Hyper-Bishops, Hyper-Rooks, and Hyper-Queens: Percentage of Safe Squares on Higher Dimensional Chess Boards

Jenna Shuffelton

jms13@williams.edu
Williams College

SMALL REU 2024
Advances in Interdisciplinary Statistics and Combinatorics,
October 12th, 2024

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What is the percentage of safe squares on an $n \times n$ board with n rooks placed? What about n bishops? Or queens? What do these problems look like with higher dimensions?

Miller, Sheng, and Turek found that when placing n rooks on an $n \times n$ board, the percentage of safe squares converged to $1/e^2$ as $n \to \infty$.

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$$S_n(\mathcal{B}) := \sum_{x_1,\ldots,x_n=1}^{n} X_{x_1,\ldots,x_n}(\mathcal{B}).$$

$$\mathbb{E}[S_n] = \sum_{x_1,\ldots,x_n=1}^{n} \mathbb{E}[X_{x_1,\ldots,x_n}(\mathcal{B})].$$

$$\mu_n := \frac{1}{n^k} \sum_{x_1,\ldots,x_n=1}^{n} \mathbb{E}[X_{x_1,\ldots,x_n}(\mathcal{B})].$$

Higher Dimension Chessboards

Definition

A k-dimensional board has k dimensions with equal integer side length n. Boards are created by stacking alternating boards in the $(k-1)$-dimensional subspace so that no two adjacent squares are the same color.

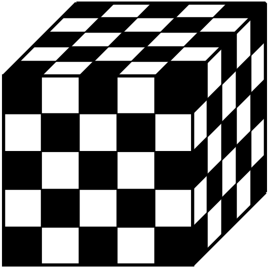

Figure: Depiction of a $5 \times 5 \times 5$ chessboard.

Combinatoric Preliminaries

Combinatorial Limit

Miller, Sheng, and Turek showed that for $a, b \in \mathbb{Z}$, with a positive,

$$\lim_{n\to\infty} \binom{n^2 - an - b}{n} \Big/ \binom{n^2}{n} = \frac{1}{e^a}.$$

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Generalized Combinatorial Limit - Cashman, Cooper, Marquez, Miller, Shuffelton

For positive integers a, k, m, c, d and any integer b, with $k > m > k - c$, we have

$$\lim_{n\to\infty} \binom{n^k - an^m + bn^{k-c}}{dn^{k-m}} \Big/ \binom{n^k}{dn^{k-m}} = \frac{1}{e^{da}}.$$

Interpreting the Combinitorial Limit

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n^k	Total spaces on chess board.
$an^m - bn^{k-c}$	Spaces attacked by piece.
dn^{k-m}	Pieces placed on board.

Count setups where a space is safe, divide by total configurations, end with probability the space is safe on a random configuration.

Combinatoric Limit Proof

We have two parts. First is

$$\binom{n^k - an^m + bn^{k-c}}{dn^{k-m}} \Big/ \binom{n^k}{dn^{k-m}} = \prod_{i=0}^{an^m - bn^{k-c} - 1} \left(1 - \frac{dn^{k-m}}{n^k - i}\right)$$

$$= \prod_{i=0}^{an^m - bn^{k-c} - 1} \left(1 - \frac{d}{n^m} - \frac{di}{n^m(n^k - i)}\right).$$

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Know that $\lim_{n\to\infty}(1 - d/n^m)^{an^m} = 1/e^{da}$. Use this to bound the limit.

Combinatoric Limit Proof Continued

Take extremes of the product, for

$$\left(1-\frac{d}{n^m}-\frac{d(an^m-bn^{k-c}-1)}{n^m(n^k-an^m+bn^{k-c}+1)}\right)^{an^m-bn^{k-c}}$$

$$\leq \prod_{i=0}^{an^m-bn^{k-c}-1}\left(1-\frac{dn^{k-m}}{n^k-i}\right) \leq \left(1-\frac{d}{n^m}\right)^{an^m-bn^{k-c}}.$$

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Both upper and lower bounds converge to $\frac{1}{e^{da}}$ after some algebra.

Variance

Theorem: Cashman, Cooper, Marquez, Miller, Shuffelton

Let $n, k, m, d, a \in \mathbb{Z}_{>0}$. Define μ_n as before, with dn^{k-m} attacking pieces placed, each of which attack an^m spaces. Then, the variance of the random variable with mean μ_n approaches 0 as n approaches infinity.

Variance Proof Sketch

- Begin with $\mathrm{Var}(S_n/n^k) = \mathrm{Var}(S_n)/n^{2k}$.
- Split into variance and covariance of the $X_{i_1,\dots,i_k}$.
- Find $\mathrm{Var}(X_{i_1,\dots,i_k}) = \mu_n - \mu_n^2$.

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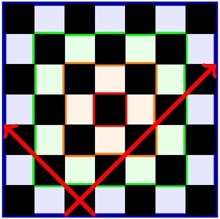
- Times that pieces can attack each other is infinitesimal as $n \to \infty$.
- Conclude that $\mathrm{Var}(S_n/n^k) \to 0$ as $n \to \infty$ for any board setup studied.

Bishop Counting

Harder to work with Bishops than Rooks, as Bishops see a variable number of squares.

A Bishop at the outer edge sees n squares, while a bishop at the center sees $2n - 1$ squares.



A bishop placed at $(3, 1)$ on a 7×7 chessboard.

Bishops and Rings

To more efficiently count, we define “rings” on the chessboard, starting with the 0^{th} ring being the center square, and working outwards. (We assume an odd n for easier calculation).

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A bishop placed in the i^{th} ring sees $2n - 2i - 1$ squares, and there are $8r$ squares in each ring.

Have $(n-1)/2$ rings, for

$$\mu_n = \frac{1}{n^2} \cdot \frac{\binom{n^2-2n+1}{n}}{\binom{n^2}{n}} + \sum_{r=1}^{(n-1)/2} \frac{4(2r)}{n^2} \frac{\binom{n^2-2n+2r+1}{n}}{\binom{n^2}{n}}.$$

2d Bishop Results

Can assume that n is odd, and the center term is an infinitesimal part of the final result. Lower terms in # of squares seen also vanish, for

$$\begin{aligned}\lim_{n\to\infty} \mu_n &= \lim_{n\to\infty} \sum_{r=1}^{(n-1)/2} \left(\frac{8r}{n^2} \frac{\binom{n^2-2n+2r}{n}}{\binom{n^2}{n}} \right) \\ &= \lim_{n\to\infty} \sum_{r=1}^{(n-1)/2} \left(\frac{8r}{n^2} \prod_{\alpha=0}^{2n-2r} \frac{n^2-n-\alpha}{n^2-\alpha} \right).\end{aligned}$$

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Then, we bound the product using extreme values of α, which lets us find bounds for the sum, giving us

$$\lim_{n\to\infty} \mu_n = \frac{2}{e^2} \approx 27.067\%.$$

Queen Results

- Queens can be modeled as combination of Bishops and Rooks.

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- Combining the two pieces gives

$$\mu_n = \sum_{r=0}^{(n-1)/2} \frac{4(2r)}{n^2} \cdot \frac{\binom{n^2-4n+2r+1}{n}}{\binom{n^2}{n}}.$$

- Evaluating gives a convergence to $\frac{2}{e^4}$ percent of squares being safe.

Line Rooks

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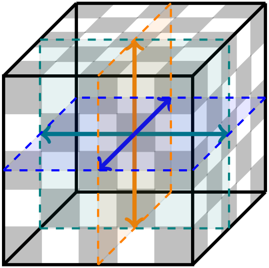

Figure: Movement of a line-rook placed at $(3, 3, 3)$ on a $5 \times 5 \times 5$ chessboard.

Line-Rooks Limit

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For n^{k-1} line-rooks on an $n \times n$ board, have

$$\lim_{n\to\infty} \binom{n^k - kn + k - 1}{n^{k-1}} \Big/ \binom{n^k}{n^{k-1}} = \frac{1}{e^k}.$$

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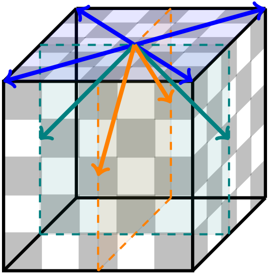

Figure: Movement of a line-bishop placed at $(3, 3, 5)$ on a $5 \times 5 \times 5$ chessboard, meaning that $r_2 = 0$ and $r_3 = 2$.

Line-Bishop Results

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In 3 dimensions,

$$\lim_{n\to\infty}\mu_n = \frac{-1+9e^2-2e^3}{3e^6} \approx 2.0929\%.$$

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Sees roughly $kn^{k-1} - an^{k-2}$ spaces, so with n hyper-rooks placed, average percentage of safe squares is

$$\mu_n = \binom{n^k - kn^{k-1} - an^{k-2}}{n} \Big/ \binom{n^k}{n}.$$

Converges to $1/e^k$.

Hyper-Bishops

Can define a regular Bishop as

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For higher dimensions, we add the new coordinates, in the possible diagonal subspaces. For example in 3 dimensions, a hyper-bishop at (i, j, k) attacks

$$\begin{aligned}(x-i)+(y-j)+(z-k) &= 0,\\ (x-i)+(y-j)-(z-k) &= 0,\\ (x-i)-(y-j)+(z-k) &= 0,\\ (x-i)-(y-j)-(z-k) &= 0.\end{aligned}$$

Hyper-Bishop Properties

Definition

In general, for a k-dimensional chessboard, a hyper-bishop at $(a_1, a_2, \ldots, a_k)$ can attack the areas defined by any possible version of

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- Only attacks spaces of the same color.
- In a 2-dimensional subspace, moves like a 2-dimensional bishop.
- Main weakness is being incredibly difficult to work with.

Hyper-Bishop Visualisation

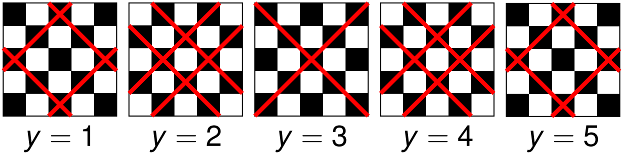

Figure: Spaces attacked by a hyper-bishop placed at $(3, 3, 3)$ on a $5 \times 5 \times 5$ board.

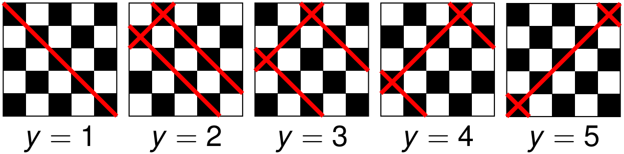

Figure: Spaces attacked by a hyper-bishop placed at $(1, 1, 1)$ on a $5 \times 5 \times 5$ board.

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Lack a good way to count spaces attacked.

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As a result, hyper-queens are similarly difficult, with bounds of

$$\frac{1}{e^6} \le \lim_{n\to\infty} \mu_n \le \frac{1}{e^{9/2}}.$$

Future Work

- Alternative ways of describing higher dimensional bishops.
- Better methods of counting spaces seen by line-bishops and hyper-bishops, and the associated better bounds and easier to calculate limits.
- Bounds on the number of line-rooks needed to dominate in k dimensions for $k > 3$.

Acknowledgments

This work was done as part of the SMALL 2024 REU Program.

I thank my coauthors, Caroline Cashman, Joseph Cooper, and Raul Marquez, as well as our advisor, Prof. Steven J. Miller.

We appreciate the support of Williams College, as well as funding from NSF grant number DMS-2241623, The William & Mary Charles Center, Emmanuel College Cambridge, and the Finnerty Fund.

Thank you!

Any questions?

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- Now have $k-1$ dimensions of rings, from r_2 to r_k, with the r_i rings existing within a i-dimensional subspace of the board.
- Increasing in $r_i \implies (i-1)^2(i-2)!$ fewer spaces attacked.
- For bishop in rings $(r_2, \ldots, r_k)$, sees

$$nk! - 2r_2 - \sum_{i=3}^{k}(i! - (i-1)!)r_i$$

spaces.

Line-Bishop Limit

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$$\mu_n = \frac{1}{n^k} \sum_{r_k=0}^{n/2} \sum_{r_{k-1}=0}^{r_k} \cdots \sum_{r_2=0}^{r_3} 2^{k-3}(k-1)k8r_2 \frac{\binom{n^k - (k! - \frac{s}{n})n}{n^{k-1}}}{\binom{n^k}{n^{k-1}}}.$$

Limit as $n \to \infty$ can be evaluated, but have not found a simplification for arbitrary k.

Line-Bishop Safe Spaces, $k = 3$

When $k = 3$, we find that the percentage of safe spaces with n^{k-1} line-bishops is

$$\lim_{n\to\infty} \mu_n = \lim_{n\to\infty} \frac{1}{n^3} \sum_{r_3=0}^{n/2} \sum_{r_2=0}^{r_3} 48 r_2 \prod_{\alpha=0}^{6n-4r_3-2r_2} \left(1 - \frac{n^2}{n^3 - \alpha}\right).$$

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- Lower percentage than line-rooks.

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$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{r_3=1}^{n/2} \sum_{r_2=1}^{r_3} 48 r_2 \prod_{\alpha=0}^{9n-4r_3-2r} \left(1 - \frac{n^2}{n^3 - \alpha}\right)$$

$$= \frac{-1 + 9e^2 - 2e^3}{3e^9} \approx 0.1042\%.$$