on an alternating cycle: its k high entries give $2k \le 4$ from the two shared triangles.

Six-cycles are impossible. Four-cycle equality contradicts the third legal label, or requires a boundary edge with two incident faces. The full extremal matrix argument is in backup.

Three variables: label literals.
A reversal forces low and high coordinate lines in every one-variable slice. Regard each line as a red or blue edge between fixed label literals in parts X, Y, Z.

Opposite-colored lines at one literal must have the same partner part: otherwise they intersect in a contradictory completion.

Each partnership needs two literals on each side. Since each part has at most three labels, three nonempty parts cannot form mutual pairs.

Heredity. The geometric lemma covers two-variable descendants. Fix null vertices legally and restore their turns by wait erasure.

Two Blank Vertices: Alternating-Cycle Bound

Geometric proof

With two blank vertices, the payoff matrix is

$$M_{ab} = C + u_a + v_b + \sum_{j=1}^{m} R(a, b, c_j), \qquad m \le 2.$$

Here u_a, v_b are unary score contributions, not probabilities. Both move orders are legal. If $L = V_F < H = V_N$, every row and column contains an entry at most L and one at least H. All terminal scores are odd, so $H - L \ge 2$.

Direct a low entry from its row to its column, and a high entry from column to row. Every vertex has an outgoing edge; a simple directed cycle has $2k = 4$ or 6 edges, since each domain has at most three labels.

The k highs minus the k lows sum to at least $2k$. Constant and unary terms cancel. A single shared triangle has only two possible nonzero matrix entries, so it contributes at most two to this difference. The full difference is at most four. Thus $k = 3$ is impossible.

If $k = 2$, equality forces $m = 2$, the same fixed third label c on both incident faces, and the off-diagonal a, b rectangle with low diagonal L and high off-diagonal $L + 2$. The remaining domain cases follow on the next page.

Appendix B78

Two Blank Vertices: Excluding Equality

Geometric proof

In the extremal rectangle, unary cancellation forces $u_a = u_b$ and $v_a = v_b$. If exactly one domain also contains the third label c, that additional row or column is constant, contradicting its required low and high entries.

If both domains contain c, put

$$t = M_{ca} = M_{cb}, \quad s = M_{ac} = M_{bc}, \quad d = M_{cc}.$$

All pair terms involving c vanish, so $d = t + s - L$.

- If d is high, both t, s must be low, giving $d \leq L$.
- If d is low, both t, s must be high, giving $d \geq 2H - L > H$.
- If d is intermediate, its row cannot contain both kinds of entry.

The last possibility is that neither domain contains c: both are $\{a, b\}$. The vertices then lie on the same open outer side. Their joining straight edge is a boundary edge, with only one incident face, contradicting $m = 2$.

This is the geometric premise needed for the hereditary three-active theorem; an arbitrary two-variable assignment game can reverse the role values. Appendix B78

Three Variables: The Literal-Part Argument

Assignment-game proof

For $f : X \times Y \times Z \to \mathbb{R}$ with each domain of size at most three, suppose the maximizing-first value is $L < H$, the minimizing-first value.

Fix any opening literal $x = a$. The minimizing reply after an FC opening gives a whole coordinate line of payoff at most L; the maximizing reply after an NFC opening gives a line at least H. This holds for every literal in every part.

Represent a line by an edge between its two fixed literals. Color low lines red and high lines blue. Every literal has neighbors of both colors. Opposite-colored incident edges cannot go to different parts: their lines intersect in a completion that would be both low and high.

One red edge fixes the partner part for all blue edges, and one blue edge then fixes it for all red edges. Each partnership needs two literals from each part, because a literal's red and blue neighbors are different.

A part of size at most three cannot serve two partner parts. Mutual partnerships would therefore partition three nonempty parts into pairs, impossible. Hence $L \geq H$ at the three-variable root. Its two-variable descendants require the separate geometric SSG lemma. Appendix B78

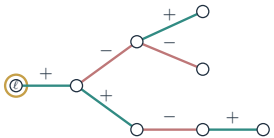
Unit Payoff Forests Admit Elimination

Theorem and proof

Suppose the position admits the binary normalized payoff $Q = (X - 1)/2$,

$$Q(x) = C + \sum_{v} a_v x_v + \sum_{uv \in E} \sigma_{uv} x_u x_v, \quad x_v \in \{0, 1\}, \quad a_v \in \mathbb{Z}, \quad \sigma_{uv} \in \{\pm 1\}.$$

The **payoff graph** has one vertex per variable and an edge for each nonzero pair term. Assume this graph is a forest; no assertion about the full triangulation's one-skeleton is being made.



leaf slice leaves a smaller forest

At a leaf ℓ with neighbor v,

$$\Delta_\ell = Q(1, x) - Q(0, x) = a_\ell + \sigma_{\ell v} x_v.$$

These are consecutive integers. They cannot have opposite strict signs, so one bit is globally preferred.

Fixing either bit deletes the leaf and only changes a neighboring integer unary coefficient. Both slices are smaller unit forests. Induction and preferred-label lifting therefore give $F \geq N$ at every follower.

The integer-unary and unit-edge hypotheses are essential. This is a comparison theorem, not an exact-value algorithm for every forest.

Four-Variable Dominance

Dominance classes

The four-object ownership draft has an exact median formula under global role dominance.

Positive single-target indicators and the specified one-versus-three dominance covers inherit next-role ordering guarantees.

Unit odd binary stars have exact root formulas. Weighted three-leaf stars have proved hereditary ordering under their stated assumptions.

These are scope-restricted results about their payoff kernels. They do not extend the unit-forest theorem to arbitrary edge weights or all four-active positions. Appendix B82

Unary Score Swings

Sufficient condition

For a payoff consisting of independent unary moves and an interacting residual term H, sort the unary swings $\delta_1 \geq \cdots \geq \delta_u \geq 0$ and put $D = \sum_{j=1}^{u} (-1)^{j+1} \delta_j$. Other mandatory moves have zero weight in this unary comparison. Then

$$V_F - V_N \geq D - \operatorname{osc}(H).$$

Unit-target forests allow a sharper structural certificate under the corresponding hypotheses.

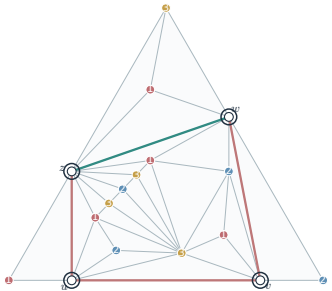
The unique sparse expansion identifies exact active support. Colored links around a vertex expose whether a label is globally preferred, in linear local work with suitable incidence data.

The oscillation bound is a sufficient certificate; a failed bound does not prove reversal. Appendix B71

Indeed each role's value lies between its unary-only value plus $\min H$ and that value plus $\max H$. Subtracting gives the displayed bound.

A Geometric Turn-Order Reversal

Counterexample and proof



Four hollow boundary vertices are playable. Bits $0/1$: $u, v : 1/2$; $w : 2/3$; $z : 3/1$. Every other label is fixed.

Counting the 25 triangles on this 17-vertex board gives

$$q = N_{-} = 3 + u + v - uv - vw + wz - zu.$$

Thus its *payoff graph*, not the triangulation, is a signed four-cycle.

For $(u, v) = (0, 0), (0, 1), (1, 0), (1, 1)$, the four expressions for $q - 3$ are

$$wz, \quad 1 - w + wz, \quad 1 - z + wz, \quad (1 - w)(1 - z).$$

Each is Boolean; every completion has $q = 3$ or 4.

NFC responds	$v = u,\ z = 1 - w$ forces $q = 3$
FC responds	$v = 1 - u,\ z = w$ forces $q = 4$

Both rules answer in disjoint pairs, in either direction. The attainable-score bounds supply the opposite inequalities.

$$\boxed{V_F = 7 < 9 = V_N}.$$

Four-Cycle Response Proof

Strategy proof

Use $X = 1 + 2q$.

$$q = 3 + u + v - uv - vw + wz - zu.$$

Conditioning on (u, v) gives $q - 3$ equal to

(u, v)	$q - 3$
$(0, 0)$	wz
$(0, 1)$	$1 - w + wz$
$(1, 0)$	$1 - z + wz$
$(1, 1)$	$(1 - w)(1 - z)$

Each expression is Boolean, so terminal fully labeled counts are exactly $7, 9$.

NFC second pairs u, v equally and w, z complementarily, giving $q = 3$. FC second reverses these choices, giving $q = 4$. Each move is answered at its untouched mate, so arbitrary interleaving is harmless.

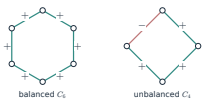
$$\boxed{F = 7, \qquad N = 9}.$$

Pure Signed Payoff Cycles: Reversal Classification

Classification and proof outline

The nonzero pair terms now form a cycle in the *payoff graph*:

$Q = C + \sum_i a_i x_i + \sum_i \sigma_i x_i x_{i+1}$, with $a_i \in \mathbb{Z}$, $\sigma_i \in \{\pm 1\}$, and $\Pi = \prod_i \sigma_i$.



Balanced: $\Pi = +1$. Unbalanced: $\Pi = -1$.

The marginal $a_i + \sigma_{i-1} x_{i-1} + \sigma_i x_{i+1}$ takes both signs exactly at

$a_i = -\frac{\sigma_{i-1}+\sigma_i}{2}$ (**marginal-critical**).

Otherwise a bit is preferred. Fixing it gives a unit path; forest elimination and lifting prove hereditary no-zugzwang.

Fully critical case. Substitution gives $Q - C = -\frac{1}{2} \sum_i \sigma_i (x_i \oplus x_{i+1})$.

Switch bits to leave zero or one negative edge. A *defect* is disagreement on $+$, agreement on $-$. Its count R is respectively even or odd; Q decreases by $R/2$.

The opening cuts the cycle to a path. Frontier play gives the unique $R \in \{r, r+1\}$ with this parity. NFC's remaining moves are $r = \lfloor n/2 \rfloor$ after FC opens, or $\lfloor (n-1)/2 \rfloor$ after NFC opens.

Comparing these counts gives $F < N$ exactly at

balanced: $n \equiv 2 \pmod 4$,
unbalanced: $n \equiv 0 \pmod 4$.

Both residue classes have genuine geometric families via a four-vertex boundary extension. Their triangulations and exact value formulas are in backup.

Critical-Cycle Value Formulas

Value formulas

Pure cycle: the n binary variables are all remaining moves. Do not silently remove an odd null wait.

Marginal criticality is $a_i = -(\sigma_{i-1} + \sigma_i)/2$. Complementing bits switches incident signs and changes the constant to C'. Let q_F, q_N denote normalized-payoff values here.

	q_F	q_N
balanced	$C' - \lfloor (n+2)/4 \rfloor$	$C' - \lfloor (n+1)/4 \rfloor$
unbalanced	$C' - \lfloor n/4 \rfloor$	$C' - \lfloor (n-1)/4 \rfloor$

For a collected $q = N_-$, recover the optimal fully labeled counts by $V_F = 1 + 2q_F$ and $V_N = 1 + 2q_N$.

A noncritical vertex has a preferred bit; fixing it opens a forest. A first move on a critical cycle leaves a path, where signed-frontier replies determine the bounds and parity selects the exact value.

The next-role reversal classes are balanced $n \equiv 2 \pmod 4$ and unbalanced $n \equiv 0 \pmod 4$. All proper followers are forests. Appendix B87

Geometric Cycle Extension

Construction

The balanced C_6 seed has

$$q = 5 + \sum_i x_i - \sum_i x_i x_{i+1}, \qquad (V_F, V_N) = (13, 15).$$

The unbalanced C_4 seed is the 17-vertex construction with values $(7, 9)$.

Choose a boundary interaction edge with the same binary convention at both endpoints and the missing third label on its incident face. Insert four new blank boundary vertices and cone the subdivided edges to that fixed third vertex.

This stretches the interaction while preserving criticality and the reversal residue. Repeating gives actual fully labeled counts

balanced C_{4k+2}	$(2k + 11, 2k + 13)$
unbalanced C_{4k}	$(2k + 5, 2k + 7)$

$(k \geq 1)$.

A Null Move Can Change a Cycle Value

Mandatory-move counterexample

The genuine four-boundary-variable position has normalized payoff

$$q = 3 + u + v - uv - vw + wz - zu, \qquad (V_F, V_N) = (7, 9).$$

Every terminal score is 7 or 9. Add one independent mandatory zero-score move W:

$$V_F(G + W) = 9, \qquad V_N(G + W) = 7.$$

FC can spend W and leave NFC next in G, attaining 9; no completion exceeds 9. NFC can spend W and leave FC next, attaining 7; no completion falls below 7.

A genuine null vertex can be inserted into a fixed monochromatic triangular pocket: its three fixed neighbors have the same label, so no incident triangle can be fully labeled, whichever legal label is chosen. It is still one move.

Thus the active cycle alone does not determine the value. Pure-cycle formulas permit no unaccounted extra moves or external components.

Finite Values and Full-Domain Collar Limits

Evidence status

Original board	Values	Evidence
K_1, K_2, K_3	$V_F = V_N = 1$	Direct proofs
K_4	$V_F = V_N = 3$	Direct lower and upper strategies
K_5	$V_F = 5$	Finite computation only
Somerville #1, $r = 0, 1, 2$	1, 1, 3 for both roles	Direct proofs
Somerville #1, $r = 3$	7 for both roles	Finite computation only

The attainable complete K_4 scores are $1, 3, 5, 7, 9$, rather than a single invariant score. No general exact K_m formula for $m \geq 5$ or full-depth Somerville #1 strategy is claimed.

Abstract full-domain face kernels require their genuine geometric collars before literal SSG values can be inferred. Appendix B29 Appendix B44

Immediate-Score Greedy Counterexample

Counterexample

Corners are $(0,0):1$, $(3,0):2$, $(0,3):3$. Fixed labels are $(0,2):1$, $(1,2):2$, $(2,0):1$, $(2,1):3$. The three blanks are

$$x=(0,1)\in\{1,3\},\qquad y=(1,0)\in\{1,2\},\qquad z=(1,1)\in\{1,2,3\}.$$

Two already fixed fully labeled triangles contribute 2. Every label at z immediately completes one further fully labeled triangle. Yet a center opening lets NFC cap the terminal score at 3.

Playing $x=3$ or $y=2$ scores nothing immediately but forces terminal score 5.

FC-first value $=5$ while every immediate-score greedy opening loses value.

This refutes immediate completed-face count as a universal optimal-move rule, even with only three remaining moves.

Completion Reduction: Source, Symbols, and Example

Reduction definitions

Source: Planar Independent Set. Given $G = (V, E)$ and integer q_0, ask whether at least q_0 vertices can be chosen with no adjacent selected pair. Write $\alpha(G)$ for the largest such size.

For a canonical completion, $S = \{v : \lambda(v) = 2\}$ is selected and label 1 means unselected. Replacing label 3 by 1 cannot lower the constructed completion score. Define $e_G(S) = |E(G[S])|$, the number of edges with both endpoints in S; $G[S]$ is the induced subgraph on S.

$$h_v = \deg_G(v) - 1, \quad h_{\text{tot}} = \sum_v h_v, \quad B = 1 + 2h_{\text{tot}}, \quad X = B + 2(|S| - 2e_G(S)).$$

Preprocess isolated source vertices and adjust q_0 before applying $h_v \geq 0$.

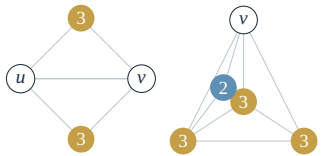
For the path $u - v - w - z$, $B = 5$. The selection $\{u, v, z\}$ has three vertices and one internal edge, giving $X = 7$. Delete v: $\{u, z\}$ has two vertices and no internal edge, giving $X = 9$. Here $\alpha(G) = q_0 = 2$ and the completion threshold is $k = 9$. This is a miniature illustration, not a restriction of the hardness source class.

NP-Completeness of SSG-COMPLETION

Reduction and proof outline

Start with planar Independent Set (G, q_0), with no isolated vertices. Put $h_v = \deg_G(v) - 1$, $h_{\text{tot}} = \sum_v h_v = 2|E| - |V|$, and $R(a, b, 3) = \mathbf{1}\{a, b, 3 \text{ distinct}\}$.

$$X = 1 + 2 \sum_{uv \in E} R(\lambda(u), \lambda(v), 3) + 2 \sum_{v} h_v \mathbf{1}\{\lambda(v) = 1\}.$$



Literal triangulated patches; hollow vertices are variables. Each face has its ordinary unit score.

Two distinct fixed-3 pins give the edge term. Two stellar insertions in an inactive $(v, 3, 3)$ face give one repeatable unary unit. A triangulated fixed Sperner annulus contributes exactly one; all filler scores are counted.

Changing label 3 to 1 cannot lower the score. For $S = \{v : \lambda(v) = 2\}$, the edge cut and unary sum give

$$\boxed{X(S) = 1 + 2\big(h_{\text{tot}} + |S| - 2e_G(S)\big)}.$$

If v lies on a selected edge, deleting it increases $|S| - 2e_G(S)$ by $-1 + 2\deg_{G[S]}(v) \geq 1$. Thus $\max X = 1 + 2(h_{\text{tot}} + \alpha(G))$, and with $k = 1 + 2(h_{\text{tot}} + q_0)$, a threshold-$k$ completion exists iff $\alpha(G) \geq q_0$. The scaffold and unary units have polynomial size; Tutte embedding gives rational coordinates. All blanks are interior with three legal labels. This proves NP-hardness for **partial positions**; the labeling certificate gives membership.

Planar Realization for Completion

Construction

For a plane graph G and nonnegative integer h_v, the construction gives

$$X(\lambda) = 1 + 2 \sum_{uv \in E(G)} R(\lambda(u), \lambda(v), 3) + 2 \sum_{v} h_v \mathbf{1}\{\lambda(v) = 1\}.$$

1. Embed and triangulate a sphere scaffold containing G.
2. Subdivide unwanted active–active edges with fixed label-3 pins; each desired edge has two fixed-third-color scored faces.
3. Add repeatable unary pockets contributing $2\mathbf{1}\{\lambda(v) = 1\}$.
4. Puncture an all-3 face. An outer $1, 2, 3$ Sperner annulus contributes exactly one fixed fully labeled triangle.

Tutte's rational barycentric embedding supplies straight-line coordinates of polynomial bit length. Blanks are interior, every triangle has at most two blanks, and the residual graph is exactly G (maximum degree at most six in the source class). Appendix B94

Completion Reduction: A Literal Unary Unit and Frame

Geometric score proof

Start with an inactive triangle (v, a, b) whose two fixed labels are 3. Insert $r = 3$, then $z = 2$ in (v, a, r). The replacement faces are

$$(v, a, z), \quad (a, r, z), \quad (r, v, z), \quad (v, r, b), \quad (a, b, r).$$

Only the first and third can be fully labeled, both exactly when $\lambda(v) = 1$. Thus the score is $2\mathbf{1}\{\lambda(v) = 1\}$, while (v, r, b) remains an inactive pocket for repetition. Every unit costs two fixed vertices.

In a triangulated spherical scaffold, each source edge has two distinct fixed-3 incident pins. Unwanted scaffold edges are subdivided by fixed vertices, removing their variable interaction. Unary pockets preserve the source-edge score.

Puncture a fixed all-3 face *abc*. Add outer corners A, B, C labeled $1, 2, 3$ and the six annular faces

$$(A, B, b), (A, b, a), (B, C, c), (B, c, b), (C, A, a), (C, a, c).$$

Exactly (A, B, b) is fully labeled. The frame contributes one, every variable is interior, and all other fixed contributions have been counted. The completed maximal planar graph has a rational straight-line realization by Tutte embedding, with polynomial coordinate bit length. Appendix B94

Independent-Set Score Identity

Reduction proof

Set $h_v = d_G(v) - 1 \geq 0$ and $h_{\text{tot}} = \sum_v h_v = 2|E| - |V|$. Replacing any label 3 by 1 cannot lower the completion score. For a binary completion, $S = \{v : \lambda(v) = 2\}$ gives

$$\sum_{uv \in E} R(\lambda(u), \lambda(v), 3) = \sum_{v \in S} d_G(v) - 2e_G(S),$$

$$\sum_v h_v \mathbf{1}\{v \notin S\} = h_{\text{tot}} - \sum_{v \in S} (d_G(v) - 1).$$

Adding cancels the degrees:

$$X = 1 + 2(h_{\text{tot}} + |S| - 2e_G(S)).$$

An internal selected edge permits a deletion with gain $-1 + 2d_{G[S]}(v) \geq 1$. Therefore the optimum is $1 + 2(h_{\text{tot}} + \alpha(G))$ and threshold $1 + 2(h_{\text{tot}} + q_0)$ encodes $\alpha(G) \geq q_0$.

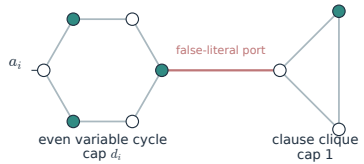
This proves completion hardness. Alternating game hardness requires the separate game reduction. Appendix B94

Source problem: Garey, Johnson, and Stockmeyer (1976)

A Pairwise Score for Planar QBF

Verifier construction

Design the score that SSG will realize. Let $\Phi = Q_1 b_1 \cdots Q_n b_n \, \varphi$ be a planar quantified 3-CNF formula with c clauses.



Verifier graph, not the triangulation. Clause vertices attach to false ports.

Replace variable b_i with an even cycle of length $2d_i$, $d_i = \max(2, \text{occurrences})$, in incidence order. A distinguished bit a_i is its master. Replace each clause by a clique of size at most three, joined to its false ports. The verifier $G_\Phi = (V_\Phi, E_\Phi)$ is planar and subcubic.

$$H_{G_\Phi}(x) = \sum_v x_v - \sum_{uv \in E_\Phi} x_u x_v, \qquad T = \sum_i d_i + c.$$

Equality certificate. On an even cycle, selected runs each contribute one, so the score is at most d_i; equality forces an alternating truth pattern. Selecting $t \le 3$ clause vertices gives $t - \binom{t}{2} \le 1$. At equality, incidence penalties vanish, so each clause selects a true literal.

$$\max_{x:\, x_{a_i} = b_i} H_{G_\Phi}(x) = T \iff \varphi(b) = 1; \qquad \varphi(b) = 0 \Rightarrow \max H_{G_\Phi} \le T - 1.$$

Thus prefix-order masters followed by FC-controlled witnesses simulate Φ.

Source problem: Lichtenstein (1982)

Planar Verifier Equality Case

Verification lemma

Let c be the clause count. For occurrence count r_i, use a length-$2d_i$ cycle with $d_i = \max(2, r_i)$, in source embedding order. The distinguished master is $a_i = t_{i,1}$.

A binary even cycle contributes at most d_i to $\sum x_v - \sum x_u x_v$. For a nonconstant word, each run of ones contributes one; equality requires an alternating pattern.

A clause clique with at most three vertices contributes $q - \binom{q}{2} \le 1$. A clause vertex attaches to the occurrence port selected exactly when its literal is false. Incidence penalties vanish at a selected clause literal exactly when that literal is true.

$$\max_{x:\, x_{a_i} = b_i} H_{G_\Phi}(x) = T \iff \varphi(b) = 1, \quad T = \sum_i d_i + c.$$

If false, the integer maximum is at most $T - 1$. Masters are assigned in the QBF prefix; all remaining verifier bits are later controlled by FC. Appendix B98

Free Vertex Choice and Quantifier Controllers

Exchange lemma and simulation

In an auxiliary ownership game, $x_v = 1$ if FC takes item v, and 0 if NFC takes it. Every unused item is selectable. Its payoff is

$$\mathcal{P}(x) = \sum_v w_v x_v + R(x), \qquad \operatorname{osc}(R) = \max R - \min R.$$

Exchange. If $w_u - w_v \geq \operatorname{osc}(R)$, replace an opening at v by one at u and transpose their names throughout every continuation. The legal trees correspond. Equal owners leave the score unchanged; different owners improve it for the first mover by at least

$$w_u - w_v - \operatorname{osc}(R) \geq 0.$$

Hence restricting optimal play to the highest remaining priority group preserves minimax value; it introduces no move rule.



Give each logical v an isolated companion v' of equal weight. Whoever starts the two-item group can give v either ownership bit by choosing v or v'. Exactly one item goes to FC, so the group's weight contribution is fixed.

A pair consumes two moves and preserves the next controller. A singleton consumes one and flips it. Put master groups in prefix order, then all witness groups under FC control. With $R = 4H_{G_\Phi}$, gaps $D = 4(|V_\Phi| + |E_\Phi|) + 2$ suffice; $w_j = D(h - j)$ is polynomial. The fixed priority total is W_0.

Priority Transposition Proof

Exchange lemma

Let π exchange two unclaimed items u, v and fix every other item. Compare a virtual strategy beginning at v with the real strategy beginning at u, translating every opponent move by π before consulting the virtual continuation.

If the two items finish with the same owner, no terminal bit changes. Otherwise, starting with u rather than v transfers the larger weight to the initiating role's advantage. The change in the residual is at most $\operatorname{osc}(R)$.

$$w_u - w_v \geq \operatorname{osc}(R) \quad \Longrightarrow \quad u \text{ weakly dominates } v \text{ for either role.}$$

Previous assignments are fixed by the transposition. Thus the comparison holds after every history and against every opposing continuation, including preemption.

Induction on remaining items permits restriction to the highest nonempty priority group without changing the unrestricted minimax value. It does not assert that every optimal strategy must respect that order. Appendix B100

Quantifier Simulation

Quantifier simulation



Let $\mathcal{W}$ contain all logical items, companions, and parity dummies. Each logical vertex v has an isolated companion v'. In a size-two group, the controller chooses which one it claims, so it can set the ownership bit of v either way. Exactly one item is claimed by FC and the group contributes its common weight.

Two moves preserve the next controller; one singleton flips it. Master groups follow the QBF prefix. All witness groups have FC as controller.

With g logical vertices, e edges, and h groups, choose

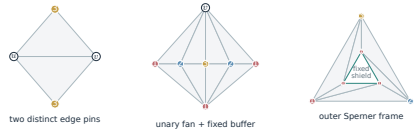
$$R = 4H_{G_\Phi}, \quad B_R = 4(g+e), \quad D = B_R + 2, \quad w_v = D(h-j).$$

Then $\mathrm{osc}(R) \le B_R < D$, and W_0 is independent of truth choices. All vertex choices remain legal. Appendix B100

Realizing the Ownership Score Geometrically

Unit-score construction

Inside a fixed label-1 shield, subdivide unwanted active edges and star faces. Only source edges retain two-variable triangles.



Local triangulated patches. The fixed buffer side and outer annulus are included in the constant score.

Two fixed-3 pins per edge give $2R(\lambda(u), \lambda(v), 3)$. Fixed-path fans correct the edge-induced unaries to

$$(U_v(1), U_v(2), U_v(3)) = (2, 2 + h_v, 2 + J),$$

$$h_v = L + w_v + 4\mathbf{1}_{v \in V_\Phi} - 2d_v, \quad L = 8(\Delta + 1),\ J = 4(\Delta + 1),$$

where $d_v = \deg_{G_\Phi}(v)$ on V_Φ and 0 otherwise and $\Delta = \max d_v$. The path's 23, 13, 12 edge counts specify these rewards. Thus $X = C_0 + \sum_v U_v + 2\sum_{uv} R$ before dominance.

Canonical labels, unchanged legality. Both unary gaps exceed $2d_v$, the largest incident pair-score change. Thus label 2 is pointwise best for FC, label 1 for NFC; *all three labels remain legal*. Their bits satisfy $R = x_u + x_v - 2x_u x_v$. Cancelling the degree terms gives

$$\boxed{X = C_{\text{geom}} + Lr + \sum_v w_v x_v + 4H_{G_\Phi}(x)},$$

where r is FC's fixed number of turns. For formula size s, the unary paths use $O(s^3)$ vertices in total, with polynomial-bit rational coordinates. Complete path counts and buffer triangulations are in backup.

Embedding: Tutte (1963); de Fraysseix, Pach, and Pollack (1990)

Geometric Priority Fans

Construction

For nonlogical items set $d_v = 0$. Put $d_v = \deg_{G_\Phi}(v)$, $\Delta = \max d_v$, $L = 8(\Delta + 1)$, $J = 4(\Delta + 1)$, and $h_v = L + w_v + 4\mathbf{1}_{v \in V_\Phi} - 2d_v$.

$$X = C_0 + \sum_v U_v(\lambda(v)) + 2 \sum_{uv \in E_\Phi} R(\lambda(u), \lambda(v), 3),$$

$$(U_v(1), U_v(2), U_v(3)) = (2, 2 + h_v, 2 + J).$$

The private fixed-label path is

$$(1, 2, 3, 2, 1), \quad (3, 1)^{a_2/2}, \quad (2, 1)^{J/2}, \qquad a_2 = 2 + L + w_v + 4\mathbf{1}_{v \in V_\Phi} - 4d_v.$$

Its counts of $23, 13, 12$ edges are $2, a_2, 2 + J$. Cone to v on one side and the fixed buffer vertex on the other. The buffer contributes a counted constant.

The inner shield is fixed to label 1. A six-face outer Sperner annulus supplies the odd constant. Rational barycenters, midpoints, and equally spaced path points preserve polynomial coordinate bit length. Appendix B103

Ternary Label Dominance and Threshold

Dominance lemma

A single vertex can change its incident pair terms by at most $2d_v$. The unary gaps satisfy

$$J > 2d_v, \qquad h_v - J \geq 2\Delta + 4 > 2d_v.$$

Thus label 2 is pointwise best for FC, and label 1 for NFC, against every continuation. All three remain legal.

Set $x_v = \mathbf{1}\{\lambda(v) = 2\}$. Then $R(\lambda(u), \lambda(v), 3) = x_u + x_v - 2x_u x_v$. Degree terms cancel, giving

$$X = C_{\text{geom}} + Lr + \sum_v w_v x_v + 4H_{G_\Phi}(x), \qquad C_{\text{geom}} = C_0 + 2|\mathcal{W}|.$$

The FC turn count r is fixed. Under the value-preserving priority restriction the weight sum is W_0. Choose

$$k = C_{\text{geom}} + Lr + W_0 + 4T.$$

True formulas have value k; false formulas at most $k - 4$. For source size s, weights are $O(s^2)$ and the actual unweighted triangulation has $O(s^3)$ size. Appendix B106

PSPACE-Completeness of SSG-VALUE

Theorem and proof outline

Theorem. SSG-VALUE is PSPACE-complete for arbitrary legal two-dimensional partial positions, even with every blank interior, at most two blanks per triangle, and a subcubic graph induced by the blanks.

1. **Labels:** pointwise unary dominance identifies legal optimal SSG play with the ownership game, plus the constant $C_{\text{geom}} + Lr$.
2. **Order:** transposition permits group-respecting optimal strategies. Choice pairs and singletons assign the masters to the QBF controllers and the later witnesses to FC. Their priority contribution is the known constant W_0.
3. **Truth:** the verifier reaches T exactly when the resulting matrix is true. Alternating control of the masters reproduces the QBF prefix.

$$k = C_{\text{geom}} + Lr + W_0 + 4T, \qquad \Phi \text{ true} \Rightarrow \text{Val} = k, \quad \Phi \text{ false} \Rightarrow \text{Val} \leq k - 4.$$

The output size is polynomial, every simplex has unit score, and every SSG vertex choice remains legal. This is a many-one reduction from planar QBF; polynomial-depth minimax supplies membership.

The fixed interior labels are part of the input. The theorem gives no hardness claim for corner-only K_m or Somerville #1 starts.

Source problem: Lichtenstein (1982)

Completion and Adversarial Value

Decision problems

Input: an explicit 2D triangulation, legal partial labels, and threshold k.

Problem	Threshold question
SSG-COMPLETION	Does a full legal labeling with $X \geq k$ exist?
SSG-VALUE	Can P_{FC} force $X \geq k$ against P_{NFC}? The next role is part of the input.

COMPLETION is in NP.
The remaining labels are a polynomial-size certificate. Check each domain and count fully labeled triangles in one scan.

VALUE is in PSPACE.
With u blanks, every play has exactly u moves. Depth-first minimax stores one position, its depth-u continuation stack, and a score of at most the face count.

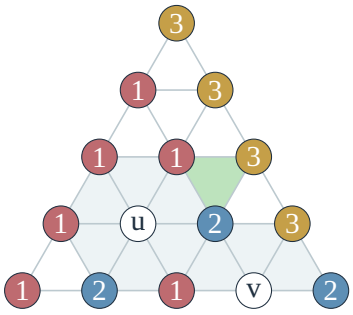
$$\exists \text{ a favorable labeling} \qquad \text{versus} \qquad \exists m_1\, \forall m_2\, \exists m_3 \cdots$$

The displayed alternation starts with FC; reverse it if NFC is next. Completion hardness alone does not prove hardness of this adversarial game.

Critical Positions: A Literal Small Example

Definition and exact counts

Critical partial position: each triangle contains at most one uncolored vertex.



	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	swing
u	0	0	4	4
v	2	0	illegal	2

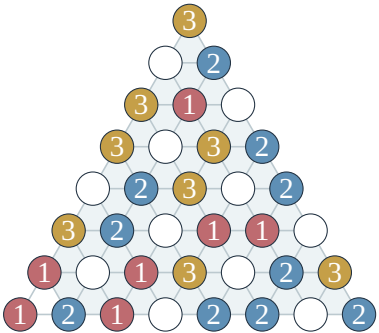
One already fully labeled triangle is fixed. Each table entry counts only the triangles incident to its uncolored vertex.

$$X = 1 + 4\mathbf{1}_{\{\lambda(u)=3\}} + 2\mathbf{1}_{\{\lambda(v)=1\}}.$$

All six completions are checked. The attainable counts are $1, 3, 5, 7$, so this position is not score-invariant. Largest swing first gives $V_F = 5$ and $V_N = 3$.

Critical Positions of Arbitrary Size

Construction and verification



This literal K_7 has 49 triangles and 11 uncolored vertices. Its fixed labels are legal; every triangle has at most one blank. Its full label and contribution tables are in the delivered certificate.

For any K_m, assign lattice vertex (i, j) the auxiliary class $i + 2j \pmod 3$. The vertices of each elementary triangle have three different classes. Keep one class uncolored except the prescribed corners, and legally fix every other vertex. No triangle can contain two blanks. This supplies critical examples for arbitrarily large m.

Why the Contributions Are Independent

Proof and scope

Pairwise nonadjacent blanks give at most one blank per triangle: each pair of vertices in a triangle shares an edge. Conversely, every edge belongs to a triangle, so adjacent blanks would violate criticality.

In an unresolved triangle, the other two labels are fixed. Its indicator depends on just its sole uncolored label. Grouping triangle terms by their blank gives the exact unary decomposition on the next page. A choice at one blank changes neither another blank's contribution nor its legal labels.

Leave an independent set of noncorners blank and legally fix the rest. This constructs supplied partial positions; it does not force those prelabels from corner-only starts.

Invariance requires the same X in all legal completions. Critical positions may have nonzero swings, as on K_4. Marginal criticality of signed cycles is a separate algebraic condition.

Independent-Vertex Optimal Strategies

Algorithm and proof

At a critical partial position (at most one blank per triangle), its other labels are fixed, so

$$X = C + \sum_{v \in U} f_v(\lambda(v)), \quad b_v = \min_{a \in A(v)} f_v(a), \quad \delta_v = \max_{a \in A(v)} f_v(a) - b_v.$$

Here $f_v(a)$ counts incident triangles made fully labeled by label a; C counts already fixed fully labeled triangles.

Local choice. Assigning an extreme label affects no other variable. After subtracting $B_{\text{ind}} = C + \sum_v b_v$, taking v contributes δ_v if FC takes it, and zero if NFC takes it.

Why largest swing first? For $\delta_i \geq \delta_j$, transpose i, j in every future response. If their final owners agree, the score is unchanged. Otherwise the first mover gains the weight difference in its preferred direction. This strategy comparison works against every opposing continuation, so both roles may always take a largest remaining swing.

$$\delta_1 \geq \cdots \geq \delta_u : \qquad V_F = B_{\text{ind}} + \sum_{j \text{ odd}} \delta_j, \quad V_N = B_{\text{ind}} + \sum_{j \text{ even}} \delta_j.$$

One face scan and a sort give value and optimal play in $\boxed{O(M + u \log u)}$, $M = |\mathcal{S}_2(K)|$. With two blanks per face, this separation can fail.

Independent-Vertex Value Formulas

Value formulas

Assume every triangle contains at most one blank. Write $X = C + \sum_v f_v(\lambda(v))$ and set

$$b_v = \min f_v, \quad \delta_v = \max f_v - b_v, \quad \delta_1 \geq \delta_2 \geq \cdots \geq \delta_u.$$

Then

$$V_F = C + \sum_v b_v + \sum_{j \text{ odd}} \delta_j, \qquad V_N = C + \sum_v b_v + \sum_{j \text{ even}} \delta_j.$$

Both roles take a largest swing. Swapping a lower-ranked choice with a larger remaining swing gives an alternating sum of nonnegative adjacent differences.

Every follower retains the independence condition, so the rule is subgame-perfect. Zero swings are real moves but may be taken after positive swings optimally. Appendix B111

Residual-Graph Algorithms

Algorithms

Restriction	Exact algorithmic conclusion
p vertices in nonsingleton residual components	FPT in p: 4^p-scale core states and polynomial board dependence. Independent vertices enter sorted queues.
Residual vertex cover c	Polynomial time for every fixed c. Outside vertices have bounded interaction types and sorted queues. The proved dependence is **XP**, rather than FPT in c.
Bounded residual component size and bounded triangle incidence	Polynomial time via component types and compressed multiplicities.

The residual graph is induced by uncolored vertices, not the dual graph. Bounding its degree does not bound full vertex degree or the unary fan sizes used by the hardness reduction. Appendix B116

Residual Vertex Cover: Role-Dependent Pruning

Algorithm proof: bounded interactions

Let $S \subseteq U$ cover every residual edge, with $|S| = c$ and $I = U \setminus S$ independent. For complete cover labels x,

$$X = C + g(x) + \sum_{v \in I} \big[u_v(\lambda(v)) + h_v(\lambda(v), x) \big].$$

Each triangle in h_v uses an edge from v to S, and each such edge has at most two incident faces. Therefore $0 \le h_v \le 2c$, irrespective of the unary score.

$$a_v = \max_{\ell} u_v(\ell), \qquad b_v = \min_{\ell} u_v(\ell).$$

FC may discard: $u_v(\ell) < a_v - 2c$,

NFC may discard: $u_v(\ell) > b_v + 2c$.

A maximizing unary label pointwise dominates the first set; a minimizing unary label pointwise dominates the second, since the interaction can change by at most $2c$. These are different, role-dependent option sets. Keep all options not eliminated by the relevant test.

Residual Vertex Cover: Types and Interleaving

Algorithm proof: XP state compression

Retain the notation of the preceding page. Expand each h_v in the cover's single-label and adjacent-pair indicators. Planarity gives $O(c)$ coordinates, and every coefficient is a bounded integer.

For FC, a surviving label has offset $\alpha = u_v(\ell) - a_v \in \{-2c, \ldots, 0\}$. For NFC, it has offset $\beta = u_v(\ell) - b_v \in \{0, \ldots, 2c\}$. A vertex's type records both role-specific sets of offsets and interaction vectors. There are finitely many types $T(c)$, each retaining at most three labels per role.

Within one type, vertices differ only by their nonnegative unary swing $a_v - b_v$. Transposition and terminal comparison permit taking the largest such swing first, even while cover vertices remain blank.

A DP stores partial cover labels, one index per sorted type queue, and the pending interaction vector. Every action advances a queue or labels the cover, preserving arbitrary interleaving. The ranges are polynomial for fixed c, but the exponent depends on c: the conclusion is XP, without an FPT claim in c.

Full proof: Complexity Research I, Section 5, especially Sections 5.3–5.5.

Complexity Boundaries and Open Families

Scope

Let the **residual graph** be the triangulation graph induced by its uncolored vertices. It is distinct from the collected payoff-interaction graph, where terms may cancel.

Input class	Result and mechanism
At most one blank per triangle	Polynomial-time value and optimal strategy: independent swings.
Residual vertex cover c	Polynomial for fixed c (**XP**, not an FPT claim): branch on cover states; group outside vertices by bounded interaction types and sorted unary queues.
Arbitrary legal 2D partial positions	COMPLETION NP-complete; VALUE PSPACE-complete by separate reductions.
Corner-only natural families	General K_m and full-depth Somerville #1 optimal strategies remain open.

The hardness instances use **precolored interior vertices**. Their subcubic residual graph does not bound the full triangulation degree: the unary fans supply large fixed neighborhoods.

The geometric question remains: which natural boards admit global optimal strategies?

Complexity Results and Scope

Scope

Result	Scope	Method
Restricted exact algorithms	Explicit legal partial positions; parameter hypotheses as stated	Exact algorithms
COMPLETION NP-complete	Arbitrary legal 2D partial positions, interior blanks, at most two blanks per face, residual degree at most six	Completion reduction
VALUE PSPACE-complete	Arbitrary legal 2D partial positions, interior blanks, at most two blanks per face, residual degree at most three	Game reduction

The SSG results are presented as results of this work.

Corner-only starts, general K_m, and full-depth Somerville #1 remain separate unresolved problems.

Research Scope and Corrections

Scope

Claim type	Examples and limits
Direct strategy proof	Interval frontiers; K_3 threat pairs; full K_4 responses; unit forests and signed cycles.
Complete symbolic certificate	D_2, D_3 residue formulas with proved contextual reduction and all-parameter symbolic coverage.
Isolated finite computation	K_5 FC-first value 5; depth-three Somerville #1 value 7.
Scope-restricted general theorem	Three-active ordering; unit-weight forest assumptions; NFC-last matching bound.
Unsettled extension	Hardness from corner-only starts; exact values for general natural meshes; unconditional FPT in residual vertex cover.

Every literal full-domain active triangle needs its geometric collar. Every hardness instance retains all standard legal labels and unrestricted vertex selection. Appendix B116

Literature Context

References

Source	Role in the talk
Sperner (1928); Kannai (2013)	Boundary labeling and oriented cancellation.
Garey–Johnson–Stockmeyer (1976)	The planar independent-set source problem.
Lichtenstein (1982)	The planar quantified-formula source problem.
Tutte (1963); de Fraysseix–Pach–Pollack (1990)	Rational straight-line and grid embeddings.

Related game: Burke–Teng, *Atropos* (2008), has different movement and terminal rules; its hardness is not used as an SSG hardness transfer. Appendix B120

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