Solutions to a Pair of Diophantine Equations

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https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

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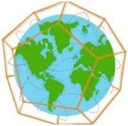
Polymath Junior Program



- Provide research opportunities to undergraduates.
- Online, runs in the spirit of the Polymath Project.
- Projects run by researcher with experience in undergraduate mentoring.
- Most 15-25 students, a main mentor, grad students / postdocs assisting.

https://geometrynyc.wixsite.com/polymathreu

Outline

- Introduction and results from Polymath Jr. 24
- What we did in Polymath Jr. 25
- Back to Polymath Jr. 23
- Future investigation

Introduction

A pair of equations

For relatively prime $a, b \in \mathbb{N}$, consider

$$\begin{aligned} ax + by &= \frac{(a-1)(b-1)}{2} \text{ and} \\ 1 + ax + by &= \frac{(a-1)(b-1)}{2}. \end{aligned}$$

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Theorem (Beiter (1964), extended by Chu (2020))

Exactly one of the equations has a nonnegative integral solution (x, y). The solution is unique.

Fibonacci numbers

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$$a \boxed{x} + b \boxed{y} \ = \ \frac{(a-1)(b-1)}{2}$$

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$$F_n \boxed{x} + F_{n+1} \boxed{y} \ = \ \frac{(F_n - 1)(F_{n+1} - 1)}{2}$$

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$$(x, y) = ?$$

Chu's six cases (2020)

$$F_{6k} \cdot \boxed{\frac{F_{6k-1}-1}{2}} + F_{6k+1} \cdot \boxed{\frac{F_{6k-1}-1}{2}} = \frac{(F_{6k}-1)(F_{6k+1}-1)}{2}$$

$$F_{6k+1} \cdot \boxed{\frac{F_{6k+1}-1}{2}} + F_{6k+2} \cdot \boxed{\frac{F_{6k-1}-1}{2}} = \frac{(F_{6k+1}-1)(F_{6k+2}-1)}{2}$$

$$F_{6k+2} \cdot \boxed{\frac{F_{6k+1}-1}{2}} + F_{6k+3} \cdot \boxed{\frac{F_{6k+1}-1}{2}} = \frac{(F_{6k+2}-1)(F_{6k+3}-1)}{2}$$

$$1 + F_{6k+3} \cdot \boxed{\frac{F_{6k+2}-1}{2}} + F_{6k+4} \cdot \boxed{\frac{F_{6k+2}-1}{2}} = \frac{(F_{6k+3}-1)(F_{6k+4}-1)}{2}$$

$$1 + F_{6k+4} \cdot \boxed{\frac{F_{6k+4}-1}{2}} + F_{6k+5} \cdot \boxed{\frac{F_{6k+2}-1}{2}} = \frac{(F_{6k+4}-1)(F_{6k+5}-1)}{2}$$

$$1 + F_{6k+5} \cdot \boxed{\frac{F_{6k+4}-1}{2}} + F_{6k+6} \cdot \boxed{\frac{F_{6k+4}-1}{2}} = \frac{(F_{6k+5}-1)(F_{6k+6}-1)}{2}$$

Fibonacci Squared (Polymath Jr. 2024)

If $n \equiv 0, 2, 3, 5 \mod 6$,

$$F_n^2 \cdot \boxed{\left(F_n^2 - \frac{F_{n-1}^2 + 1}{2}\right)} + F_{n+1}^2 \cdot \boxed{\frac{F_{n-1}^2 - 1}{2}} = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}.$$

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If $n \equiv 1 \mod 6$,

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If $n \equiv 4 \mod 6$,

$$1 + F_n^2 \cdot \boxed{\frac{F_n^2 + 1}{2}} + F_{n+1}^2 \cdot \boxed{\frac{F_n^2 - F_{n-1}^2 - 1}{2}} = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}.$$

Fibonacci Cubed (Polymath Jr. 2024)

For $n \geq 2$,

$$F_{2n-1}^3 \cdot \boxed{\sum_{i=1}^{2n-1} (-1)^{i-1} F_i^3} + F_{2n}^3 \cdot \boxed{\sum_{i=2}^{2n-2} F_i^3} = \frac{(F_{2n-1}^3 - 1)(F_{2n}^3 - 1)}{2};$$

$$1 + F_{2n}^3 \cdot \boxed{\left(\sum_{i=1}^{2n} (-1)^i F_i^3 - 1\right)} + F_{2n+1}^3 \cdot \boxed{\sum_{i=2}^{2n-1} F_i^3} = \frac{(F_{2n}^3 - 1)(F_{2n+1}^3 - 1)}{2}.$$

Problem 1

For $(i, j) \in \mathbb{N}^2$, find the nonnegative integral solution (x, y) to

$$F_n^i \cdot x + F_{n+1}^j \cdot y = \frac{(F_n^i - 1)(F_{n+1}^j - 1)}{2}$$

or

$$1 + F_n^i \cdot x + F_{n+1}^j \cdot y = \frac{(F_n^i - 1)(F_{n+1}^j - 1)}{2}.$$

Balancing numbers and Lucas-balancing numbers

A positive integer n is called balancing if there is $d \geq 0$ with

$$1 + 2 + \cdots + (n-1) \;=\; (n+1) + \cdots + (n+d).$$

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Balancing numbers:

$$B_1 \ = \ 1, \quad B_2 \ = \ 6, \quad B_n \ = \ 6B_{n-1} - B_{n-2}$$

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Lucas-balancing numbers:

$$C_1 \; = \; 3, C_2 \; = \; 17, C_n \; = \; 6C_{n-1} - C_{n-2}$$

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$(B_n)_{n=1}^{\infty} \sim (C_n)_{n=1}^{\infty}$ resembles $(F_n)_{n=1}^{\infty} \sim (L_n)_{n=1}^{\infty}$

Davala (2023)

$$B_{4n-3} \cdot \boxed{\sum_{i=1}^{n-1} C_{4i}} + B_{4n-1} \cdot \boxed{\sum_{i=1}^{n-1} C_{4i}} = \frac{(B_{4n-3}-1)(B_{4n-1}-1)}{2}$$

$$1 + B_{4n-1} \cdot \boxed{\sum_{i=1}^{n} C_{4i}} + B_{4n+1} \cdot \boxed{\sum_{i=1}^{n-1} C_{4i}} = \frac{(B_{4n-1}-1)(B_{4n+1}-1)}{2}$$

$$\frac{B_{4n-2}}{6} \cdot \boxed{\sum_{i=1}^{n-1} C_{4i}} + \frac{B_{4n}}{6} \cdot \boxed{\sum_{i=1}^{2n-2} (-1)^i C_{2i}} = \frac{(B_{4n-2}/6-1)(B_{4n}/6-1)}{2}$$

$$1 + \frac{B_{4n}}{6} \cdot \boxed{\sum_{i=1}^{2n} (-1)^i C_{2i}} + \frac{B_{4n+2}}{6} \cdot \boxed{\sum_{i=1}^{n-1} C_{4i}} = \frac{(B_{4n}/6-1)(B_{4n+2}/6-1)}{2}$$

What we did in Polymath Jr. 25

Fibonacci-like sequences

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Define $(t_n^{(u,v)})_{n=1}^{\infty} : t_1^{(u,v)} = u, \quad t_2^{(u,v)} = v, \quad t_n^{(u,v)} = t_{n-1}^{(u,v)} + t_{n-2}^{(u,v)}.$

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$t_n^{(u,v)} = F_{n-2}u + F_{n-1}v$

$\gcd(t_n^{(u,v)}, t_{n+1}^{(u,v)}) = 1$

$$t_n^{(u,v)}x + t_{n+1}^{(u,v)}y = \frac{(t_n^{(u,v)} - 1)(t_{n+1}^{(u,v)} - 1)}{2}$$

$$t_n^{(u,v)}x + t_{n+1}^{(u,v)}y + 1 = \frac{(t_n^{(u,v)} - 1)(t_{n+1}^{(u,v)} - 1)}{2}$$

Nonnegative integral $(x, y) = ?$

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Nonnegative integral $(x, y) = ?$ Depend on n modulo 6.

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Cassini's identity (2): $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$

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Fibonacci pairs (3): $(F_{6n}, F_{6n+3}), (F_{6n+1}, F_{6n+4}), (F_{6n+2}, F_{6n+5})$

The solution (x, y)

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Fibonacci pairs (3): $(F_{6n}, F_{6n+3}), (F_{6n+1}, F_{6n+4}), (F_{6n+2}, F_{6n+5})$

In each case, the solution (x, y) depends further on (u, v).

Sample case: $n \equiv 4 \mod 6$

Polymath Jr. 25 Diophantine Group

Given $(u, v, n, r) \in \mathbb{Z}^4$ with even n, it holds that

$$1 + \frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n - 1\right)t_n^{(u,v)} + \frac{1}{2}\left(rF_{n-2} + \frac{vr-1}{u}F_{n-1} - 1\right)t_{n+1}^{(u,v)} = \frac{(t_n^{(u,v)}-1)(t_{n+1}^{(u,v)}-1)}{2}$$

and

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and

$$\frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v-1}{u}F_n - 1\right)t_n^{(u,v)} + \frac{1}{2}\left(rF_{n-2} + \frac{vr+1}{u}F_{n-1} - 1\right)t_{n+1}^{(u,v)} = \frac{(t_n^{(u,v)}-1)(t_{n+1}^{(u,v)}-1)}{2}.$$

Sample case: $n \equiv 4 \mod 6$

For even n,

$$1 + \frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n - 1\right)t_n^{(u,v)} +$$
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Proof:

Sample case: $n \equiv 4 \mod 6$

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Proof:

$$t_n^{(u,v)} = F_{n-2}u + F_{n-1}v$$

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$$1 + \frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n - 1\right)t_n^{(u,v)} + \frac{1}{2}\left(rF_{n-2} + \frac{vr-1}{u}F_{n-1} - 1\right)t_{n+1}^{(u,v)} = \frac{(t_n^{(u,v)}-1)(t_{n+1}^{(u,v)}-1)}{2}.$$

Proof:

$$t_n^{(u,v)} = F_{n-2}u + F_{n-1}v$$

$$F_{n-1}F_{n+1} - F_n^2 = 1 \text{ (for even } n)$$

Sample case: $n \equiv 4 \mod 6$

$$1 + \frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n - 1\right)t_n^{(u,v)} + \frac{1}{2}\left(rF_{n-2} + \frac{vr-1}{u}F_{n-1} - 1\right)t_{n+1}^{(u,v)}$$

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$$= \frac{1}{2} + \frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n\right)(F_{n-2}u + F_{n-1}v) + \frac{1}{2}\left(rF_{n-2} + \frac{vr-1}{u}F_{n-1}\right)(F_{n-1}u + F_n v) - \frac{1}{2}(t_n^{(u,v)} + t_{n+1}^{(u,v)} - 1)$$

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$$1+\frac{1}{2}\left((u-r)F_{n-1}+\frac{(u-r)v+1}{u}F_n-1\right)t_n^{(u,v)}+ \frac{1}{2}\left(rF_{n-2}+\frac{vr-1}{u}F_{n-1}-1\right)t_{n+1}^{(u,v)}$$

$$=\frac{1}{2}+\frac{1}{2}\left((u-r)F_{n-1}+\frac{(u-r)v+1}{u}F_n\right)(F_{n-2}u+F_{n-1}v)+ \frac{1}{2}\left(rF_{n-2}+\frac{vr-1}{u}F_{n-1}\right)(F_{n-1}u+F_nv)-\frac{1}{2}(t_n^{(u,v)}+t_{n+1}^{(u,v)}-1)$$

$$=\frac{1}{2}\left(1+F_{n-2}F_n-F_{n-1}^2\right)+\frac{1}{2}\underbrace{(uF_{n-2}+vF_{n-1})}_{t_n^{(u,v)}}\underbrace{(uF_{n-1}+vF_n)}_{t_{n+1}^{(u,v)}}-\frac{1}{2}(t_n^{(u,v)}+t_{n+1}^{(u,v)}-1)$$

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$$=\frac{1}{2}(t_n^{(u,v)}-1)(t_{n+1}^{(u,v)}-1)$$

Sample case: $n \equiv 4 \mod 6$

Polymath Jr. 25 Diophantine Group

$$1 + t_n^{(u,v)} \cdot \boxed{\frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n - 1\right)} + t_{n+1}^{(u,v)} \cdot \boxed{\frac{1}{2}\left(rF_{n-2} + \frac{vr-1}{u}F_{n-1} - 1\right)} = \frac{(t_n^{(u,v)} - 1)(t_{n+1}^{(u,v)} - 1)}{2}$$

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$$\begin{cases} x &= \frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n - 1\right) \\ y &= \frac{1}{2}\left(rF_{n-2} + \frac{vr-1}{u}F_{n-1} - 1\right) \end{cases} ?$$

NOT YET!

Sample case: $n \equiv 4 \mod 6$

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$$1 + t_n^{(u,v)} \cdot \boxed{\frac{1}{2}\left((u-r)F_{n-1} + \frac{(u-r)v+1}{u}F_n - 1\right)} +$$

$$t_{n+1}^{(u,v)} \cdot \boxed{\frac{1}{2}\left(rF_{n-2} + \frac{vr-1}{u}F_{n-1} - 1\right)} = \frac{(t_n^{(u,v)}-1)(t_{n+1}^{(u,v)}-1)}{2}$$

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Need r such that the boxed are nonnegative integers

Choose r when $n \equiv 4 \mod 6$

Lemma

Given $(u, v) \in \mathbb{N}^2$ with $\gcd(u, v) = 1$ and odd u,

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Denote $\boldsymbol{r}$ by $\mathbb{O}(u, v)$.

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$\gcd(u, v) = 1 \implies \{1 \cdot v, 2 \cdot v, \ldots, u \cdot v\}$ is a complete modulo system of u.

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$\implies u \mid v(x_1 + x_2) \implies u | (x_1 + x_2) \implies x_1 + x_2 = u.$

Solutions when $n \equiv 4 \mod 6$

If u is odd and $v\mathbb{O}(u,v) \equiv 1 \mod u$ or u is even and $v\mathbb{O}(u,v) \equiv 1 \mod 2u$,

$$1 + t_n^{(u,v)} \cdot \boxed{\frac{1}{2}\left((u-\mathbb{O}(u,v))F_{n-1} + \frac{(u-\mathbb{O}(u,v))v+1}{u}F_n - 1\right)} + t_{n+1}^{(u,v)} \cdot \boxed{\frac{1}{2}\left(\mathbb{O}(u,v)F_{n-2} + \frac{v\mathbb{O}(u,v)-1}{u}F_{n-1} - 1\right)} = \frac{(t_n^{(u,v)}-1)(t_{n+1}^{(u,v)}-1)}{2}.$$

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If u is odd, $u \geq 3$, and $v\mathbb{O}(u,v) \equiv -1 \mod u$ or u is even and $v\mathbb{O}(u,v) \equiv -1 \mod 2u$,

$$t_n^{(u,v)} \cdot \boxed{\frac{1}{2}\left((u - \mathbb{O}(u,v))F_{n-1} + \frac{(u-\mathbb{O}(u,v))v-1}{u}F_n - 1\right)} +$$

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Nonnegative, integral solutions for $n \equiv 4 \mod 6$

u is odd and $v\mathbb{O}(u,v) \equiv 1 \mod u$:

$$1 + t_n^{(u,v)} \cdot \frac{1}{2}\left((u - \mathbb{O}(u,v)) \underbrace{F_{n-1}}_{even} + \underbrace{\frac{(u - \mathbb{O}(u,v))v + 1}{u}}_{odd} \underbrace{F_n}_{odd} - 1 \right) +$$

$$t_{n+1}^{(u,v)} \cdot \frac{1}{2}\left(\underbrace{\mathbb{O}(u,v)}_{odd} \underbrace{F_{n-2}}_{odd} + \frac{v\mathbb{O}(u,v) - 1}{u} \underbrace{F_{n-1}}_{even} - 1 \right) = \frac{(t_n^{(u,v)} - 1)(t_{n+1}^{(u,v)} - 1)}{2}$$

Application: $u = v = 1$ (Fibonacci) and $n = 6k + 4$

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This matches Chu's (2020) ✓ :

$$1 + F_{6k+4} \cdot \boxed{\frac{F_{6k+4} - 1}{2}} + F_{6k+5} \cdot \boxed{\frac{F_{6k+2} - 1}{2}} = \frac{(F_{6k+4} - 1)(F_{6k+5} - 1)}{2}$$

Problem 2

Find the formula for the solutions (x, y) to

$$a \cdot x + b \cdot y = \frac{(a-1)(b-1)}{2}$$

or

$$1 + a \cdot x + b \cdot y = \frac{(a-1)(b-1)}{2},$$

where a and b are taken from other recursively defined sequences.

Back to Polymath Jr. 23

Which equation to use

Define $\Gamma : \mathbb{N}^2 \to \{0,1\}$ as follows: $\Gamma(a,b) = 0$ if

$$\frac{a}{\gcd(a,b)}x + \frac{b}{\gcd(a,b)}y \;=\; \frac{1}{2}\left(\frac{a}{\gcd(a,b)} - 1\right)\left(\frac{b}{\gcd(a,b)} - 1\right)$$

has a nonnegative integral solution, and $\Gamma(a,b) = 1$ if

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Given an integer sequence $(a_n)_{n=1}^{\infty}$, what is the sequence $(\Gamma(a_n, a_{n+1}))_{n=1}^{\infty}$?

Which equation to use

Theorem (Polymath Jr. 23)

Let $a, b \in \mathbb{N}$. If $a|b$ or $b|a$, then $\Gamma(a, b) = 0$. Otherwise:

a) When $a/\gcd(a, b)$ is odd, then $\Gamma(a, b) = 0$ if and only if $\Theta(b, a)$ is odd.

b) When $a/\gcd(a, b)$ is even, then $\Gamma(a, b) = 0$ if and only if $\Theta(a, b)$ is odd.

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Problem 3

Let $\mathcal{F} = \{(a_n)_{n=1}^{\infty} : (\Gamma(a_n, a_{n+1}))_{n=1}^{\infty}$ eventually alternates between 0 and 1$\}$. Characterize sequences that are in $\mathcal{F}$.

Problem 4 (from Polymath Jr. 23)

$$H(x) := \frac{\#\{(a,b) \in \mathbb{N}^2 : 1 \leqslant a \leqslant b \leqslant x, \Gamma(a,b) = 1\}}{\#\{(a,b) \in \mathbb{N}^2 : 1 \leqslant a \leqslant b \leqslant x\}}$$

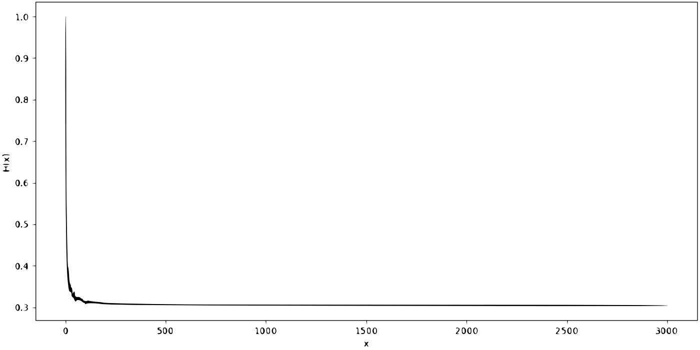

Figure: Plots of $H(x)$ for $1 \leq x \leq 3000$ with 2000 sample points. In particular, $H(3000) \approx 0.30423059$.

The sequence $(\Gamma(k,n))_{n=1}^{\infty}$ and Problem 5

Theorem (Polymath Jr. 23)

Let $k \in \mathbb{N}$. The following hold.

1. If k is odd, $(\Gamma(k,n))_{n=1}^{\infty}$ has period k. In each period, the number of 0's is one more than the number of 1's.
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Problem 5: Fix $k \in \mathbb{N}$. For which sequences $(a_n)_{n=1}^{\infty}$ is the sequence $(\Gamma(k,a_n))_{n=1}^{\infty}$ periodic?

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