THE I LOVE RECTANGLES GAME

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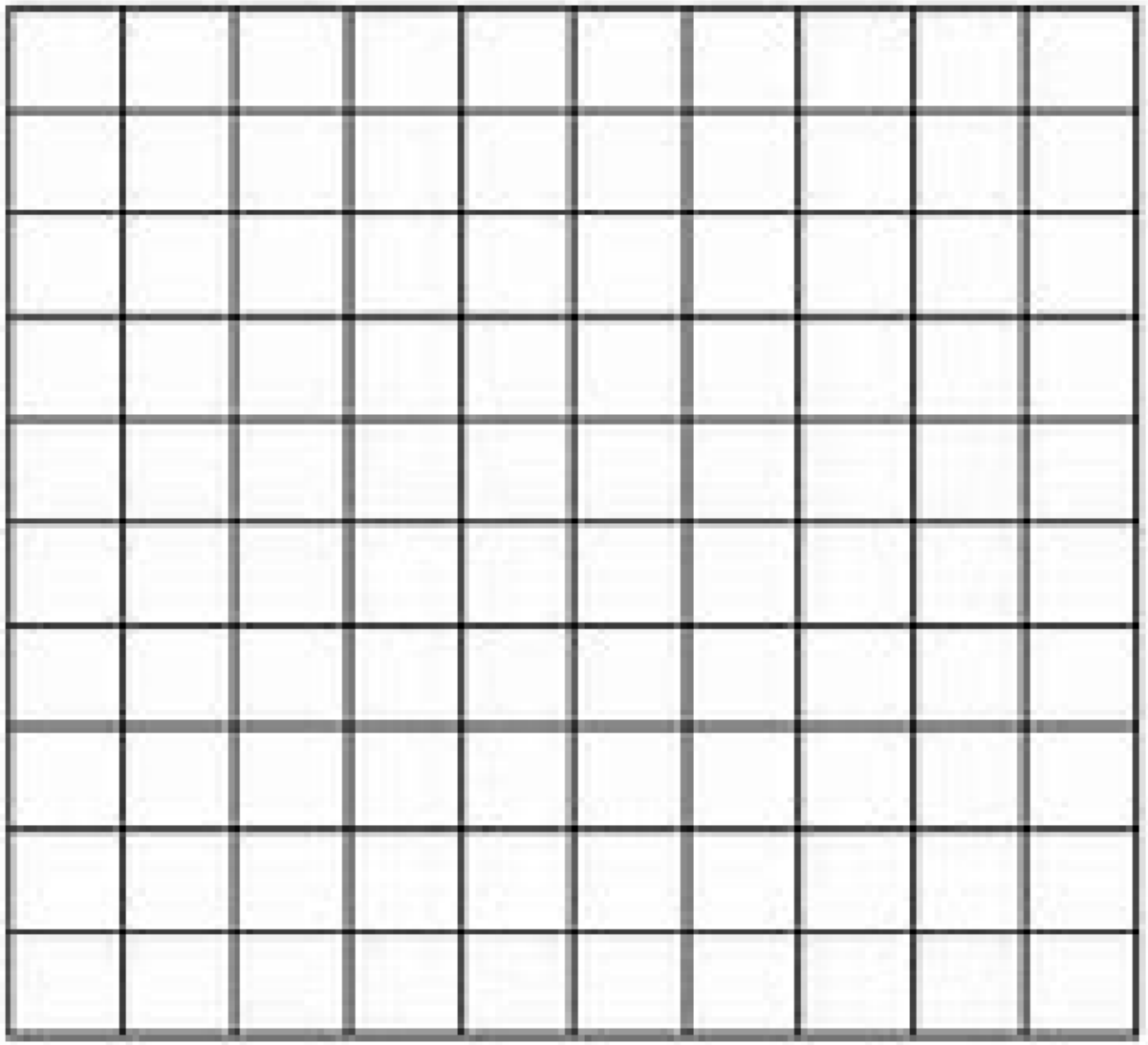
Our goal is to explore **tilings**. What is a tiling?

We have a collection of objects and we want to place them down to cover a space.

For example, imagine you want to cover the floor and the floor is a giant square, say 10 feet by 10 feet. What would be a good shape to use to cover it? We want the shape to be smaller that the floor, and we want all the pieces to fit together with no gaps.



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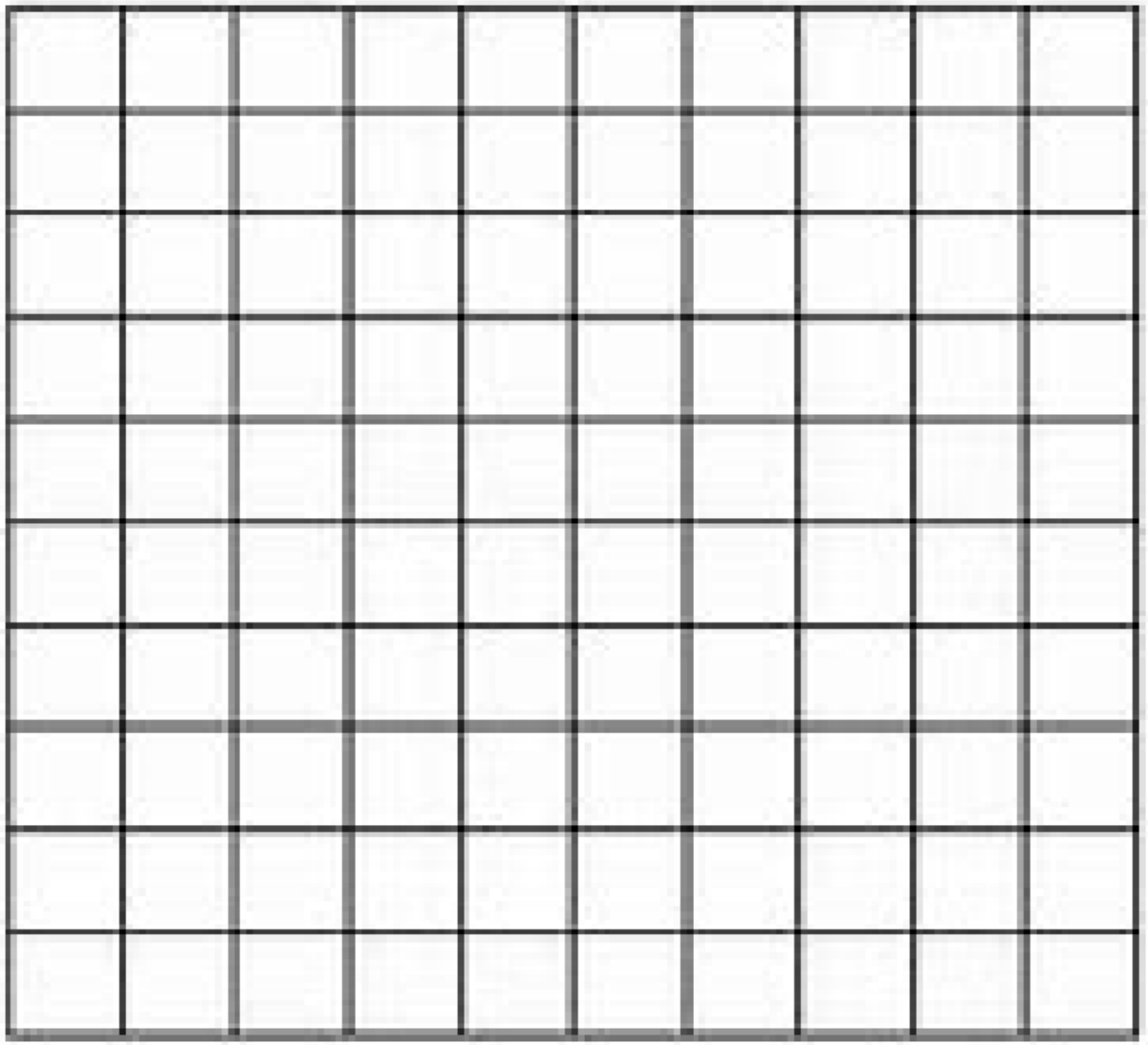
For example, imagine you want to cover the floor and the floor is a giant square, say 10 feet by 10 feet. What would be a good shape to use to cover it? We want the shape to be smaller that the floor, and we want all the pieces to fit together with no gaps. Answer: 1 foot by 1 foot squares!



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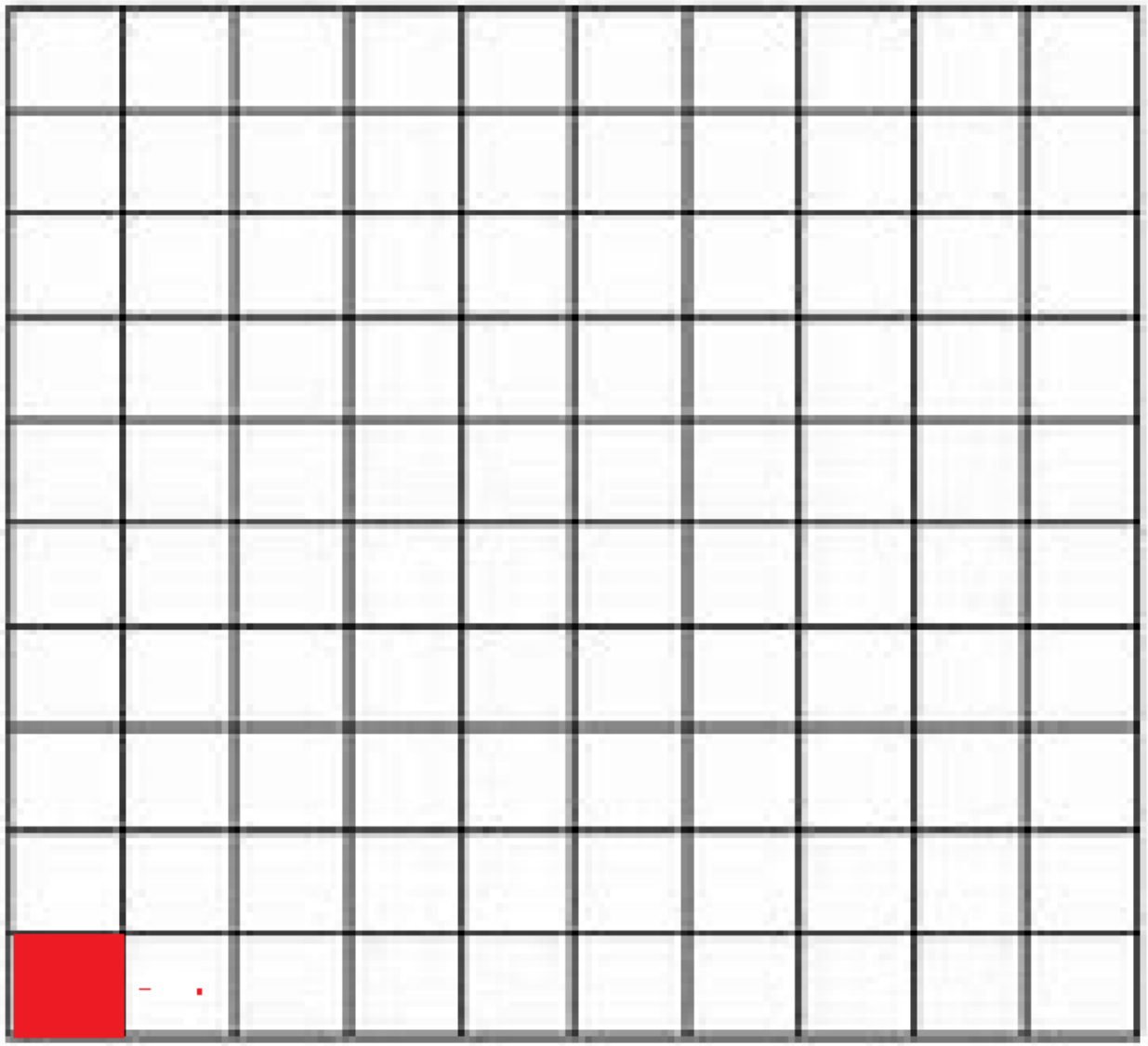

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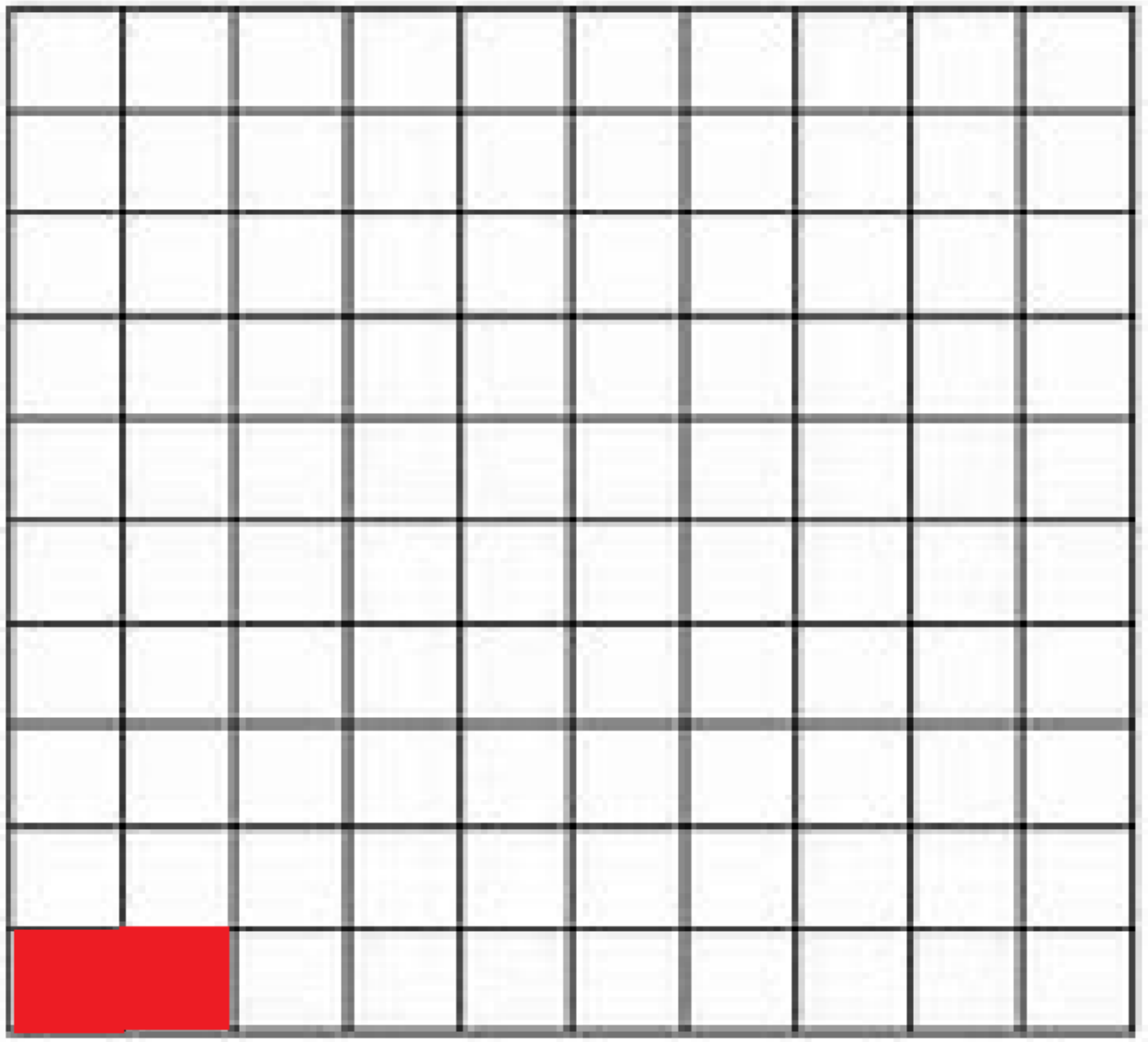

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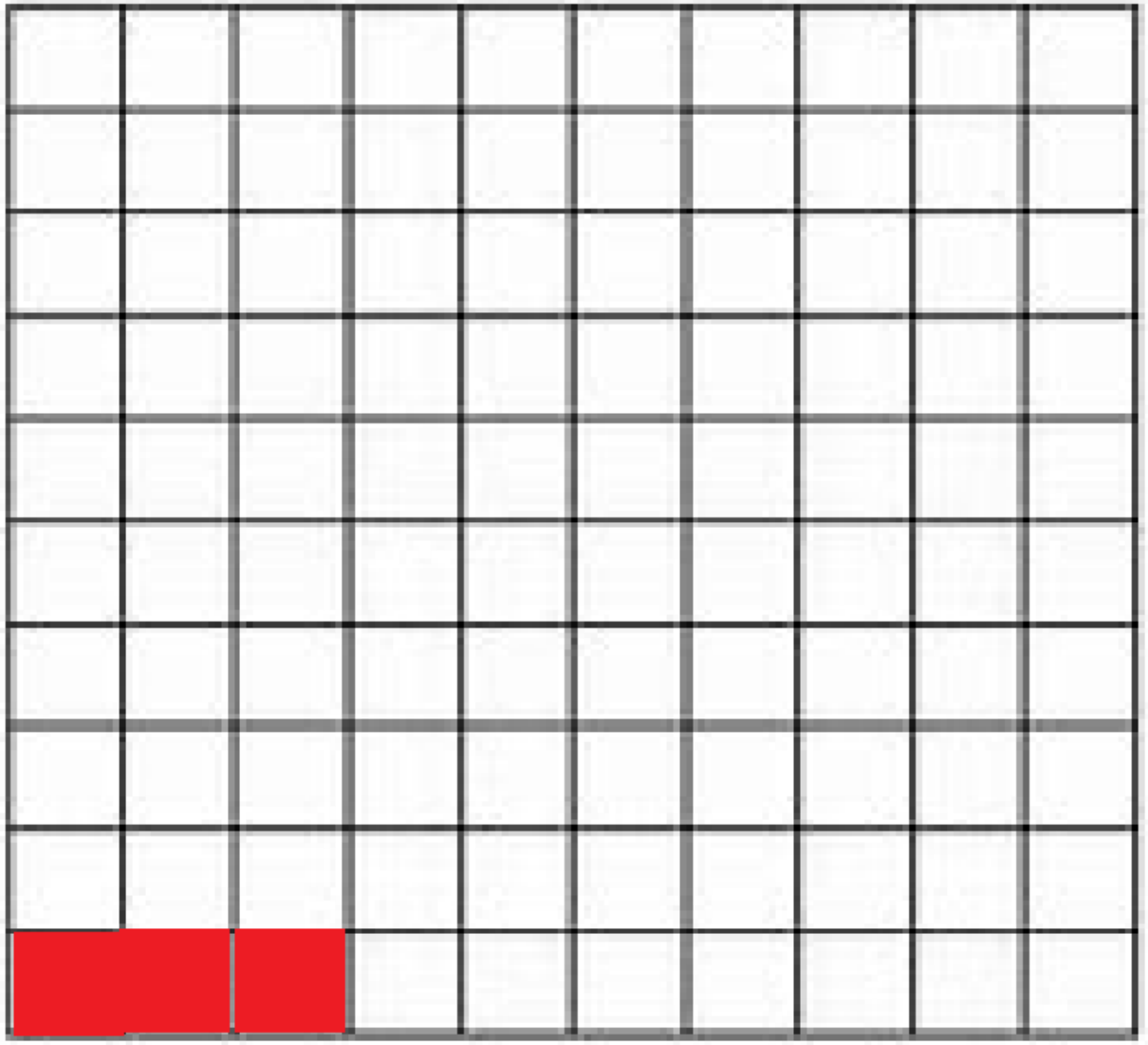
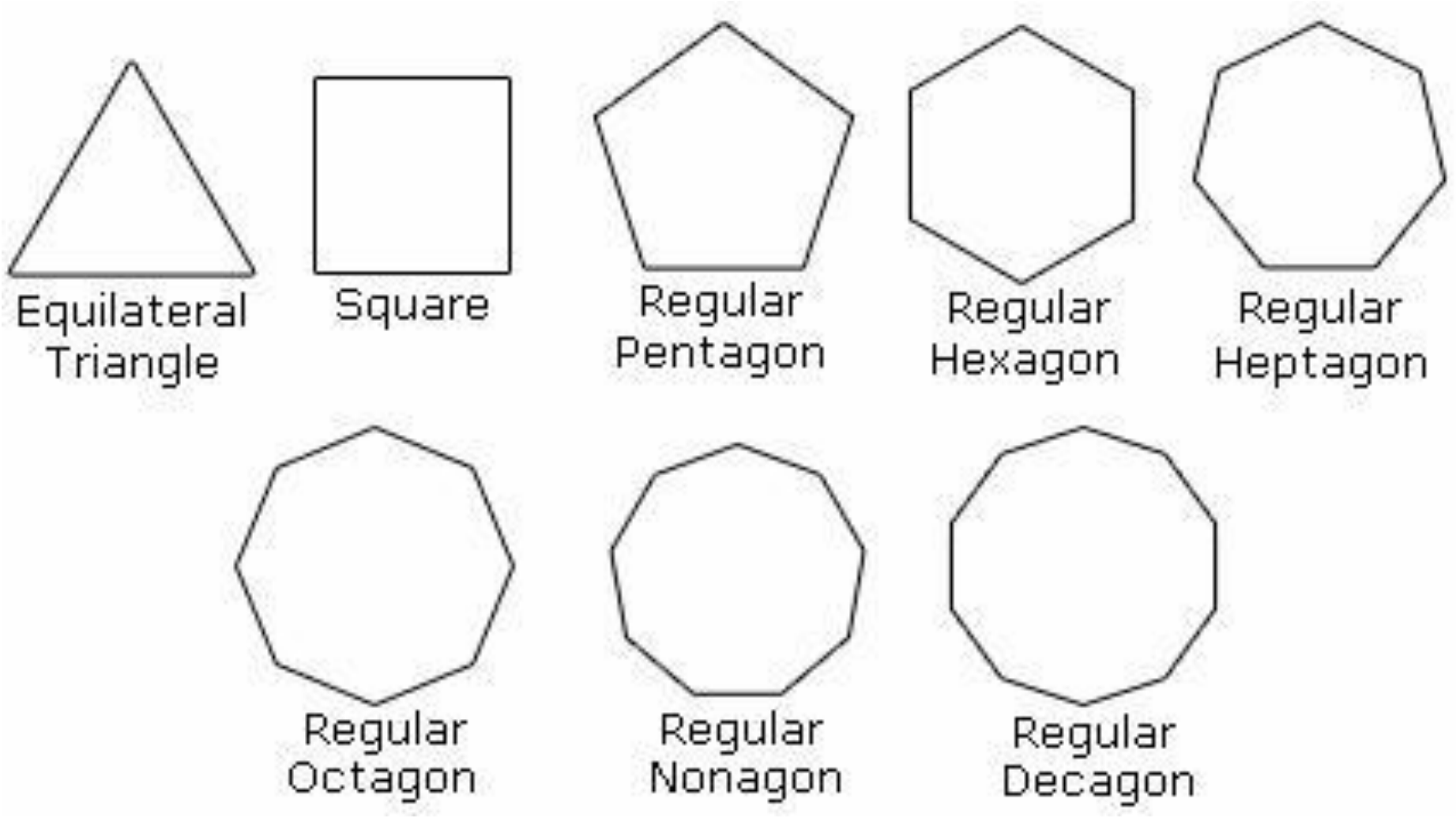


We just continue adding the smaller squares…..

Building on our success, as a fun problem see if you can tile larger and larger regions, with no gaps, with the following shapes.



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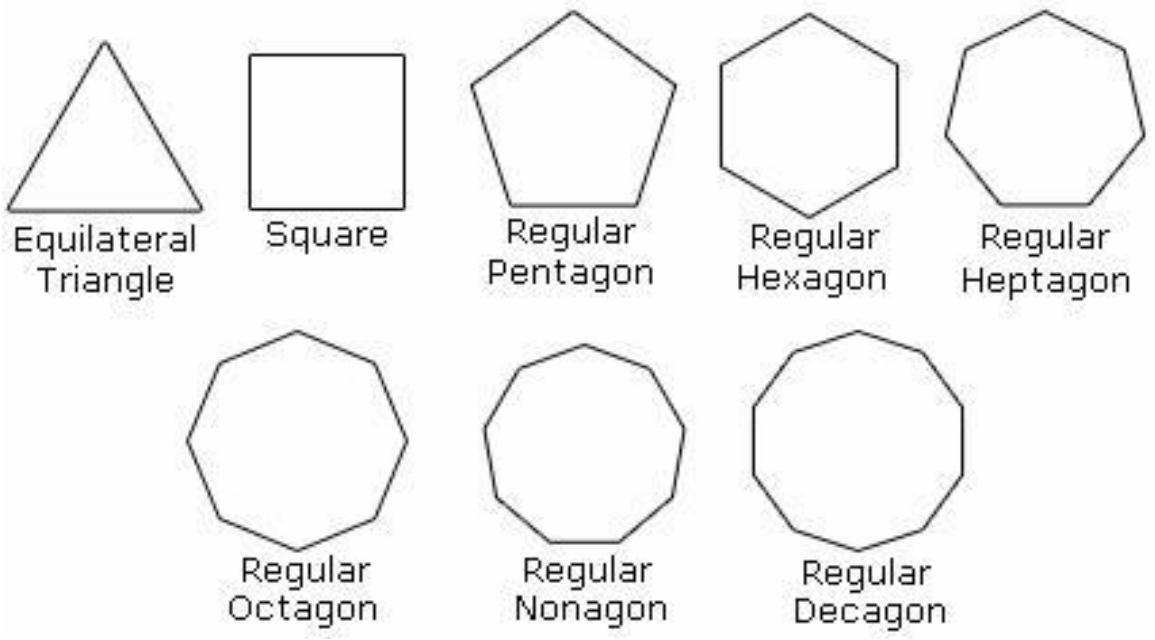


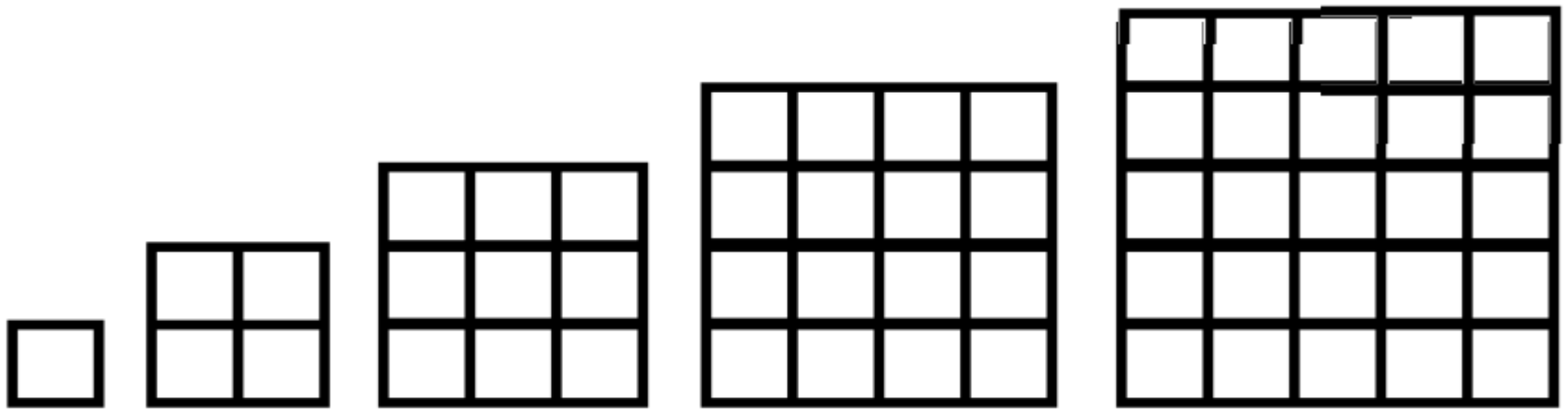
Note each shape above has all sides of the same length. We saw we can do it with the square. What about the triangle? What about the pentagon?

GOOD LUCK!

The I LOVE RECTANGLES Game

If we have an unlimited supply of 1 foot by 1 foot squares, we can cover larger and larger rectangles.

Let's make it more interesting. Imagine now we have EXACTLY ONE of each size square. We have one 1 by 1 rectangle, one 2 by 2 rectangle, one 3 by 3 rectangle, one 4 by 4 rectangle, and so on.



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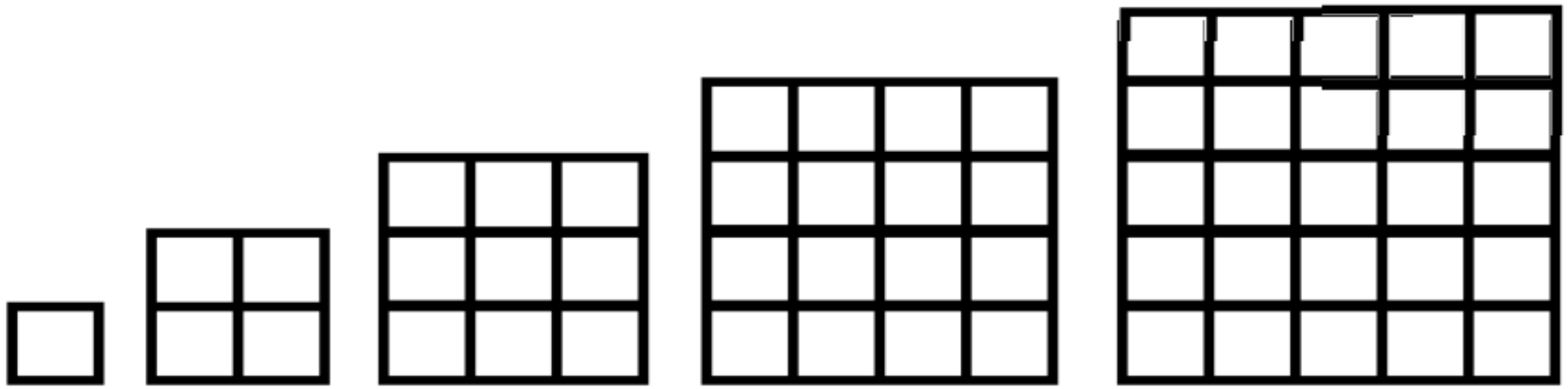
Here's the rule: we put these squares down ONE AT A TIME, and at EVERY MOMENT IN TIME our shape MUST be a rectangle. Can it be done? Note a square IS a rectangle.



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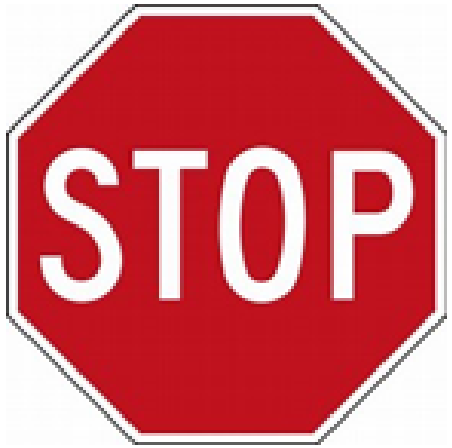
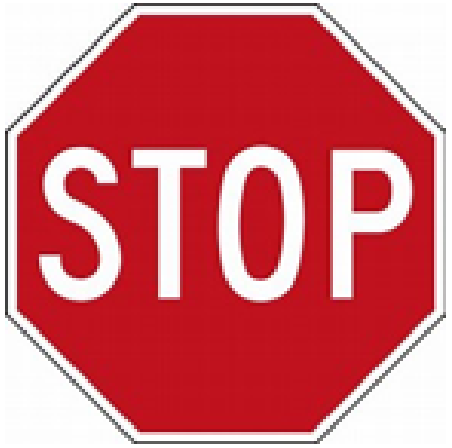
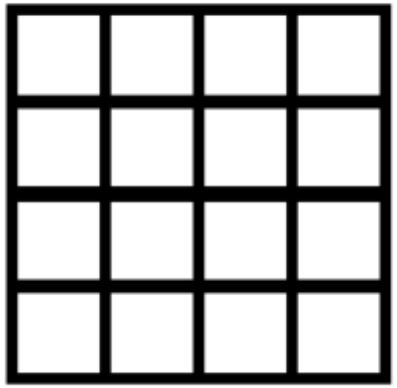


SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

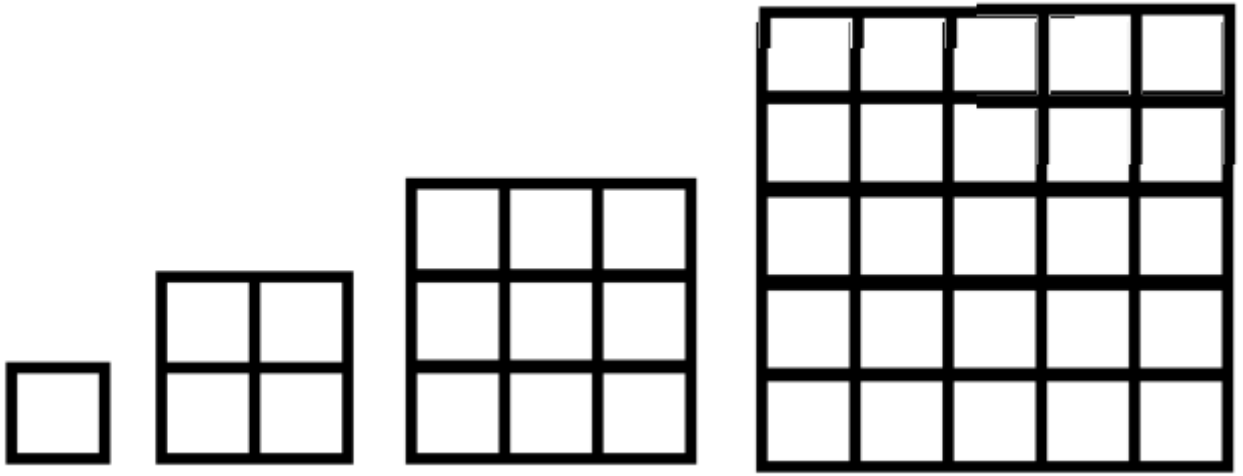


The I LOVE RECTANGLES Game

Imagine we put the 4 by 4 square down. That gives us a rectangle, so far so good. Can we put down anything else next to it and still have a rectangle?



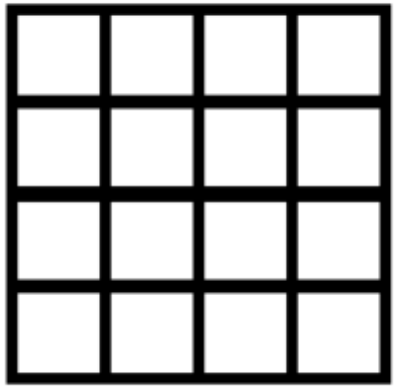
We have placed a 4 by 4 square. This is a rectangle!



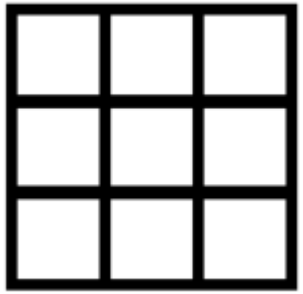
These are the squares we have left. We have a 1 by 1, a 2 by 2, a 3 by 3, a 5 by 5, a 6 by 6 (not drawn) and so on. Can we place anything next to the 4 by 4 and still have a rectangle?

The I LOVE RECTANGLES Game

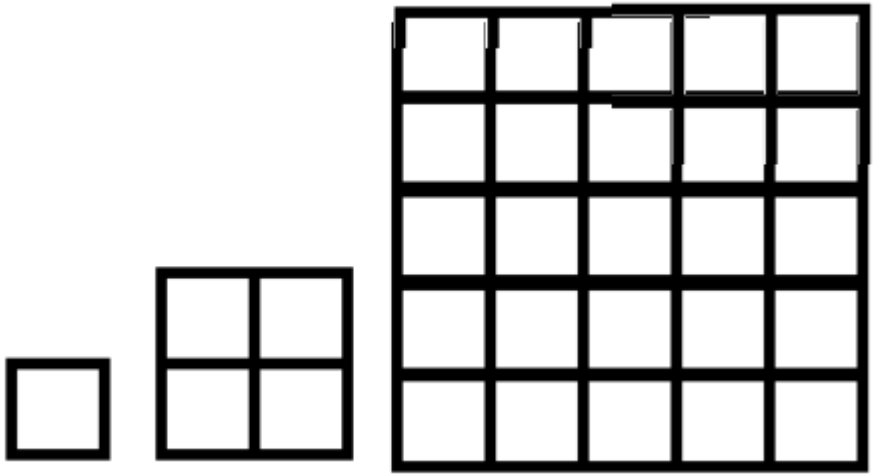
Imagine we put the 4 by 4 square down. That gives us a rectangle, so far so good. Can we put down anything else? Let's try putting down the 3 by 3.



We have placed a 4 by 4 square. This is a rectangle!



We see the 3 by 3 will not fit next to the 4 by 4 and still give a rectangle!

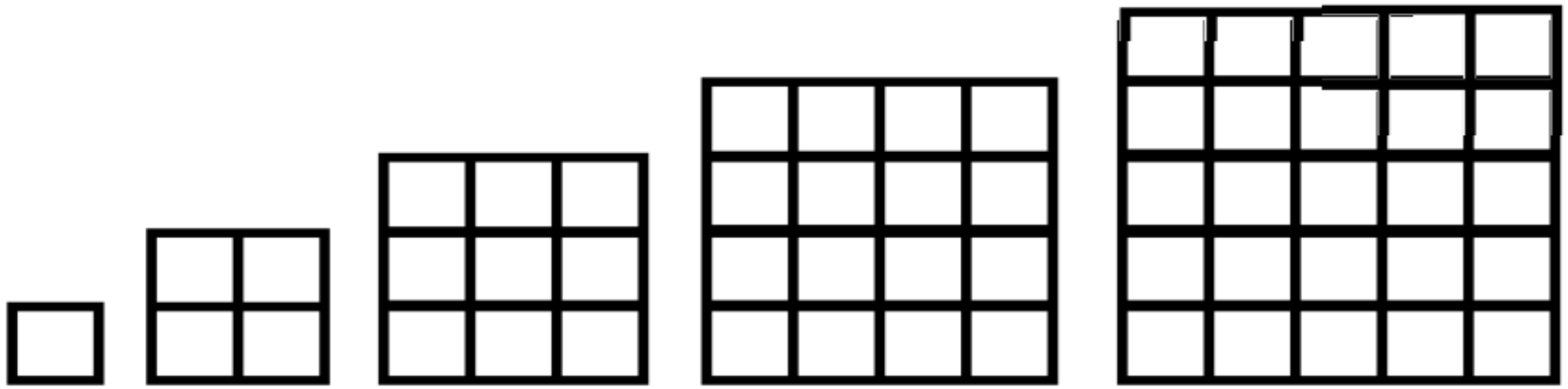


These are the squares we would have left if we try to use a 3 by 3. We would have a 1 by 1, a 2 by 2, a 5 by 5, a 6 by 6 (not drawn) and so on.

The I LOVE RECTANGLES Game

In fact, no matter WHAT square we put down first, we cannot put any more down! If we put down a 5 by 5, to keep it a rectangle we would need something that has a side of length 5, but we only have ONE of each square!

We have to modify the game. We need to give at least ONE more square. What is the smallest square we can give?



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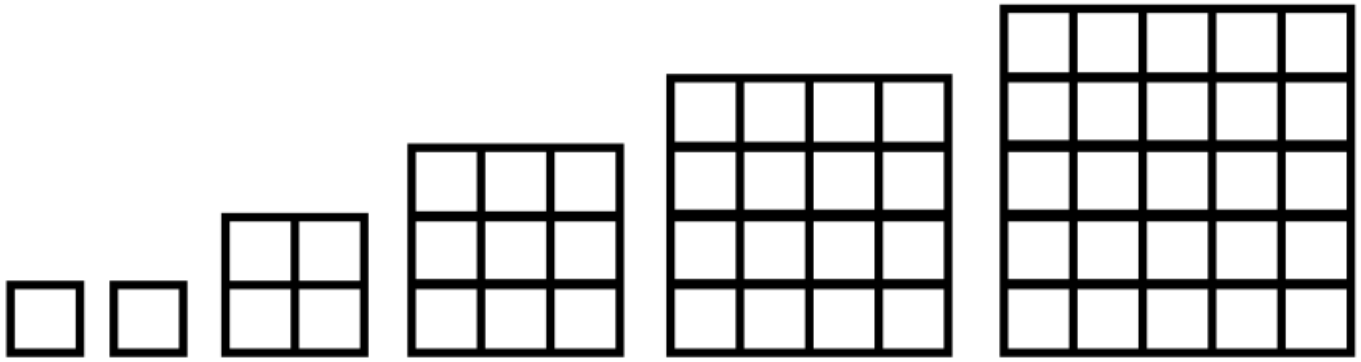
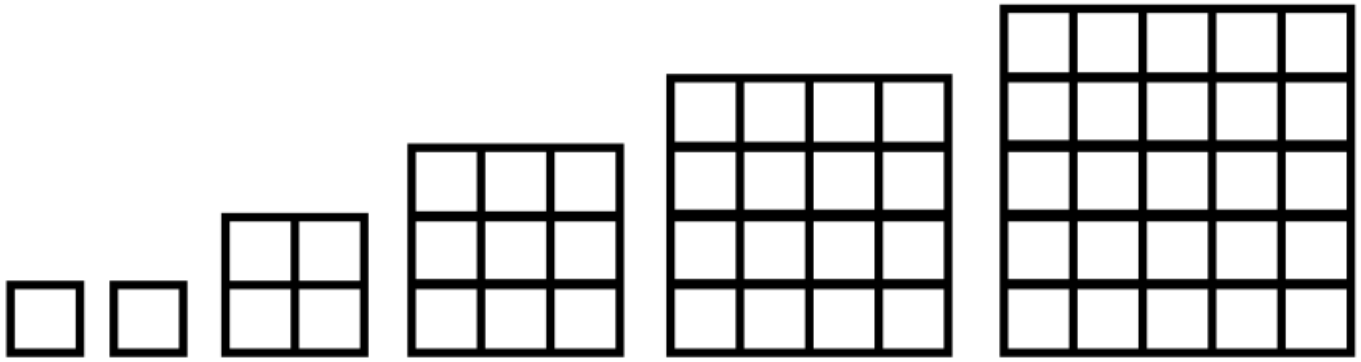
We have to modify the game. We need to give at least ONE more square. What is the smallest square we can give? Answer: a 1 by 1 square! Can we do it now?



The I LOVE RECTANGLES Game

OK, we want to put the squares down one at a time so that we always have a rectangle. We cannot put a square on top of a square. Which should we put down first? Which should we put down second?



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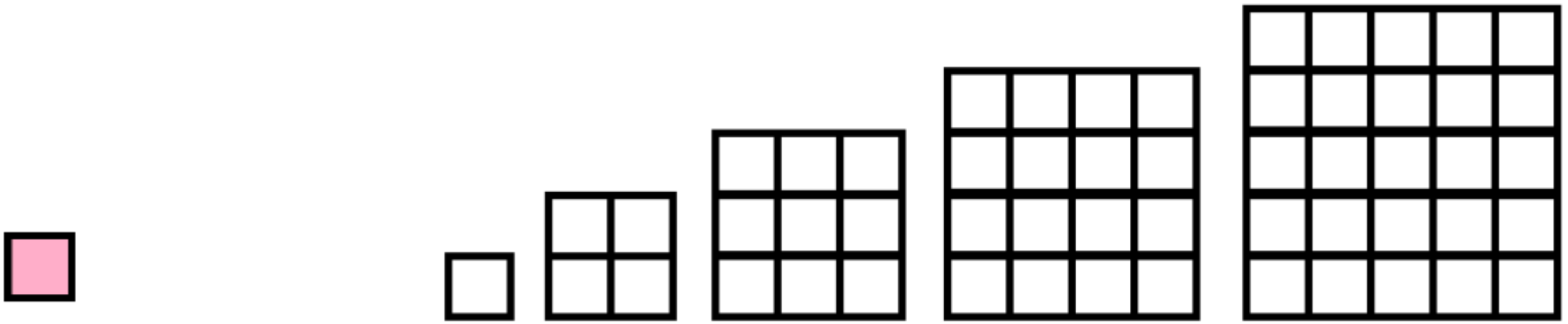
OK, we want to put the squares down one at a time so that we always have a rectangle. We cannot put a square on top of a square. Which should we put down first? Which should we put down second?

Makes sense to start with the two 1 by 1 squares, as they fit! Here is placing the first 1 by 1 square. Now we have one 1 by 1, one 2 by 2, one 3 by 3, and so on.



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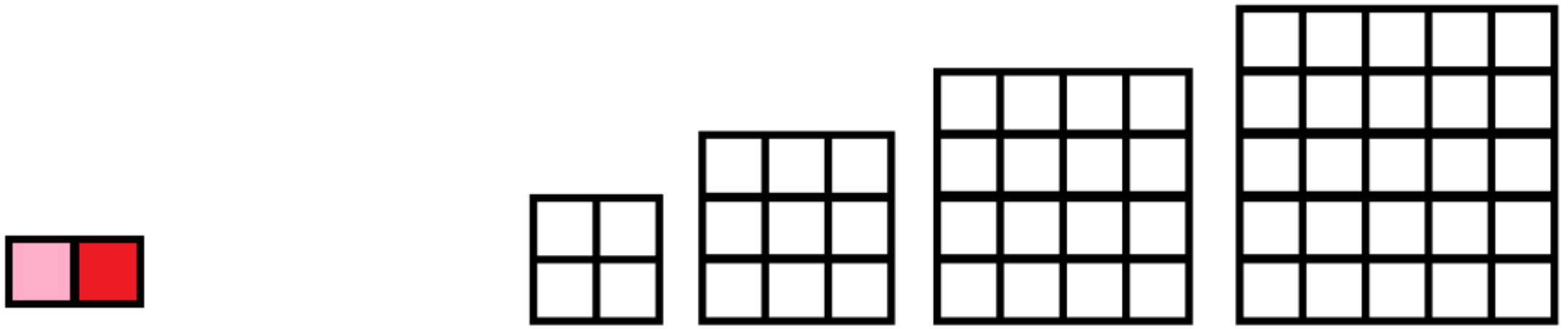
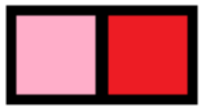
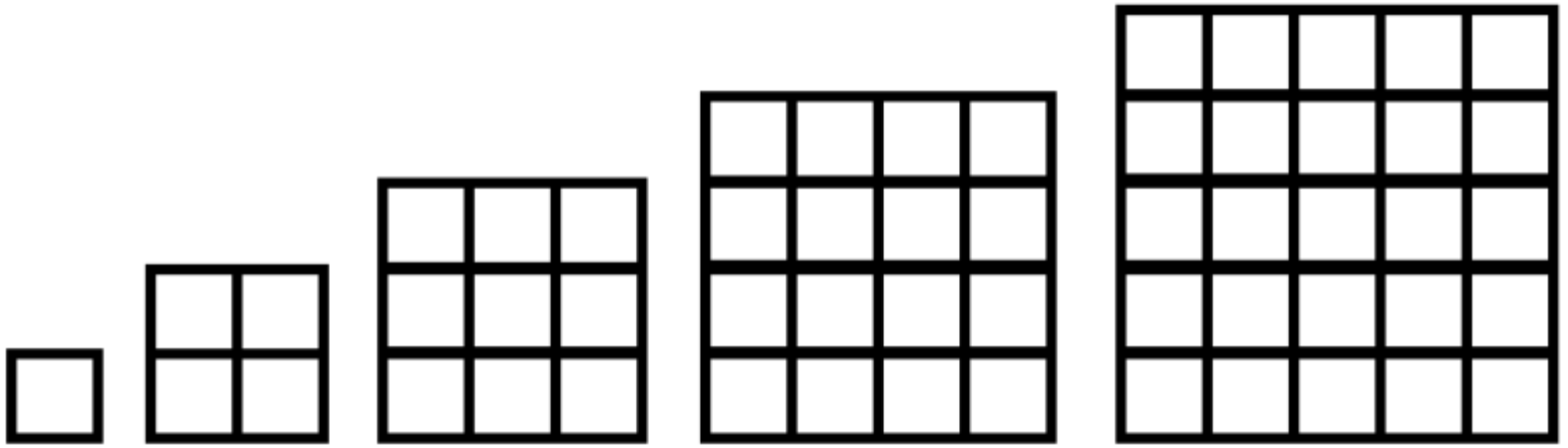


The I LOVE RECTANGLES Game

We have placed the two 1 by 1 squares, we have a 2 by 2, a 3 by 3, a 4 by 4, a 5 by 5 and so on. What should we place next to the two 1 by 1 squares so that we still have a rectangle? Note the two 1 by 1 squares have formed a 1 by 2 rectangle…..



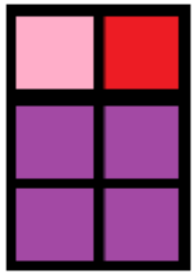


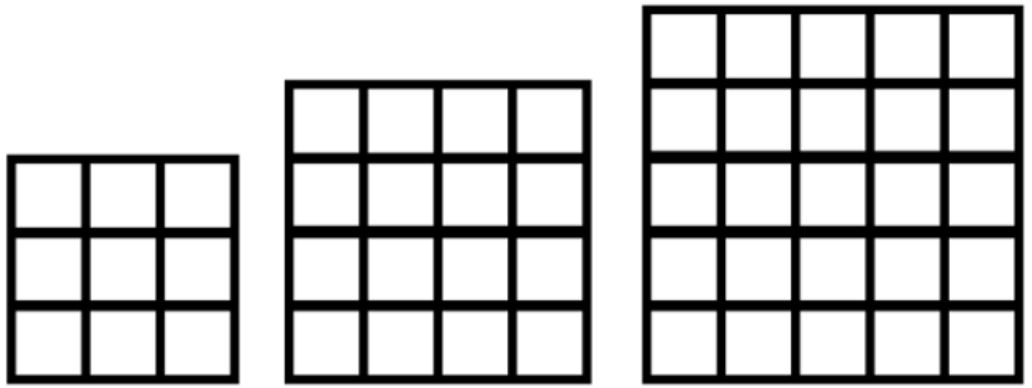
SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

The I LOVE RECTANGLES Game

We had a 1 by 2 rectangle, so we need a square that has a side of length 1 or a side of length 2. Looking at our squares, we see we can use the 2 by 2 square!

Building on this success, what should we put down next? Note we now have a rectangle that is 2 by 3….



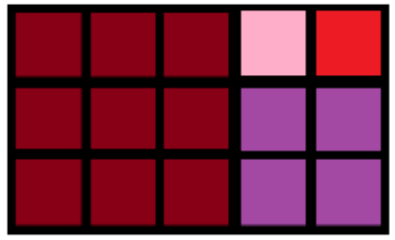


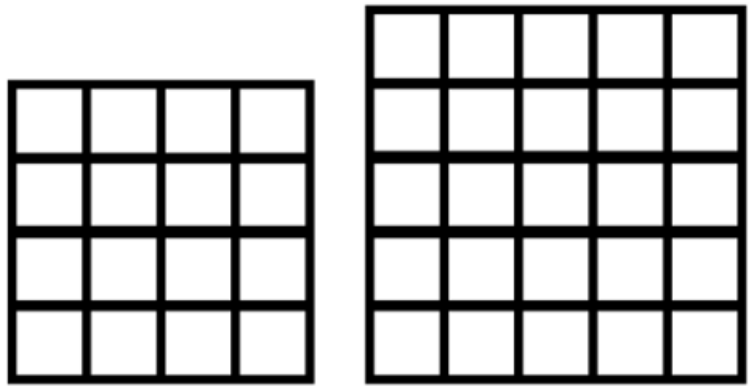
SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

The I LOVE RECTANGLES Game

We had a 2 by 3 rectangle, so we need a square that has a side of length 2 or a side of length 3. Looking at our squares, we see we can use the 3 by 3 square!

Building on this success, what should we put down next? Note we now have a 3 by 5 rectangle.





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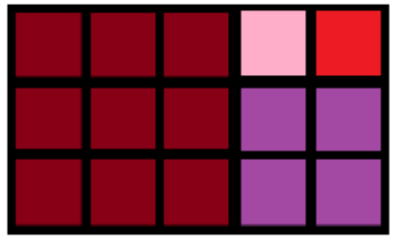
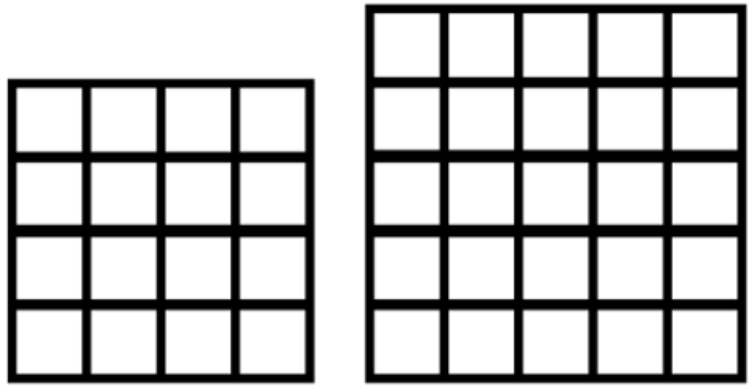
The I LOVE RECTANGLES Game

We had a 2 by 3 rectangle, so we need a square that has a side of length 2 or a side of length 3. Looking at our squares, we see we can use the 3 by 3 square!

Building on this success, what should we put down next? Note we now have a 3 by 5 rectangle. Hint: the 4 by 4 square does not fit!



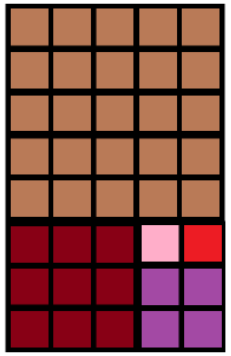


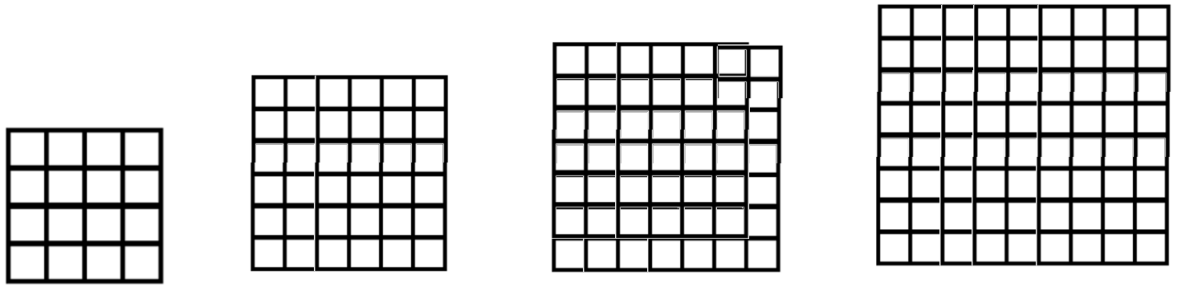
SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

The I LOVE RECTANGLES Game

We had a 3 by 5 rectangle. Looking at our squares, we see we can use the 5 by 5 square!

Building on this success, what should we put down next? Note we now have a 5 by 8 rectangle. The 4 by 4 is too small, we still have a 6 by 6, …..



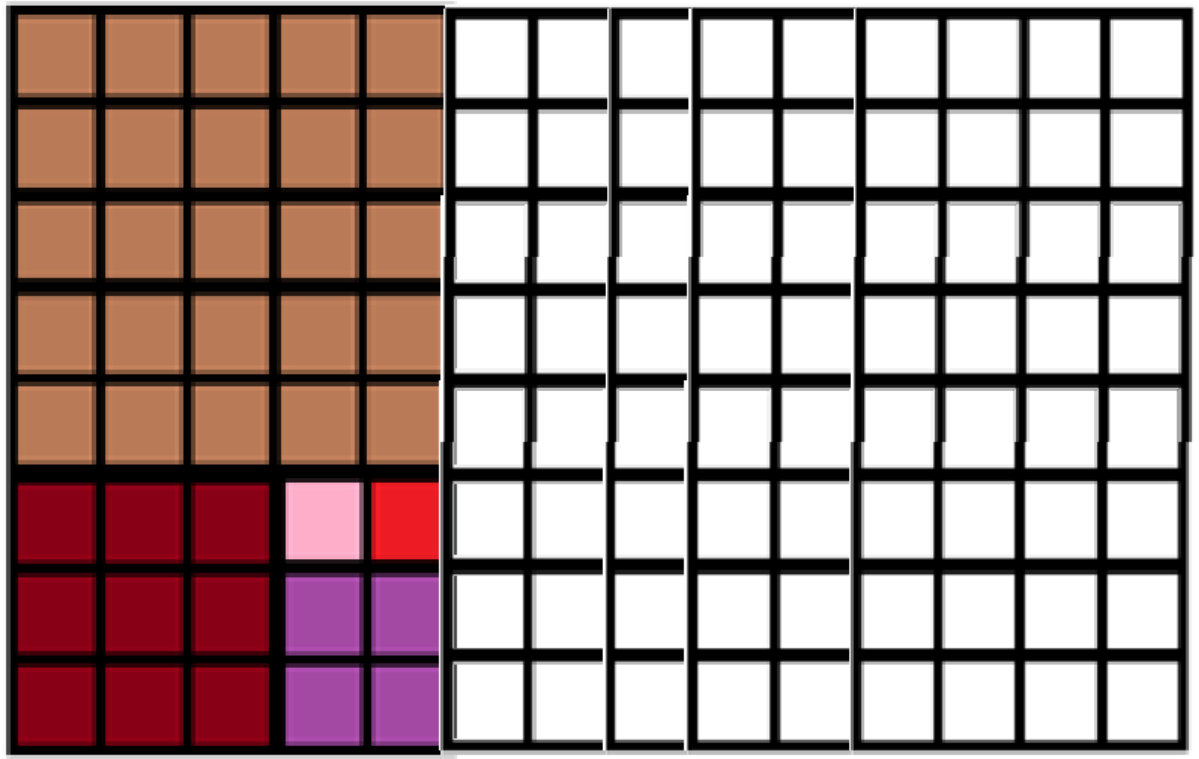


We still have a 6 by 6, a 7 by 7, an 8 by 8, a 9 by 9 (not drawn), a 10 by 10 (not drawn), and so on…..

SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

The I LOVE RECTANGLES Game

We had a 5 by 8 rectangle. We need to add something with a side of length 5 or 8. Thus we won't use the 4 by 4, the 6 by 6 or the 7 by 7, but we will use the 8 by 8……



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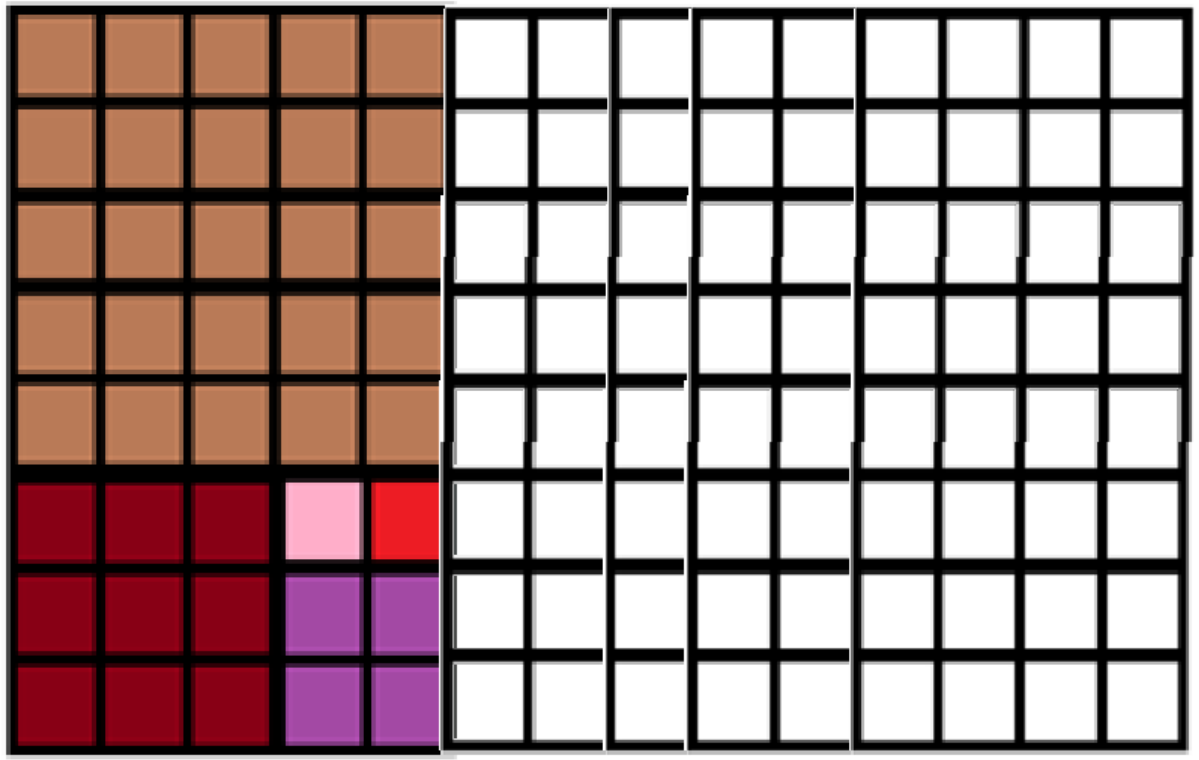
We write down the squares used in the order used:

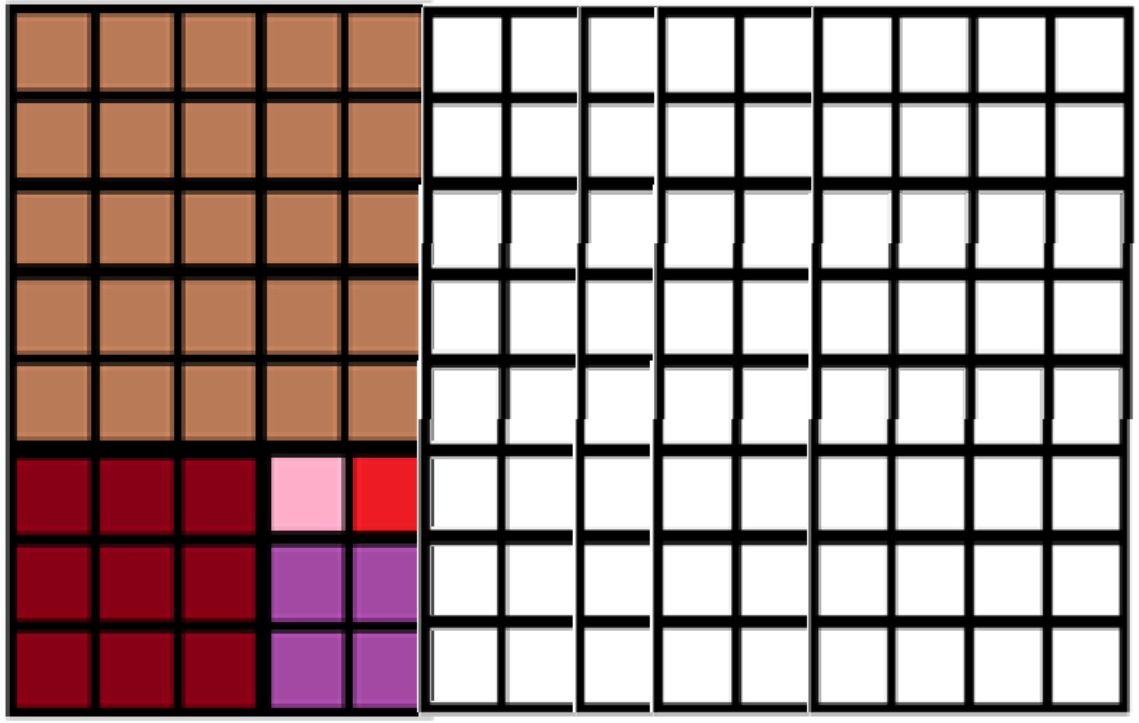
1 by 1, 1 by 1, 2 by 2, 3 by 3, 5 by 5, 8 by 8, …..



The I LOVE RECTANGLES Game

Let's just write down the side lengths of the squares in the order used:

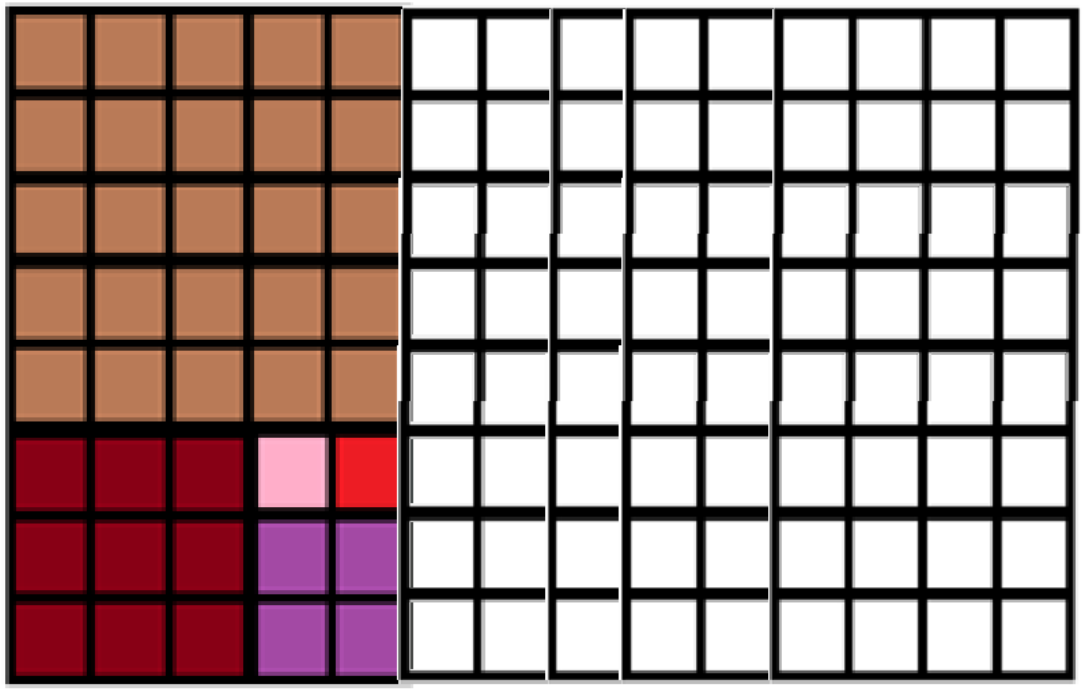
1, 1, 2, 3, 5, 8, …. DO YOU NOTICE A PATTERN?



The I LOVE RECTANGLES Game

Let's just write down the side lengths of the squares in the order used (we'll add a few more terms to the sequence):

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, …. DO YOU NOTICE A PATTERN?



SPEND A MOMENT AND SEE IF YOU CAN ANSWER THIS!

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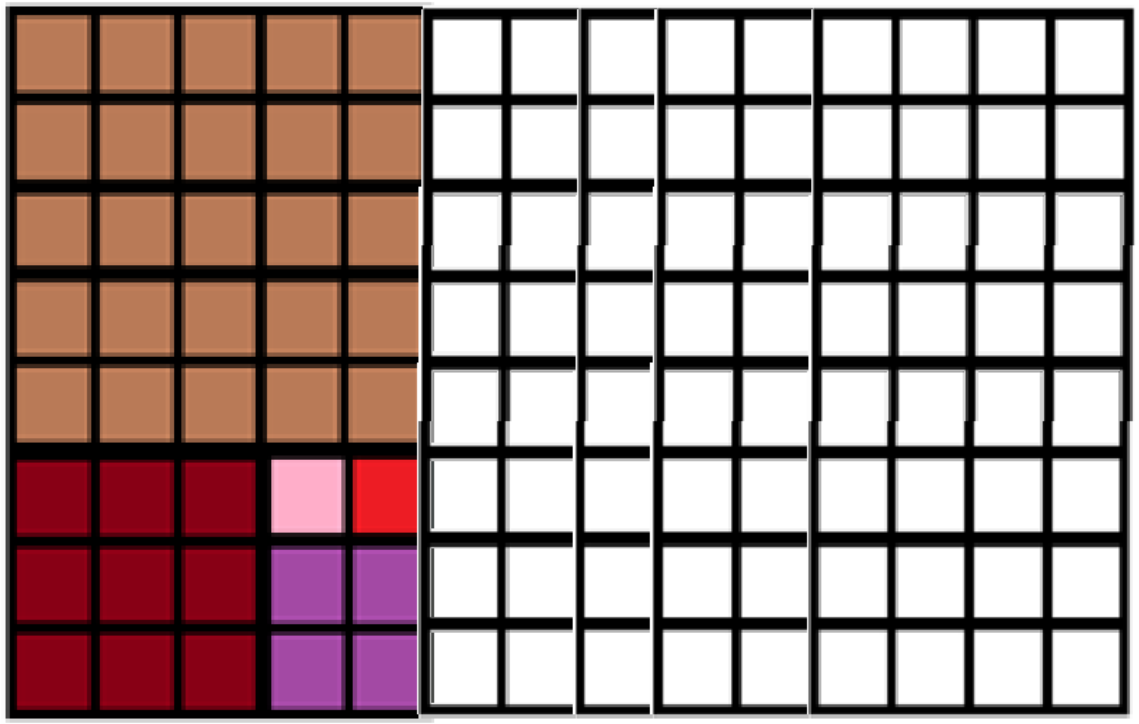
Let's just write down the side lengths of the squares in the order used:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ….

We start 1, 1, and then after that each term is the sum of the previous two terms! 2 = 1 + 1, 3 = 2 + 1, 5 = 3 + 2, 8 = 5 + 3, and so on. Can you continue the pattern?

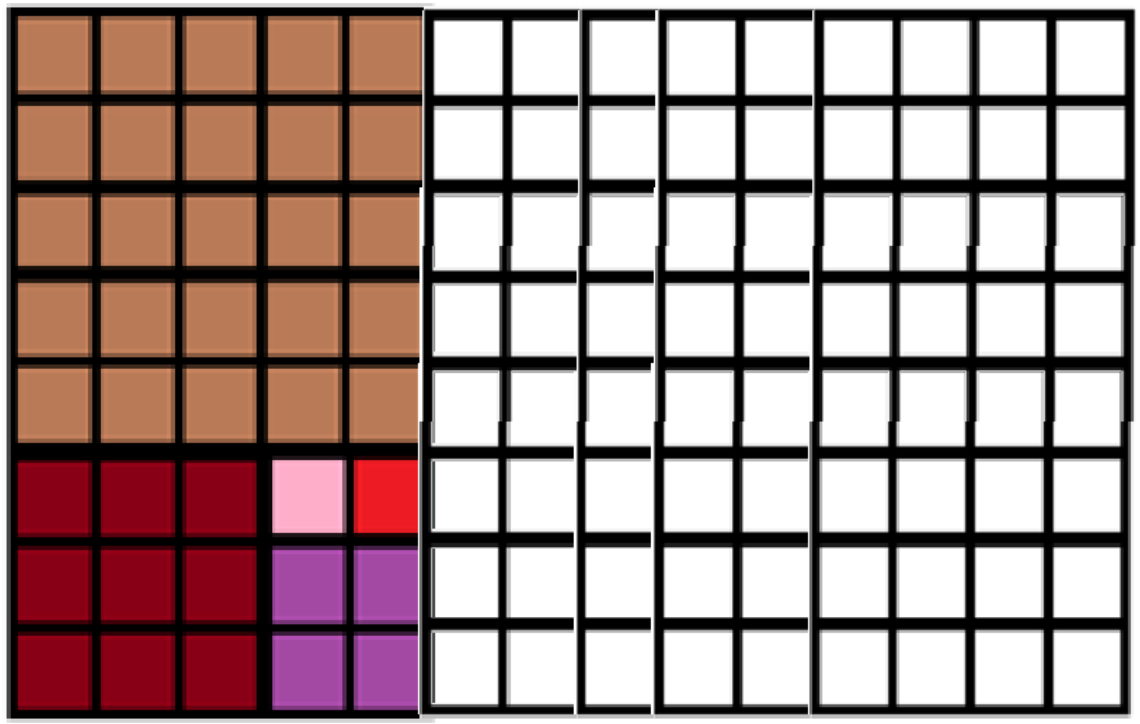


The Fibonacci Sequence

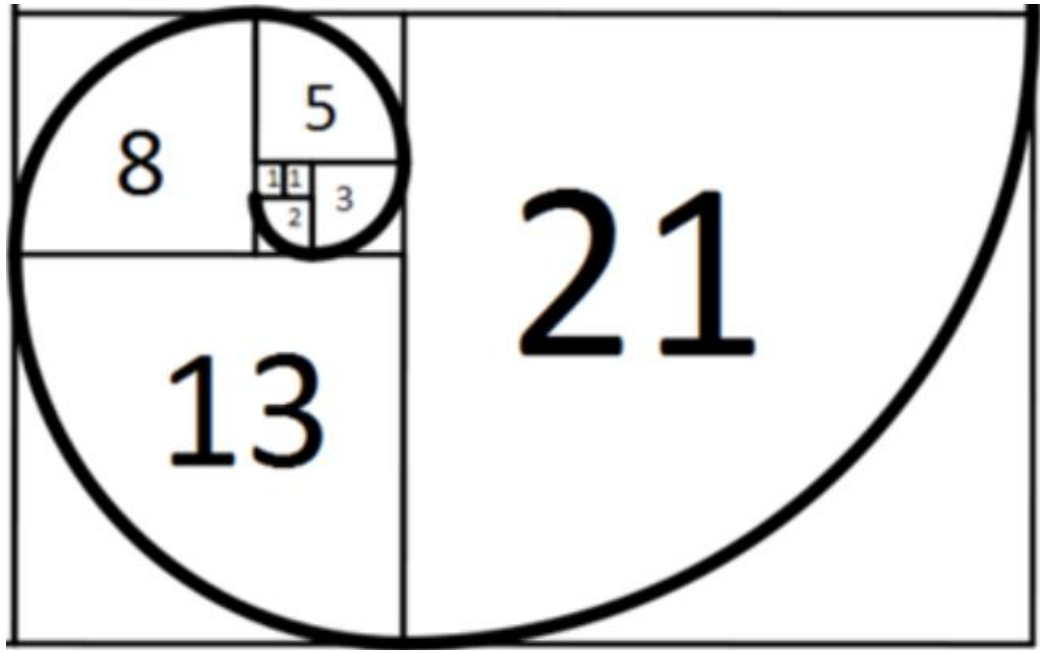
The numbers

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, ….

are called the Fibonacci numbers, and have many wondrous properties. See for example https://www.youtube.com/watch?v=me6Dnl2DOtM .



ADVANCED TOPIC!



Advanced: you can calculate area two ways. It is length times width, which here is 21 by 34. It is also the sum of the areas of each square, which is $1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 8^2 + 13^2 + 21^2$. These are equal! You can thus prove the sum of the squares of the first n Fibonacci numbers is the n^{th} Fibonacci number times the $(n+1)^{st}$ Fibonacci number!