JOURNAL OF NUMBER THEORY

Teachers As Scholars
henry.bolter@gmail.com

These math outreach lectures are supported in part by the Journal of Number Theory and the Teachers as Scholars program; it is a pleasure to thank them for their support.

Introduction to Probability

Steven J Miller, Williams College

sjm1@Williams.edu

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

Part I: Factorials

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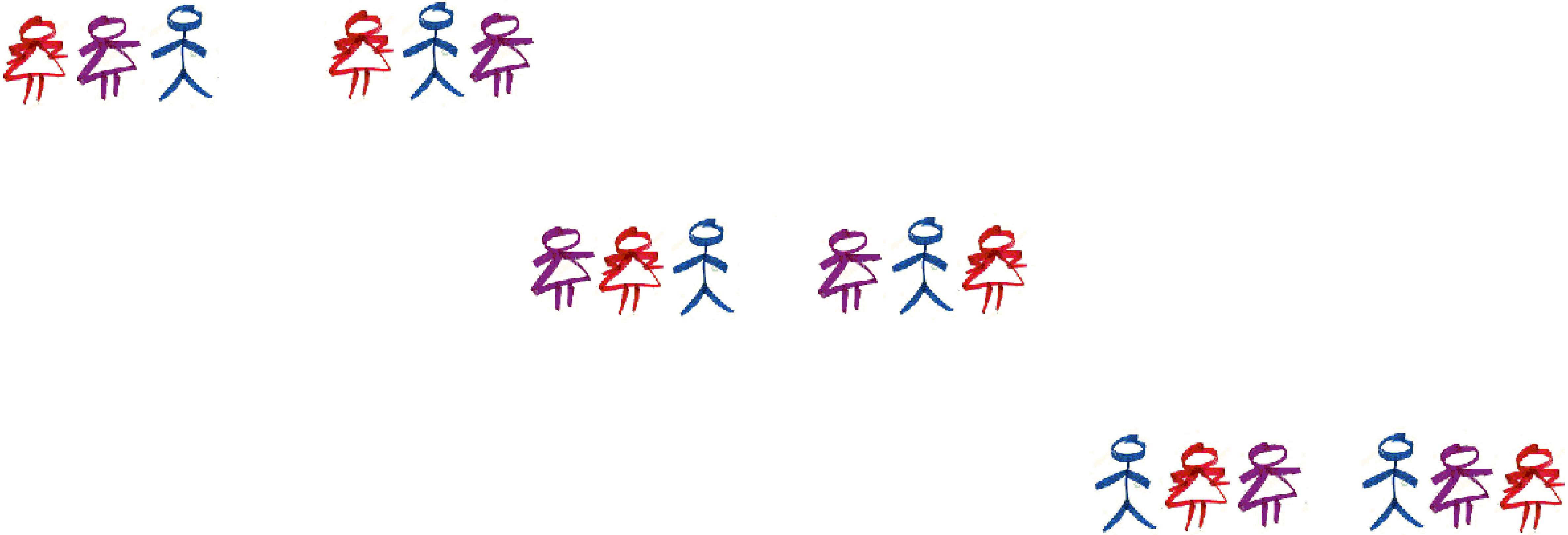

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Basics

- Will explore simple probabilities.
- Problems from card games, dice, ….
- In many situations each event is equally likely, so the probability an event A happens is the number of ways A can happen, divided by the number of ways something can happen.

- Example: Imagine we roll a fair die; our result is in the set {1, 2, 3, 4, 5, 6}.
- What is the probability we roll a 6?
- What is the probability we roll an even number?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

Basics

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- In many situations each event is equally likely, so the probability an event A happens is the number of ways A can happen, divided by the number of ways something can happen.

- Example: Imagine we roll a fair die; our result is in the set {1, 2, 3, 4, 5, 6}.
- What is the probability we roll a 6? 1/6 (six possible rolls, only one is a 6)
- What is the probability we roll an even number? 3/6 or 1/2 (six possible rolls, and 2, 4 and 6 are even while the rest are odd).

Factorial Function n!

One of the most useful functions in probability is the **factorial function**. We denote the factorial of an integer n by writing **n!** (so n followed by an exclamation point).

It has a nice meaning: n! is the number of ways to arrange n objects when order matters.

Thus if we want to order the elements of {a} there is just one way.

If we want to order the elements of {a,b} there are two ways: ab and ba.

If we want to order the elements of {a,b,c} there are six ways:

abc, acb, bac, bca, cab, cba (why did we list this way?)

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Factorial Function n!

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If we want to order the elements of {a,b,c} there are six ways:

abc, acb, bac, bca, cab, cba (why did we list this way?)

We want to make sure we do not miss any cases. We first do all the ways with a as the first element, and then use the PREVIOUS result about how to order a set of two elements. Then we do all the ways with b as the first element, then all the ways with c first.

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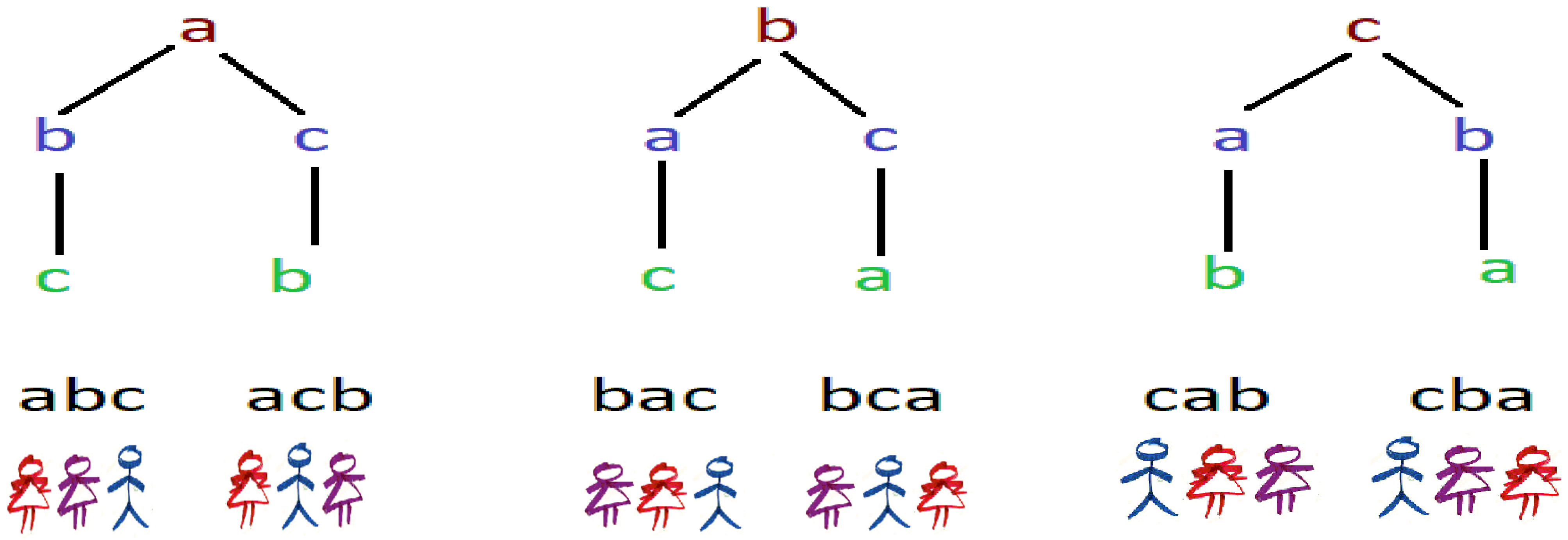

Factorial Function n!

It has a nice meaning: n! is the number of ways to arrange n objects when order matters.

How many ways are there to order {a,b,c,d}?

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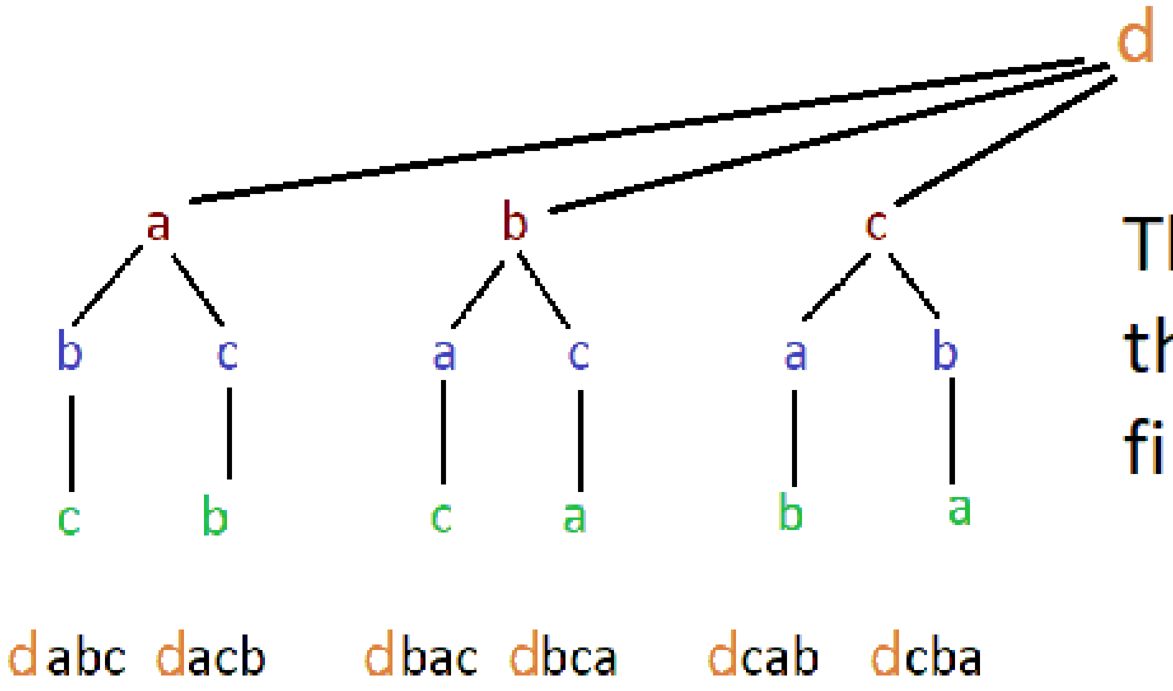
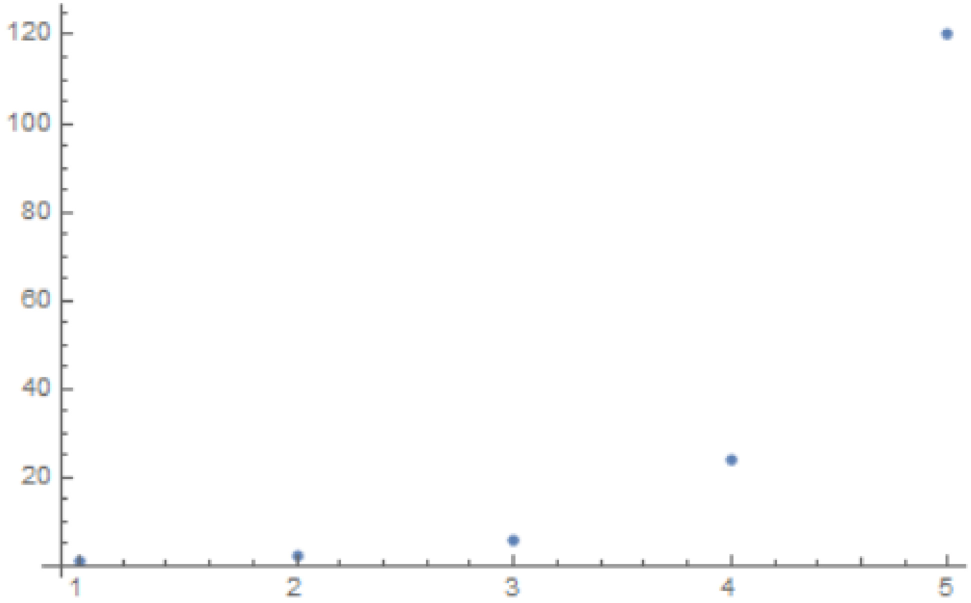
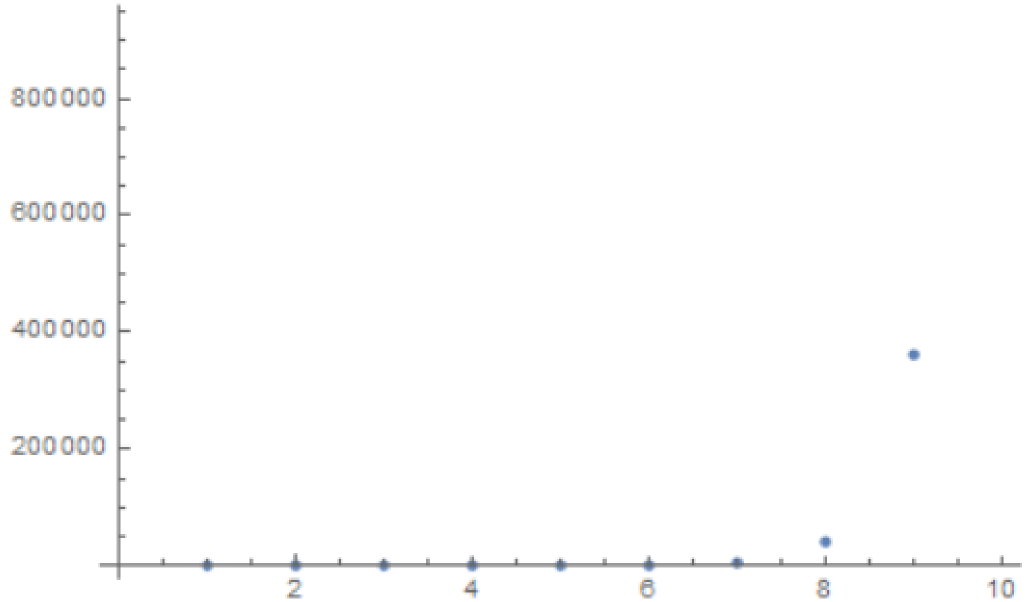


This is one-fourth of the tree. Also have a first, b first, c first.

Factorial Function n!

The factorial function grows very rapidly.





- The number of ways to order a deck of 52 cards is 52!, which is 80658175170943878571660636856403766975289505440883277824000000000000 (or about 10^{68}). To put in perspective, there are about 10^{80} to 10^{90} objects in the universe!
- The number of ways to order the standard alphabet is 26! which is 403291461126605635584000000 (or about 10^{26}).

Factorial Function n!

What should 0! equal?

4! = 24 3! = 6 2! = 2 1! = 1

We have the interpretation that n! is the number of ways to order n objects when order matters.

How many ways are there to order zero objects?

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Factorial Function n!

What should 0! equal?

4! = 24 3! = 6 2! = 2 1! = 1

We have the interpretation that n! is the number of ways to order n objects when order matters.

How many ways are there to order zero objects?

We DEFINE 0! to be 1; there is one way to do nothing (you can't do nothing in more than one way). This turns out to be a VERY useful definition.

Factorial Function and Recursion

Note $4! = 4 * 3 * 2 * 1 = 4 * (3 * 2 * 1) = 4 * 3!$.

Similarly $5! = 5 * 4 * 3 * 2 * 1 = 5 * (4 * 3 * 2 * 1) = 5 * 4!$.

If you know n! it is very easy to find (n+1)!

$$
\begin{aligned}
(n+1)! &= (n+1) * n * \ldots * 3 * 2 * 1 \\
&= (n+1) * (n * (n-1) * \ldots * 3 * 2 * 1) \\
&= (n+1) * n!
\end{aligned}
$$

(If you have seen Fibonacci numbers, another example of a recurrence.)

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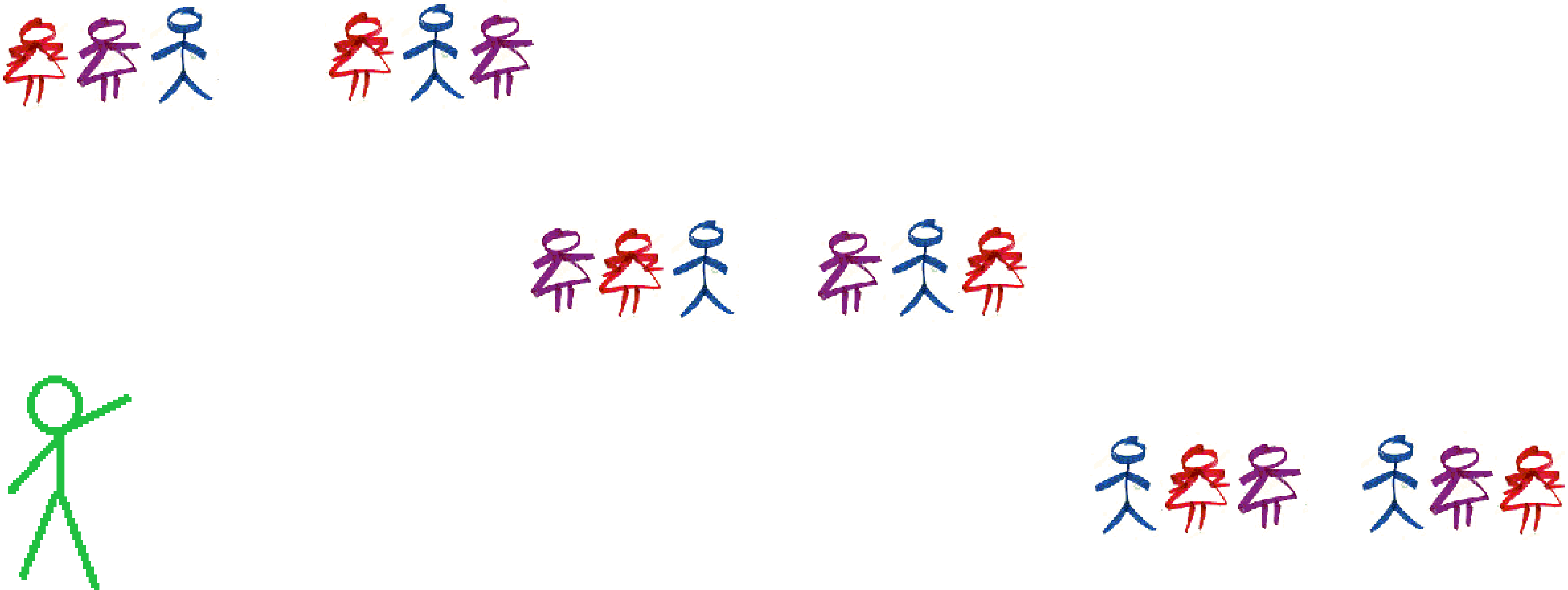
Part II: Permutations



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Permutations: nPr or ${}_nP_r$

The factorial function is great for ordering n objects; what if we only want to order some of them?

For example, imagine we have five people: {a, b, c, d, e}. How many ways are there to choose two people where order matters (so the first chosen is president, the second is vice-president)? How many ways are there to choose three people where order matters?

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Permutations: nPr

The factorial function is great for ordering n objects; what if we only want to order some of them? nPr or ${}_nP_r$ is the number of ways to choose r people from n when order matters. P stands for permutations.

For example, imagine we have five people: {a, b, c, d, e}.

How many ways are there to choose two people where order matters (so the first chosen is president, the second is vice-president)?

- 20 ways: We have 5 choices for the first spot, and then 4 for the second: 5 * 4 = 20.

How many ways are there to choose three people where order matters?

- 60 ways: We have 5 choices for the first spot, 4 for the second, then 3 for the third: 5 * 4 * 3 = 60.

We denote the first 5P2 or ${}_5P_2$ and the second 5P3 or ${}_5P_3$.

Permutations: nPr

The factorial function is great for ordering n objects; what if we only want to order some of them? nPr or ${}_nP_r$ is the number of ways to choose r people from n when order matters. P stands for permutations.

What if we want to choose 4 people from 11, order matters?

We have 11 choices for the first, then 10 for the second, then 9 for the third, then 8 for the fourth, or $11 \ * 10 \ * 9 \ * 8$.

How can you write using factorials? Hint: you must do one of the most important algebraic attacks in mathematics: you multiply by 1. This doesn't change the answer, but allows you to re-write the algebra.

STOP STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION. STOP

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HINT: Note $11 * 10 * 9 * 8$ looks a lot like 11!; what would we need to multiply it by to get 11!? You can't multiply by anything other than 1 or you change the value, so what should we multiply it by?

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HINT: Note $11 * 10 * 9 * 8$ looks a lot like 11!; what would we need to multiply it by to get 11!? You can’t multiply by anything other than 1 or you change the value, so what should we multiply it by? Multiply by 7!/7!, so

$$11 * 10 * 9 * 8 = 11 * 10 * 9 * 8 * \frac{7!}{7!} = \frac{11 * 10 * 9 * 8 * 7!}{7!} = \frac{11!}{7!}$$

Permutations: nPr

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More generally, what if we want to choose r people from n, order matters?

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What if we want to choose 4 people from 11?

$$11 * 10 * 9 * 8 = 11 * 10 * 9 * 8 * \frac{7!}{7!} = \frac{11 * 10 * 9 * 8 * 7!}{7!} = \frac{11!}{7!}$$

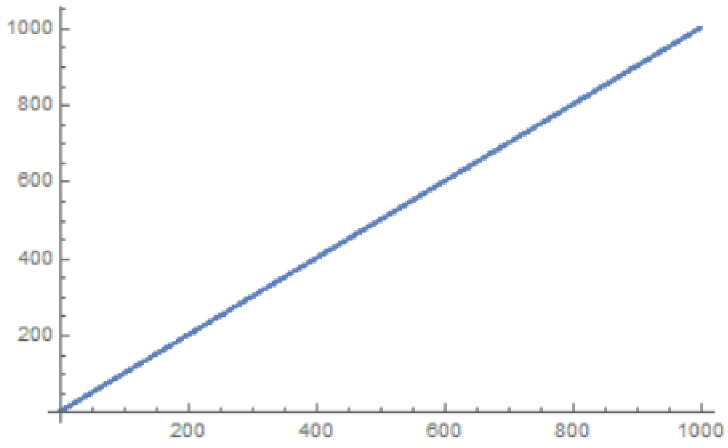
More generally, what if we want to choose r people from n, order matters? It is

$$n * (n-1) * \dots * (n-(r-1)) = n * (n-1) * \dots * (n-(r-1)) * \frac{(n-r)!}{(n-r)!}$$

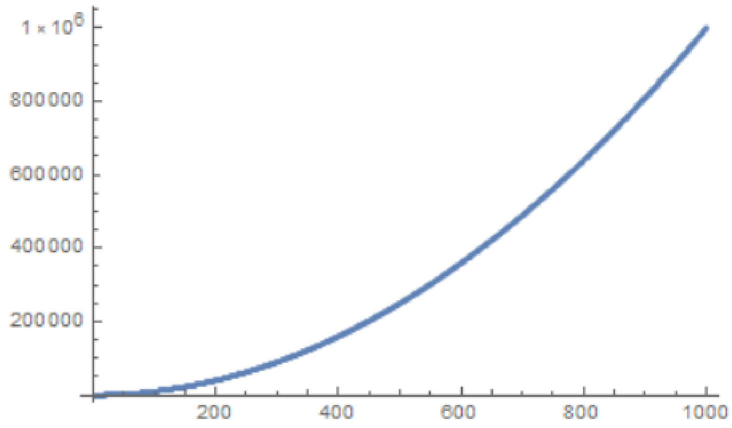
We thus find nPr or ${}_nP_r$ equals $\frac{n!}{(n-r)!}$

Permutations: nPr

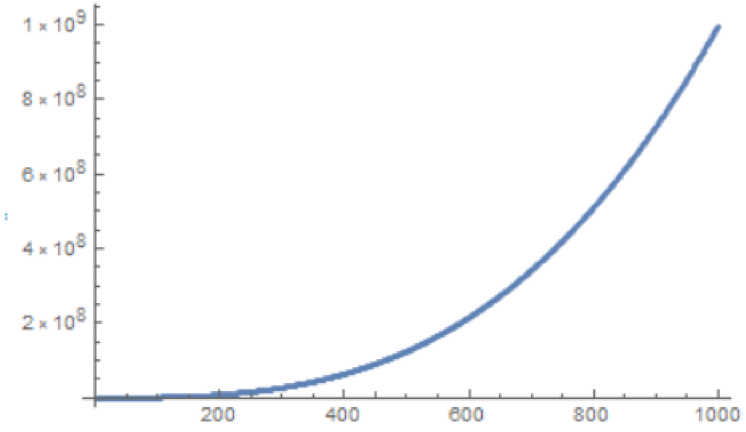
We thus find nPr or ${}_nP_r$ equals $\frac{n!}{(n-r)!}$. Grows a lot slower than n!



r = 1



r = 2



r = 3

What do these growth rates look like? They look like …..

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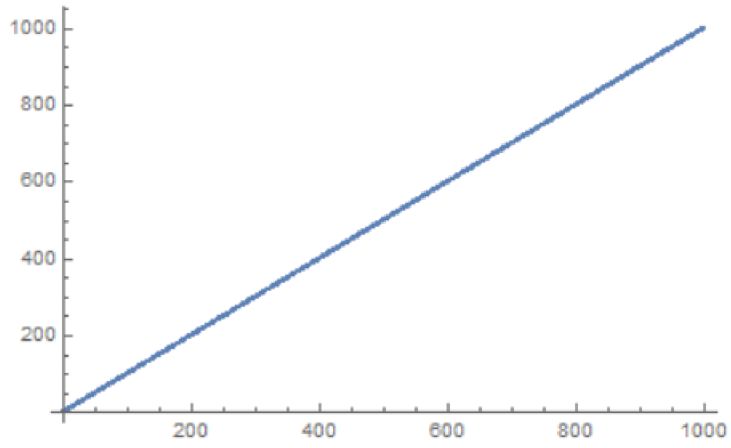
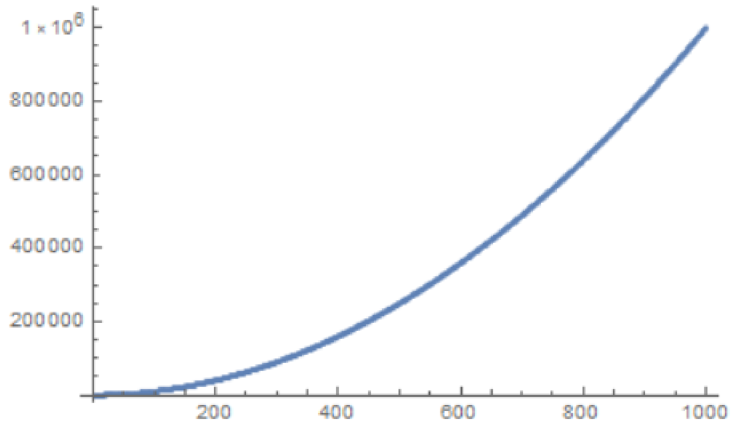
Permutations: nPr

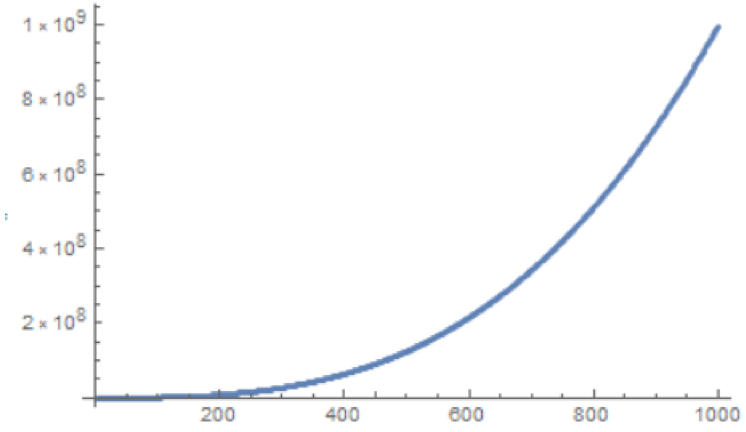
We thus find nPr or ${}_nP_r$ equals $\frac{n!}{(n-r)!}$. Grows a lot slower than n!



r = 1



r = 2



r = 3

What do these growth rates look like? They look like polynomials.

First looks linear, others are harder but are quadratics and then cubics.

Can we prove? Can we find formulas for these?

Permutations: nPr

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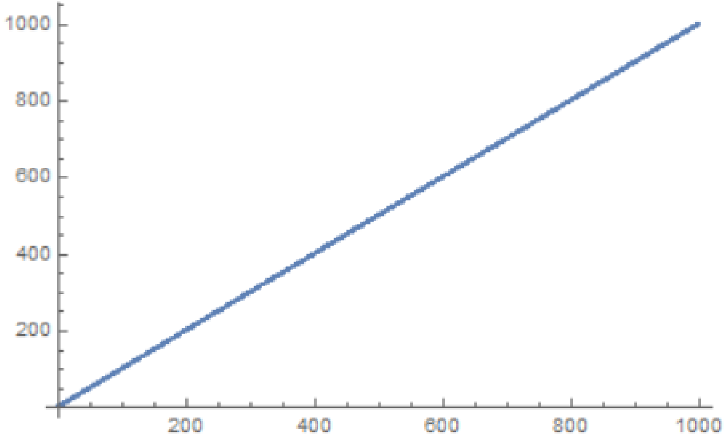
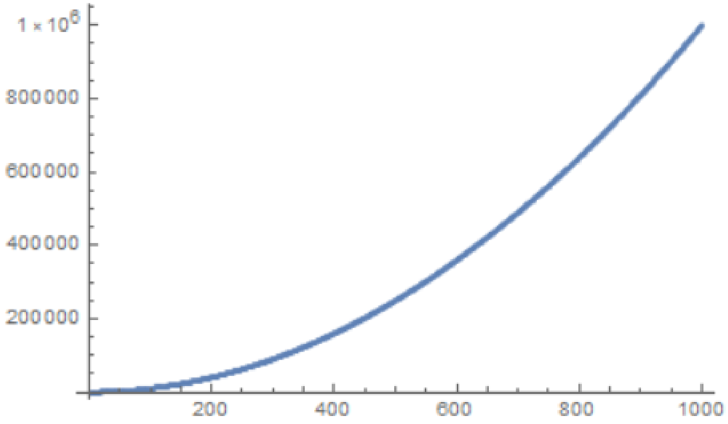
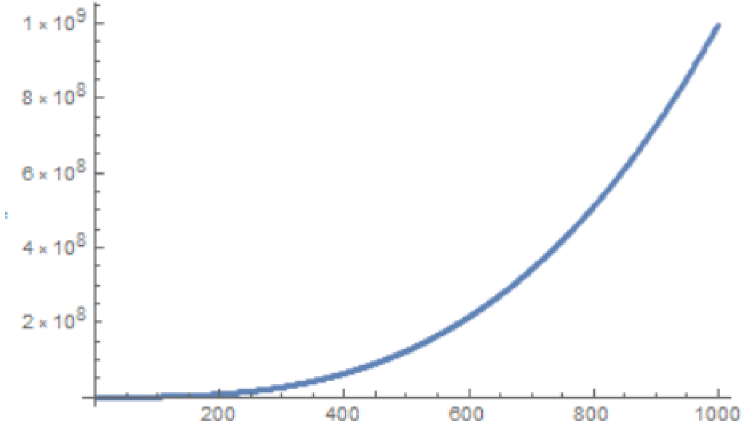


r = 1



r = 2

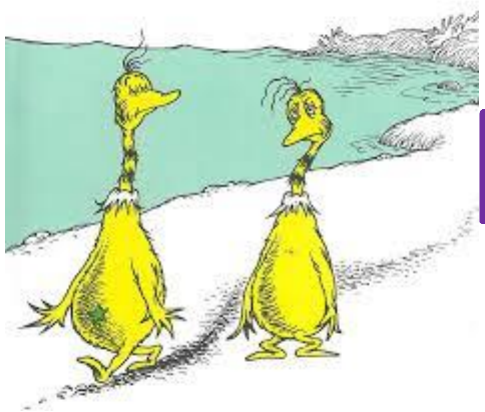


r = 3

nP1 or ${}_nP_1$ is n!/(n-1)!. As n! = n * (n-1)! we have it equals n, so linear!

nP2 or ${}_nP_2$ is n!/(n-2)!. As n! = n * (n-1) * (n-2)! we have it equals n * (n-1), which equals $n^2 - n$, so quadratic!

In general, nPr is a polynomial of degree r. Good exercise to prove this.



Part III: Combinations

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1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

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Permutation Review: nPr or $_nP_r$

We saw n! is the number of ways of order n objects when order matters.

Then nPr or $_nP_r$ is the number of ways of choosing r objects from n, when order matters. Note n! is nPr for some r – what r is it?

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Permutation Review: nPr or ${}_nP_r$

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We saw we could write 11P4 as $11 * 10 * 9 * 8 \quad OR \quad \frac{11!}{7!}$.

Both have advantages. We can see what is going on with $11 * 10 * 9 * 8$, BUT if we have a factorial function defined then we can compute 11!/7! faster; imagine having to write out the product of 2020P1010.

Additionally, the ratio of factorials will help us see connections later in nCr.

Combinations: nCr or ${}_nC_r$

We saw n! is the number of ways of order n objects when order matters.

Then nPr or ${}_nP_r$ is the number of ways of choosing r objects from n, when order matters. Note n! is nPr for some r – it is r = n.

We now introduce a new function: nCr or ${}_nC_r$ is the number of ways to choose r objects from n when order DOES NOT matter; the C stands for combinations.

Let's evaluate ${}_nC_r$ for some r; can you think of some r where it is easy to figure out what ${}_nC_r$ is? Try r = ….

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Combinations: nCr or ${}_nC_r$

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Let's evaluate ${}_nC_r$ for some r; can you think of some r where it is easy to figure out what ${}_nC_r$ is? We find ${}_nC_0 = {}_nC_n = 1$.

What would ${}_nC_r$ be when $r = 1$ or $n-1$? Note this is different than most presentations; we have NOT given the formula for ${}_nC_r$ but instead are trying to find it....

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Combinations: nCr or ${}_nC_r$

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Let's evaluate ${}_nC_r$ for some r. We find ${}_nC_0 = {}_nC_n = 1$.

What would ${}_nC_r$ be when r = 1 or n-1?

- If r = 1 we choose just one person. There are n ways to choose one person so ${}_nC_1 = n$.
- If r = n-1 we choose everyone but one person for our set. There are n ways to choose a person to exclude, so ${}_nC_{n-1} = n$.

Hmm. Notice ${}_nC_0 = {}_nC_n = 1$ and ${}_nC_1 = {}_nC_{n-1} = n$. Any thoughts on what might be true more generally? Maybe compare the values at r and at what?

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Combinations: nCr or ${}_nC_r$

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Hmm. Notice ${}_nC_0 = {}_nC_n = 1$ and ${}_nC_1 = {}_nC_{n-1} = n$. Any thoughts on what might be true more generally? Maybe compare the values at r and n-r. Are these equal?

Combinations: nCr or ${}_nC_r$

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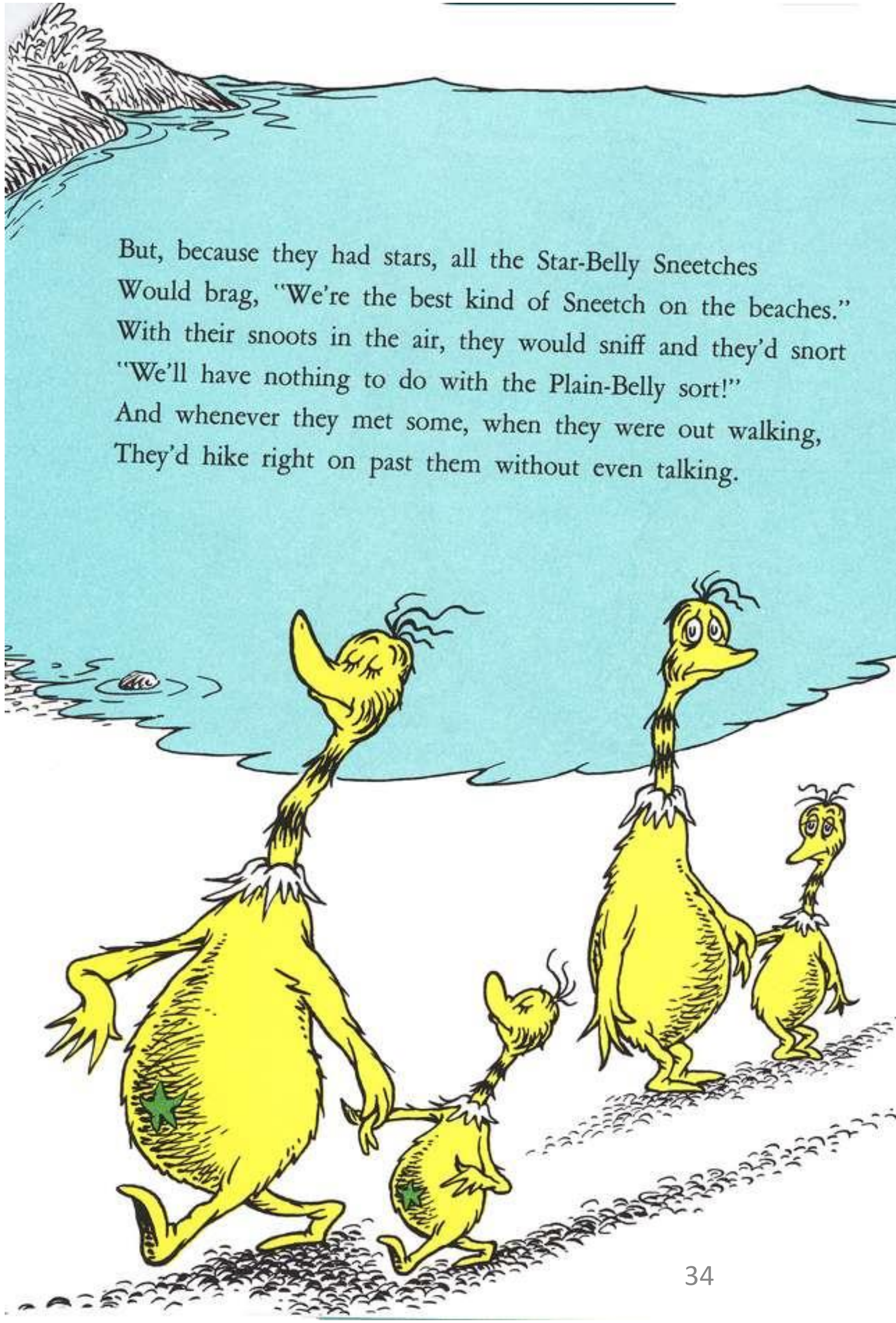
Compare the values at r and n-r. Are these equal?

Note choosing r from n to BE in the group is the same as choosing n-r to EXCLUDE.



Combinations: nCr or $_nC_r$

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The factorial function is great for ordering n objects; what if we only want to order some of them? nPr or $_nP_r$ is the number of ways to choose r people from n when order matters. P stands for permutations.

How do you think $nCr,\quad r!\ \ and\ \ nPr = \frac{n!}{(n-r)!}$ are related?

- r! is the number of ways to order r objects.
- nPr is the number of ways to choose r objects from n when order matters.
- nCr is the number of ways to choose r objects from n when order doesn't matter.

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Combinations: nCr or ${}_nC_r$

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The factorial function is great for ordering n objects; what if we only want to order some of them? nPr or ${}_nP_r$ is the number of ways to choose r people from n when order matters. P stands for permutations.

Claim: $nCr\ * r! = nPr = \frac{n!}{(n-r)!}$, thus $nCr = \frac{n!}{r!(n-r)!}$.

Proof: nPr is the number of ways to choose r objects from n when order MATTERS.

- How many ways are there to order r objects? This is just r!.
- Thus each UNORDERED group of size r contributes r! ORDERED sets.

Comparing nCr and nPr

nCr or $_nC_r$ is the number of ways to choose r objects from n when order DOES NOT matter, while nPr or $_nP_r$ is the number of ways to choose r people from n when order matters. P stands for permutations.

- How many ways are there to choose 13 cards from a deck of 52 cards when order matters?

- How many ways are there to choose 13 cards from a deck of 52 cards when order does not matter?

- What is the probability of getting a given set of 13 cards?

Comparing nCr and nPr

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- How many ways are there to choose 13 cards from a deck of 52 cards when order matters? 52P13 = 3,954,242,643,911,239,680,000 (about $10^{21.6}$).

- How many ways are there to choose 13 cards from a deck of 52 cards when order does not matter? 52C13 = 635,013,559,600 (about $10^{11.8}$)

- What is the probability of getting a given set of 13 cards? 1/52C13 or about $1/10^{11.8}$.

Distinct Deals in Bridge

nCr or $_{n}C_{r}$ is the number of ways to choose r objects from n when order DOES NOT matter, while nPr or $_{n}P_{r}$ is the number of ways to choose r people from n when order matters. P stands for permutations.

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

- How many ways are there to deal the cards?

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Distinct Deals in Bridge

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In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

- How many ways are there to deal the cards?

  Answer: $52C13 \ * 39C13 \ * 26C13 \ * 13C13$

  Equals 53,644,737,765,488,792,839,237,440,000 (about $10^{28.7}$)

Do you think in all of human history there have ever been two deals the same?

Distinct Deals in Bridge

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

- How many ways are there to deal the cards?

  Answer: $52C13 \ * 39C13 \ * 26C13 \ * 13C13$

  Equals 53,644,737,765,488,792,839,237,440,000 (about $10^{28.7295}$)

Do you think in all of human history there have ever been two deals the same?

Number of seconds since the universe began:

$60 * 60 * 24 * 366 \ * 14{,}000{,}000{,}000$ or about $10^{17.6}$.

About 108,000,000,000 people have been born, if each deals a hand a second since the dawn of time get up to about $10^{28.6795}$.

Part IV:

May the Fourth Probabilities

JOURNAL OF NUMBER THEORY

Teachers As Scholars
henry.bolter@gmail.com

These math outreach lectures are supported in part by the Journal of Number Theory and the Teachers as Scholars program; it is a pleasure to thank them for their support.

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

Happy Star Wars Day!

In honor of today being Star Wars Day (May the Fourth be with you), will discuss some probabilities inspired by Star Wars.

Pre-requisite: **The Geometric Series Formula** (covered in an earlier lecture: Induction and Sums: Part III: From the Geometric Series Formula to Primes https://youtu.be/UWNM8EtzoMI, assuming algebra I, 22 minutes):

$$If\ \ |r| < \ 1\ \ then\ 1 + r + r^2 + r^3 + r^4 + \ \cdots = \frac{1}{1-r}$$

The Geometric Series Formula (Review)

The Geometric Series Formula is one of the most important in mathematics. It is one of the few sums we can evaluate exactly.

If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots = \frac{1}{1-r}$.

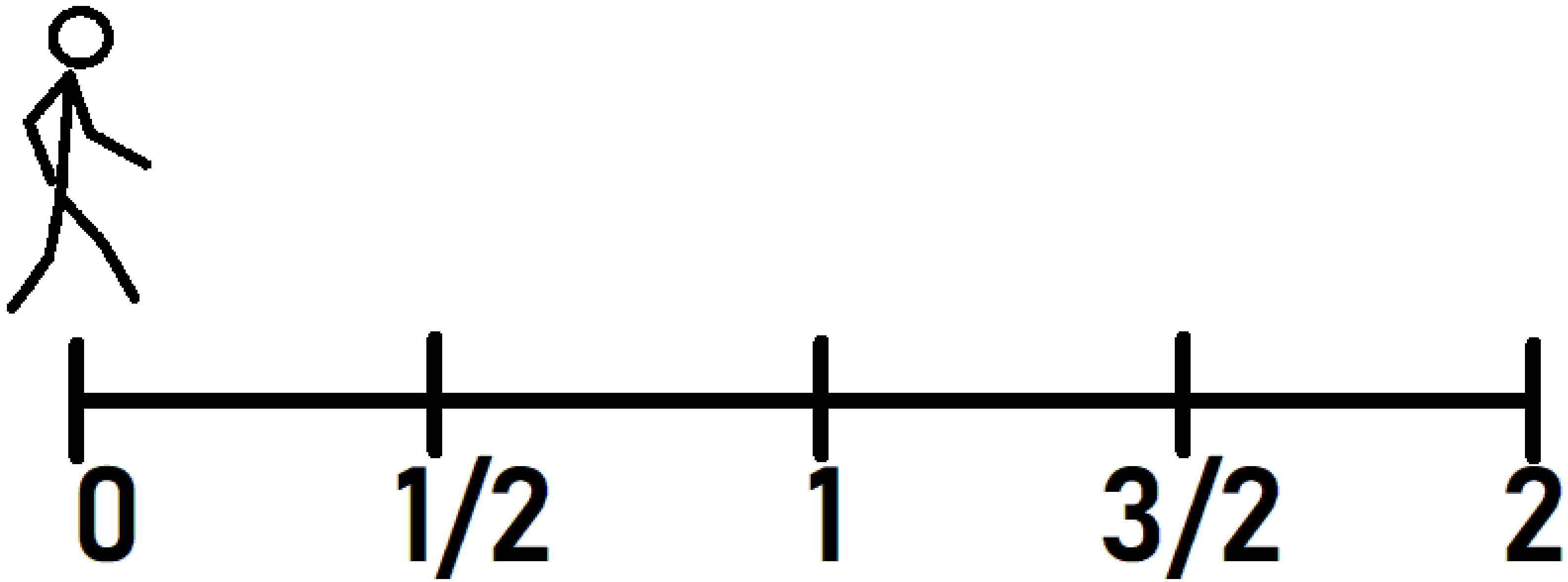
This is often proved by first computing the finite sum, up to r^n, and taking a limit. Note since $|r| < 1$ that each term r^n gets small fast.....

The Geometric Series Converges if |r| < 1: Review

$$1 + r + r^2 + r^3 + r^4 + \cdots = \frac{1}{1-r}.$$

Why does this converge? Take r = ½. We then have 1 + ½ + ¼ + …. = $\frac{1}{1-\frac{1}{2}} = 2$,

and we can view this as we start at 0, and each step covers half the distance to 2. We thus never reach it in finitely many steps, but we cover half the ground each time.



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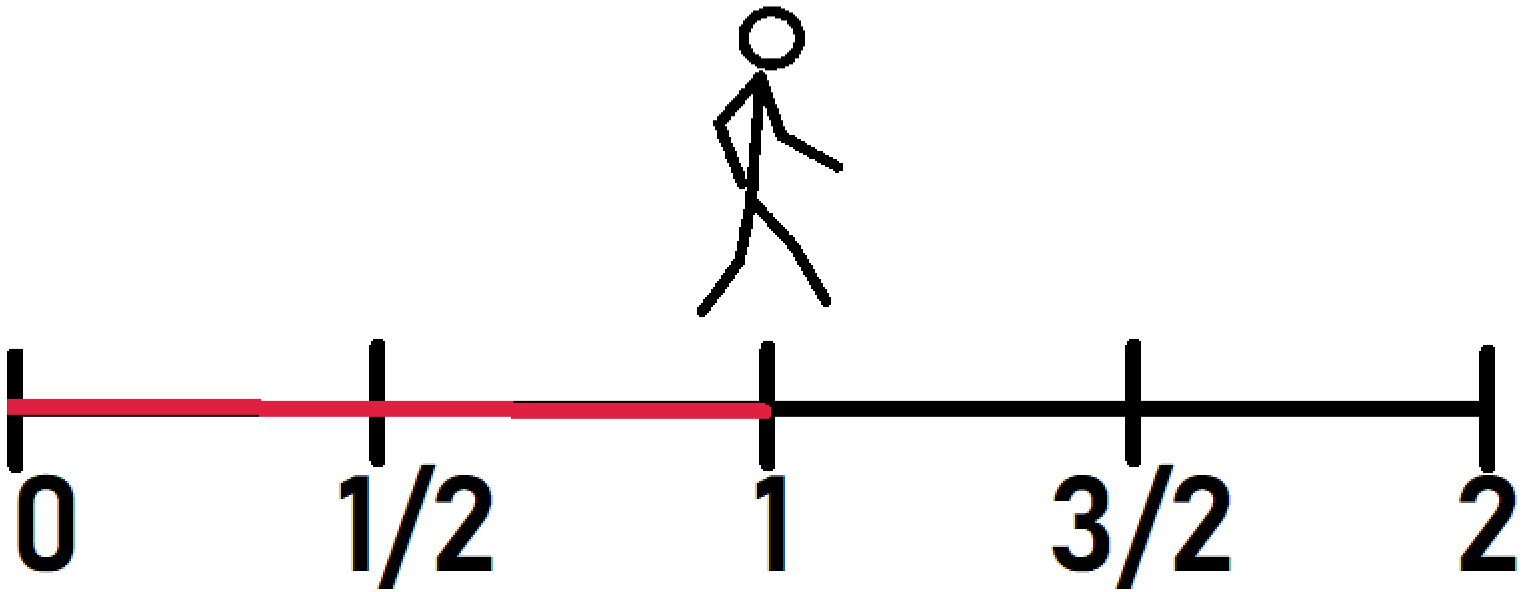

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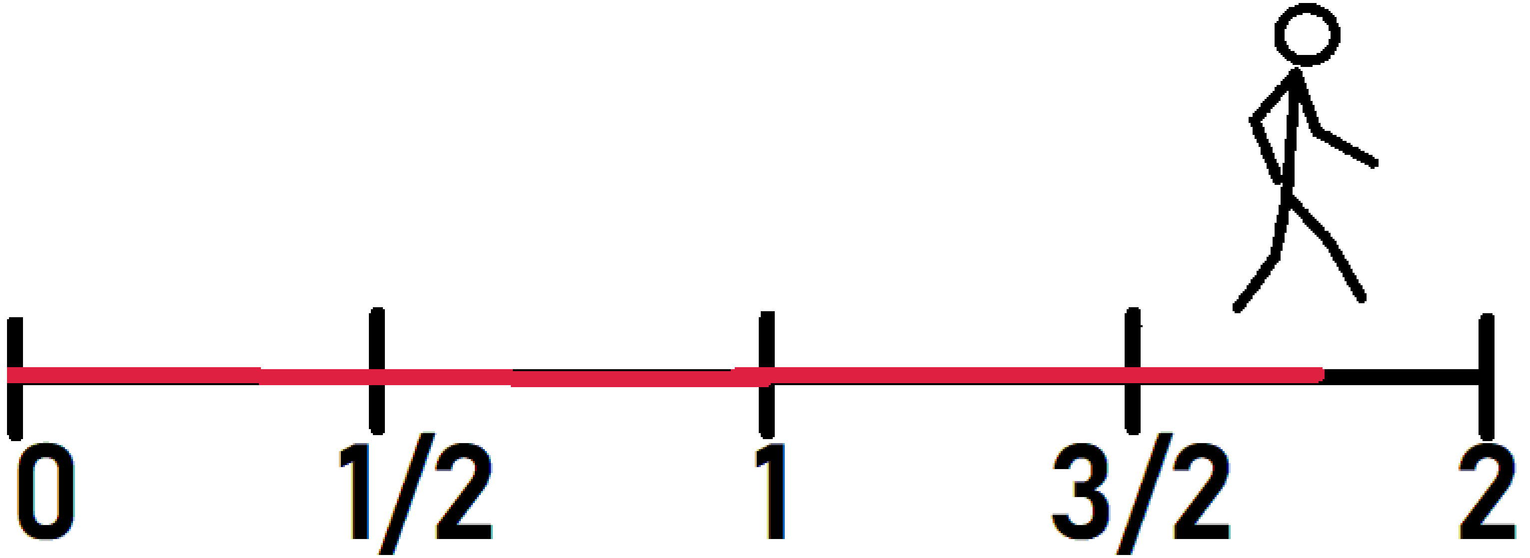

The Geometric Series Formula: Review

The Geometric Series Formula is one of the most important in mathematics. It is one of the few sums we can evaluate exactly.

Lemma: If $|r| < 1$ then $1 + r + r^2 + r^3 + r^4 + \ldots + r^n = \frac{1 - r^{n+1}}{1-r}$.

Proof: Let $S_n = 1 + r + r^2 + r^3 + r^4 + \ldots + r^n$

Then $\quad r S_n = \quad r + r^2 + r^3 + r^4 + \ldots + r^n + r^{n+1}$

What should we do now?

The Geometric Series Formula: Review

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Then $r\,S_n = \quad r + r^2 + r^3 + r^4 + \ldots + r^n + r^{n+1}$

Subtract: $S_n - r\,S_n = 1 - r^{n+1}$,

So $(1-r)\,S_n = 1 - r^{n+1}$, or S_n

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If we let n go to infinity, we see r^{n+1} goes to

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Subtract: $S_n - r\,S_n = 1 - r^{n+1}$,

So $(1-r)\,S_n = 1 - r^{n+1}$, or $S_n = \frac{1 - r^{n+1}}{1-r}$.

If we let n go to infinity, we see r^{n+1} goes to 0, so we get the infinite sum is $\frac{1}{1-r}$.

The Darth Vader Problem

Only the Emperor is less forgiving than Darth Vader; one mistake and you are dead! No one seems to fail him twice….

If your probability of failing him on a task is p, how many tasks till you die?

The Darth Vader Problem

If your probability of failing him on a task is p,

how many tasks till you die?

Could be unlucky and fail at the first task and die.

Could be very lucky and never fail and live a long, long time….

- What is the probability your first failure is on your first task?
- What is the probability your first failure is on your second task?
- What is the probability your first failure is on your third task?
- What is the probability your first failure is on your n^{th} task?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.


The Darth Vader Problem

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The Darth Vader Problem

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- What is the probability your first failure is on your n^{th} task? $(1-p)^{n-1}$ p

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

Here it is : $1 * Prob(first\ fail\ at\ 1) + 2 * Prob(first\ fail\ at\ 2) + \cdots$

The Darth Vader Problem

If your probability of failing him on a task is p,

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- What is the probability your first failure is on your first task? p
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The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

Here it is : $1 * p + 2 * (1-p)p + 3 * (1-p)^2 p + \cdots + n * (1-p)^{n-1} p + \cdots$

The Darth Vader Problem: LOWER BOUND

If your probability of failing a task is p, how many tasks till you die?

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)^2 \ + 4(1 - p)^3 + \cdots)$$

Note $p(1 + (1 - p) + (1 - p)^2 \ + (1 - p)^3 + \cdots) \leq S(p)$

Using the Geometric Series formula with r = ???

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The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1-p) + 3 * (1-p)^2 + 4(1-p)^3 + \cdots)$$

Note $p(1 + (1-p) + (1-p)^2 + (1-p)^3 + \cdots) \leq S(p)$

Using the Geometric Series formula with r = 1-p we get $p\frac{1}{1-(1-p)} \leq S(p)$

Gives the useless lower bound of S(p) is at least 1.

The Darth Vader Problem: UPPER BOUND

If your probability of failing a task is p, how many tasks till you die?

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1-p) + 3 * (1-p)^2 + 4(1-p)^3 + \cdots)$$

Note $p(1 + 2(1-p) + 2^2(1-p)^2 + 2^3(1-p)^3 + \cdots) \geq S(p)$

If (1-p) < ??? then we can use the geometric series with ratio r = ???.

The Darth Vader Problem: UPPER BOUND

If your probability of failing a task is p, how many tasks till you die?

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1-p) + 3 * (1-p)^2 \ + 4(1-p)^3 + \cdots)$$

Note $p(1 + 2(1-p) + 2^2(1-p)^2 \ + 2^3(1-p)^3 + \cdots) \geq S(p)$

If (1-p) < ½ then 2(1-p) < 1 so can use the Geometric Series formula and get

$p \ \frac{1}{1-2(1-p)} \geq S(p)$

For example, if p = ¾ gives an upper bound of 3/2 or 1.5.

The Darth Vader Problem

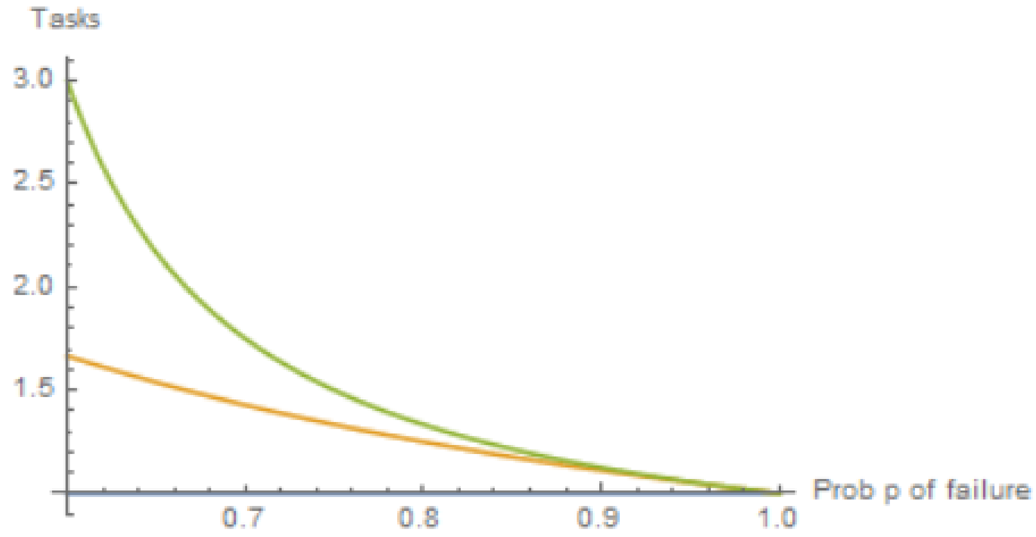
If your probability of failing a task is p, how many tasks till you die?

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1-p) + 3 * (1-p)^2 \; + 4(1-p)^3 + \cdots)$$

Bounds: $If\ (1-p) < \frac{1}{2}\ so\ p > \frac{1}{2}$

$Then\ 1 \ \leq S(p) \leq \frac{p}{1 - 2(1-p)}$.



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If your probability of failing a task is p, how many tasks till you die?

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)^2 \ + 4(1 - p)^3 + \cdots)$$

Using Calculus one can show S(p) = 1/p; is this formula reasonable?

Look at extreme cases: what happens as p goes to 0 or 1?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

The Darth Vader Problem

If your probability of failing a task is p, how many tasks till you die?

The EXPECTED VALUE of a random variable is the sum of the product of each value it takes on times the probability it takes on that value.

$$S(p) = p(1 + 2 * (1-p) + 3 * (1-p)^2 \; + 4(1-p)^3 + \cdots)$$

Using Calculus one can show S(p) = 1/p; is this formula reasonable?

Look at extreme cases: what happens as p goes to 0 or infinity?

- *As p goes to 1 you are a complete failure, and only do one tasks.*
- *As p goes to 0 you never fail, and tasks goes to infinity!*

The Darth Vader Problem

Probability of failing a task is p, how many tasks till you die?

$$S(p) = p(1 + 2 * (1-p) + 3 * (1-p)^2 \ + 4(1-p)^3 + \cdots)$$

Let q = 1-p. Note this is $p(1 + 2q + 3q^2 + 4q^3 + \ \cdots)$.

We can rewrite: It is

$p(1 + q + q^2 + q^3 + \ \cdots) + p(q + q^2 + q^3 + \ \cdots) + p\ (q^2 + q^3 + q^4 + \ \cdots) + \cdots$

Each is a geometric series with ratios ???

The Darth Vader Problem

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$p(1 + q + q^2 + q^3 + \cdots) + p(q + q^2 + q^3 + \cdots) + p\,(q^2 + q^3 + q^4 + \cdots) + \cdots$

Each is a geometric series with ratios q, q, q, … but different starting terms.

$S(p) = p\,(1 + q + q^2 + \cdots) + pq\,(1 + q + q^2 + \cdots) + pq^2\,(1 + q + q^2 + \cdots) + \cdots$

$S(p) = (p + pq + pq^2 + pq^3 + \cdots)\,\frac{1}{1-q} \; =$

The Darth Vader Problem

Probability of failing a task is p, how many tasks till you die?

$$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)^2 + 4(1 - p)^3 + \cdots)$$

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Each is a geometric series with ratios q, q, q, … but different starting terms.

$S(p) = p\,(1 + q + q^2 + \cdots) + pq\,(1 + q + q^2 + \cdots) + pq^2\,(1 + q + q^2 + \cdots) + \cdots$

$S(p) = (p + pq + pq^2 + pq^3 + \cdots)\frac{1}{1-q} = p\,(1 + q + q^2 + q^3 + \cdots)\frac{1}{1-q} = p\,\frac{1}{1-q}\,\frac{1}{1-q}$

Thus $S(p) = 1/p$ as claimed! And without calculus!

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Part V: Die Another Game

https://youtu.be/tBz2GIxfYXA?t=2

The Darth Vader Problem: Review

Probability of failing a task is p, how many tasks till you die?

Answer: Expect 1/p.

Equivalently, if the probability of a success is p, the number of tasks or tries you need before the first success is 1/p.

The Sixes Game

Probability of failing a task is p, how many tasks till you die?

Answer: Expect 1/p.

Equivalently, if the probability of a success is p, the number of tasks or tries you need before the first success is 1/p.

We can use this to study a new game!

The sixes game: you roll a fair die until you get a 6. How many rolls do you expect before this happens?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

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Answer: Expect 1/p

Equivalently, if the probability of a success is p, the number of tasks or tries you need before the first success is 1/p.

We can use this to study a new game!

The sixes game: you roll a fair die until you get a 6. How many rolls do you expect before this happens?

Answer: As the probability of rolling a 6 is $p = 1/6$ (all six outcomes are equally likely) we expect it will take 6 rolls.

The Double Sixes Game

You have two fair die.

On each turn you can roll one or both of the die.

The goal is to have both show a 6.

Thus once one of the die lands on a 6 you can stop rolling it.

Questions:

- How many rolls do you expect before you have double sixes?
- What is the probability you win on your first turn? On your second? On your n^{th}?

Can we use the Darth Vader Theorem here? Why or why not?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

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Questions:

- How many rolls do you expect before you have double sixes?
- What is the probability you win on your first turn? On your second? On your n^{th}?

Can we use the Darth Vader Theorem here? Why or why not?

Hard to use: the difficulty is that our probability of a success is NOT constant; it depends on whether or not we rolled a 6 earlier…. Need a new method.

The Double Sixes Game

You have two fair die.

On each turn you can roll one or both of the die.

The goal is to have both show a 6.

Thus once one of the die lands on a 6 you can stop rolling it.

We will first find the probability of winning after a given number of rolls.

It is easy to find the probability of winning on the first roll: It is ???.

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The Double Sixes Game

You have two fair die.

On each turn you can roll one or both of the die.

The goal is to have both show a 6.

Thus once one of the die lands on a 6 you can stop rolling it.

We will first find the probability of winning after a given number of rolls.

It is easy to find the probability of winning on the first roll: It is 1/36.

What is the probability you win on the second roll? It is ???.

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

The Double Sixes Game

You have two fair die.

On each turn you can roll one or both of the die.

The goal is to have both show a 6.

Thus once one of the die lands on a 6 you can stop rolling it.

We will first find the probability of winning after a given number of rolls.

It is easy to find the probability of winning on the first roll: It is 1/36.

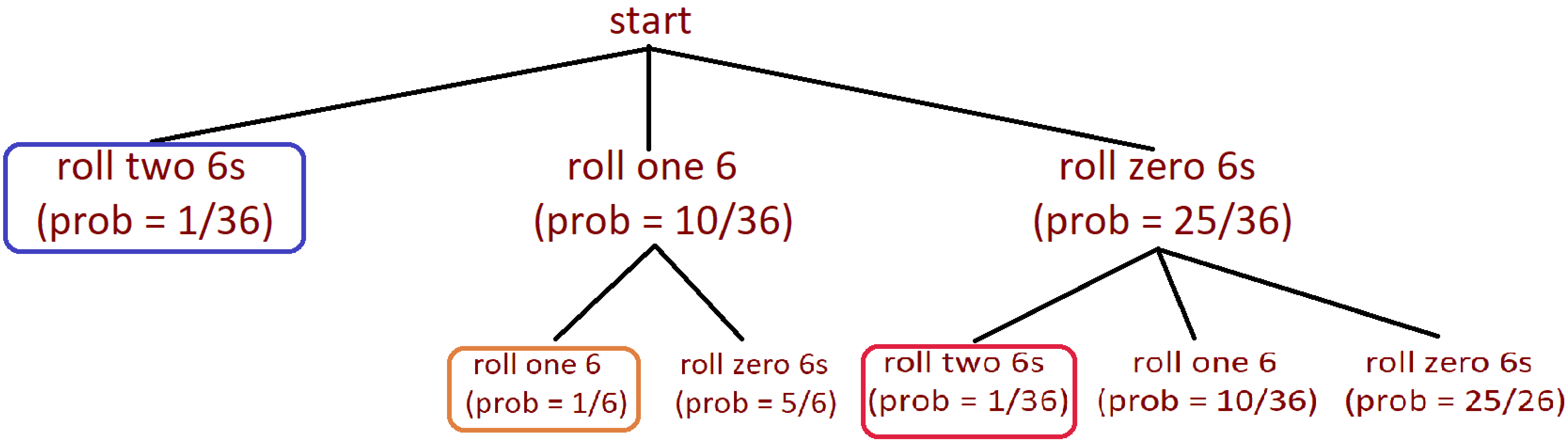
What is the probability you win on the second roll? It is

$10/36 * 1/6 + 25/36 * 1/36$. But why???

The Double Sixes Game

You have two fair die. On each turn you can roll one or both of the die.

The goal is to have both show a 6. Thus once one of the die lands on a 6 you can stop rolling it.



Prob(win first roll) = 1/36. Prob(win second roll) = 10/36*1/6 + 25/36*1/36 = 85/1296

Great Probability Results

We can continue the analysis, but there are more and more branches as we go down.

We introduce a WONDERFUL idea in probability:

The Law of Complementary Events: If the probability something happens is p, then the probability it does not happen is ????.

Great Probability Results

We can continue the analysis, but there are more and more branches as we go down.

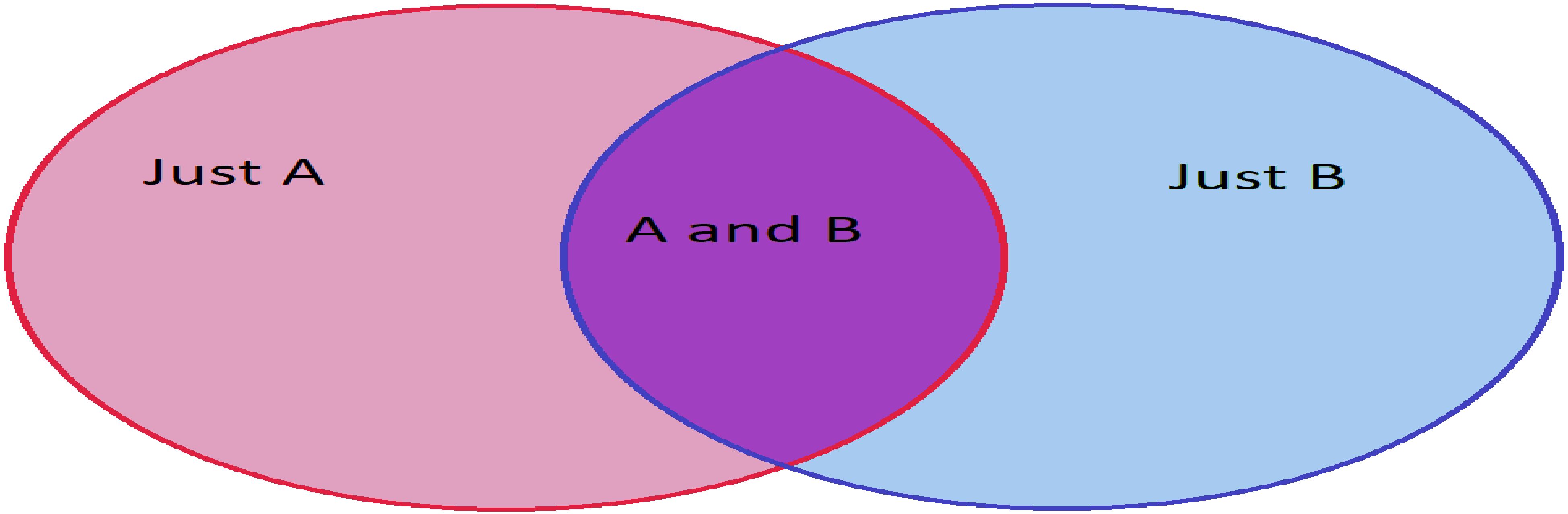
We introduce a WONDERFUL idea in probability:

The Law of Complementary Events: If the probability something happens is p, then the probability it does not happen is 1-p.

Great Probability Results

We introduce another WONDERFUL idea in probability:

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen: Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).



The Double Sixes Game

You have two fair die. On each turn you can roll one or both of the die.

Want both to show a 6. Once one of the die lands on a 6 you can stop rolling it.

The Law of Complementary Events: If the probability something happens is p, then the probability it does not happen is 1-p.

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen: Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).

What is the probability we win by the n^{th} turn?

It is 1 minus the probability we have NOT won.

What is the probability we haven't won? It is ???.

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The Double Sixes Game

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Want both to show a 6. Once one of the die lands on a 6 you can stop rolling it.

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What is the probability we win by the n^{th} turn?

It is 1 minus the probability we have NOT won.

What is the probability we haven't won? It is $(5/6)^n + (5/6)^n - (25/36)^n$.

Where did this come from? It is the probability the first die is never a 6 PLUS the probability the second is never a six, MINUS the probability neither die is ever a 6 (we must subtract as we we DOUBLE COUNTED that that probability).

The Double Sixes Game

You have two fair die. On each turn you can roll one or both of the die.

The goal is to have both show a 6. Thus once one of the die lands on a 6 you can stop rolling it.

The Law of Complementary Events: If the probability something happens is p, then the probability it does not happen is 1-p.

What is the probability we win BY the n^{th} turn? $1 - 2*(5/6)^n + (25/36)^n$.

It is 1 minus the probability we have NOT won.

What is the probability we haven't won? It is $(5/6)^n + (5/6)^n - (25/36)^n$.

So…, what is the probability we win ON the n^{th} turn?

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The Double Sixes Game

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So…, what is the probability we win ON the n^{th} turn?

It is the probability we win BY the n^{th} turn MINUS the probability we win BY the $(n-1)^{st}$ turn! $(1 – 2*(5/6)^n + (25/36)^n) – (1 – 2*(5/6)^{n-1} + (25/36)^{n-1})$

The Double Sixes Game

You have two fair die. On each turn you can roll one or both of the die.

The goal is to have both show a 6. Thus once one of the die lands on a 6 you can stop rolling it.

The Law of Complementary Events: If the probability something happens is p, then the probability it does not happen is 1-p.

What is the probability we win BY the n^{th} turn? $1 - 2*(5/6)^n + (25/36)^n$.

It is 1 minus the probability we have NOT won.

What is the probability we haven't won? It is $(5/6)^n + (5/6)^n - (25/36)^n$.

So…, what is the probability we win ON the n^{th} turn?

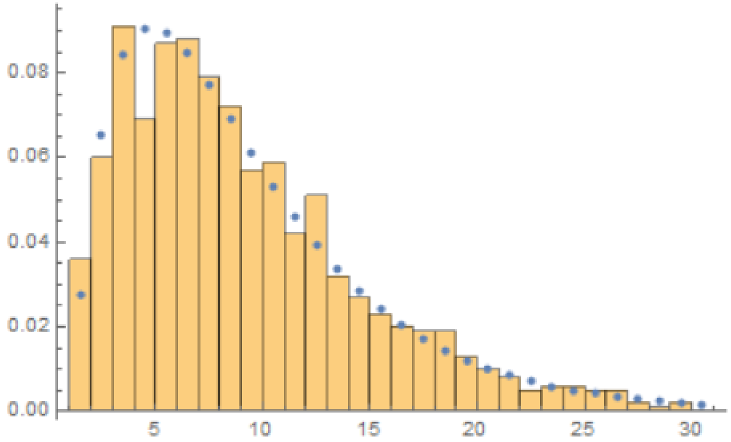
It is the probability we win BY the n^{th} turn MINUS the probability we win BY the $(n-1)^{st}$ turn! $(2/6)(5/6)^{n-1} - (11/36)(25/36)^{n-1}$.

The Double Sixes Game

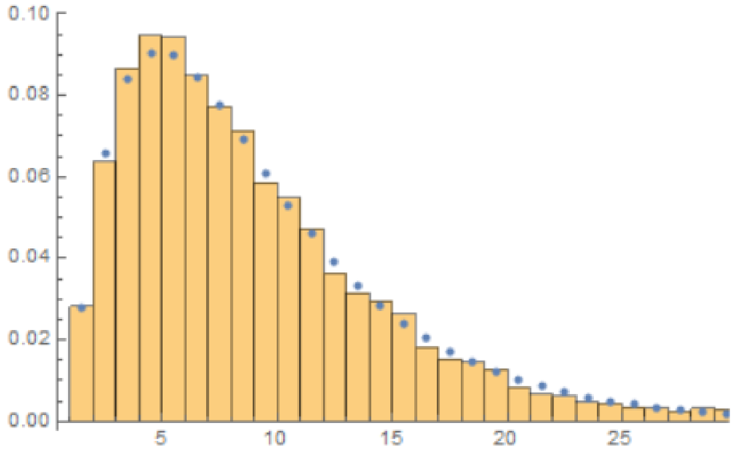
You have two fair die. On each turn you can roll one or both of the die.

The goal is to have both show a 6. Thus once one of the die lands on a 6 you can stop rolling it.

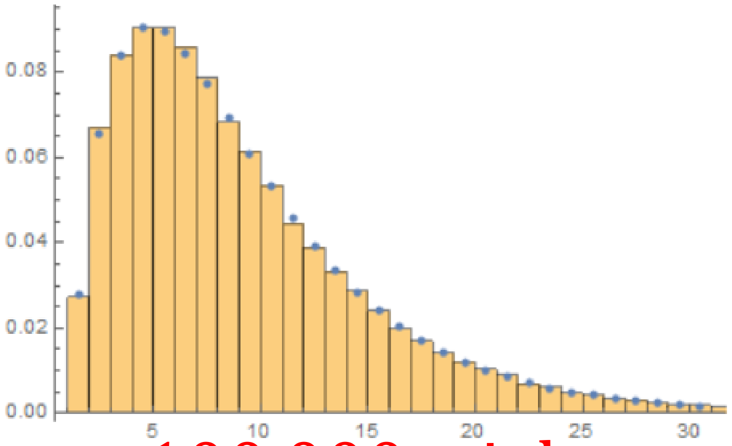
Probability win on n^{th} turn: $(2/6)(5/6)^{n-1} - (11/36)(25/36)^{n-1}$.



100 trials



10,000 trials



100,000 trials

The Double Sixes Game: Code

Mathematica code to simulate

```
In[68]:= f[n_] := 2 (5 / 6) ^n - (25 / 36) ^n
         g[n_] := 1 - f[n] (* probability succeed by n *)
         success[n_] := g[n] - g[n - 1];
         (* probability succeed at n *)

In[71]:= doublesixes[numdo_] := Module[{},
           count = {};
           For[m = 1, m ≤ numdo, m++,
            {
             firstdie = 0; seconddie = 0; rolls = 0;
             While[firstdie + seconddie < 12,
              {
               rolls = rolls + 1;
               die1 = RandomInteger[{1, 6}];
               die2 = RandomInteger[{1, 6}];
               If[die1 == 6, firstdie = 6];
               If[die2 == 6, seconddie = 6];
              }];
             count = AppendTo[count, rolls];
            }];
           theory = {};
           For[k = 1, k ≤ 30, k++, theory = AppendTo[theory, {k + .5, success[k]}]];
           Print[Show[Histogram[count, Automatic, "Probability"], ListPlot[theory]]];
          ]
```

The Double Sixes Game: Expected Value

Need the FULL strength of the Darth Vader Theorem (friendly version).

The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore:
$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)^2 + 4(1 - p)^3 + \cdots) = 1/p.$

The Double Sixes Game: Expected Value

Need the FULL strength of the Darth Vader Theorem (friendly version).

The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore:
$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)^2 + 4(1 - p)^3 + \cdots) = 1/\text{p}.$

To computed the expected number of rolls until the Double Sixes game ends we need to compute the sum of n * Prob(takes exactly n rolls), n from 1 to infinity.

As Prob(takes exactly n rolls) = $(2/6)(5/6)^{n-1} - (11/36)(25/36)^{n-1}$.

Notation: $\sum_{n=1}^{\infty} a_n \quad means \;\; a_1 + a_2 + a_3 + \cdots$ (using a Greek Sigma for Sum)

We have $\sum_{n=1}^{\infty} n((2/6)(5/6)^{n-1} - (11/36)(25/36)^{n-1})$.

First term: $\frac{2}{6}\left(1 + 2\left(\frac{5}{6}\right) + 3\left(\frac{5}{6}\right)^2 + 4\left(\frac{5}{6}\right)^3 + \cdots\right)$

The Double Sixes Game: Expected Value

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The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore:
$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)^2 + 4(1 - p)^3 + \cdots) = 1/\mathrm{p}$.

Notation: $\sum_{n=1}^{\infty} a_n \quad means \;\; a_1 + a_2 + a_3 + \cdots$ (using a Greek Sigma for Sum)

We have $\sum_{n=1}^{\infty} n((2/6)(5/6)^{\mathrm{n}-1} - (11/36)(25/36)^{\mathrm{n}-1})$.

Equals $\frac{2}{6}\sum_{n=1}^{\infty} n\,(5/6)^{\mathrm{n}-1}$ - $\frac{11}{36}\sum_{n=1}^{\infty} n\,(25/36)^{\mathrm{n}-1}$.

Each looks a lot like the Darth Vader Theorem – need to adjust a bit. What should p be for the first? For the second?

STOP STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION. STOP

The Double Sixes Game: Expected Value

Need the FULL strength of the Darth Vader Theorem (friendly version).

The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore:
$S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)^2 + 4(1 - p)^3 + \cdots) = 1/\text{p}$.

Notation: $\sum_{n=1}^{\infty} a_n$ $\quad$ *means* $a_1 + a_2 + a_3 + \cdots\cdot$ (using a Greek Sigma for Sum)

We have $\sum_{n=1}^{\infty} n((2/6)(5/6)^{n-1} - (11/36)(25/36)^{n-1})$.

Equals $\frac{2}{6}\sum_{n=1}^{\infty} n\,(5/6)^{n-1}$ - $\frac{11}{36}\sum_{n=1}^{\infty} n\,(25/36)^{n-1}$.

Each looks a lot like the Darth Vader Theorem – need to adjust a bit. What should be for the first? p = 1/6 (want 1-p = 5/6)

For the second? p = 11/36 (want 1-p = 25/36)

The Double Sixes Game: Expected Value

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We have $\sum_{n=1}^{\infty} n((2/6)(5/6)^{\text{n}-1} - (11/36)(25/36)^{\text{n}-1})$.

Equals $\frac{2}{6}\sum_{n=1}^{\infty} n\,(5/6)^{\text{n}-1}$ - $\frac{11}{36}\sum_{n=1}^{\infty} n\,(25/36)^{\text{n}-1}$.

Equals $2 * \frac{1}{6}\sum_{n=1}^{\infty} n\,(1-1/6)^{\text{n}-1}$ - $\frac{11}{36}\sum_{n=1}^{\infty} n\,(1-11/36)^{\text{n}-1}$.

What is the first term? What is second?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

The Double Sixes Game: Expected Value

Need the FULL strength of the Darth Vader Theorem (friendly version).

The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore: $S(p) = p(1 + 2 * (1 - p) + 3 * (1 - p)^2 + 4(1 - p)^3 + \cdots) = 1/p$.

Notation: $\sum_{n=1}^{\infty} a_n \quad means \;\; a_1 + a_2 + a_3 + \cdots$ (using a Greek Sigma for Sum)

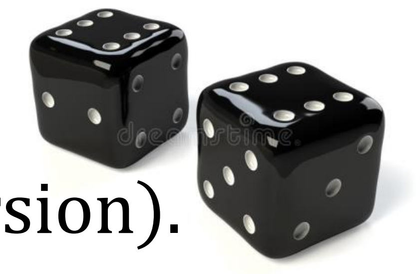
We have $\sum_{n=1}^{\infty} n((2/6)(5/6)^{n-1} - (11/36)(25/36)^{n-1})$.

Equals $\frac{2}{6} \sum_{n=1}^{\infty} n\,(5/6)^{n-1} - \frac{11}{36} \sum_{n=1}^{\infty} n\,(25/36)^{n-1}$.

Equals $2 * \frac{1}{6} \sum_{n=1}^{\infty} n\,(1 - 1/6)^{n-1} - \frac{11}{36} \sum_{n=1}^{\infty} n\,(1 - 11/36)^{n-1}$.

What is the first term? $2 * \frac{1}{1/6}$ What is second? $\frac{1}{11/36}$. Answer is ....

The Double Sixes Game: Expected Value



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Notation: $\sum_{n=1}^{\infty} a_n \quad means \ \ a_1 + a_2 + a_3 + \cdots$ (using a Greek Sigma for Sum)

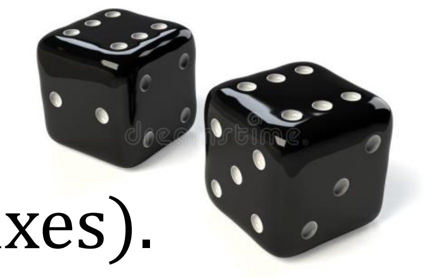
We have $\sum_{n=1}^{\infty} n((2/6)(5/6)^{\mathrm{n}-1} - (11/36)(25/36)^{\mathrm{n}-1})$.

Equals $2 * \frac{1}{6} \sum_{n=1}^{\infty} n \ (1 - 1/6)^{\mathrm{n}-1}$ - $\frac{25}{11} \ \frac{11}{36} \sum_{n=1}^{\infty} n \ (1 - 11/36)^{\mathrm{n}-1}$.

What is the first term? $2 * \frac{1}{1/6}$ What is second? $\frac{1}{11/36}$.

Answer is $2 * 6 \ - \frac{36}{11} = \frac{96}{11}$ (or about 8.7 rolls until you get both sixes).

The Double Sixes Game: Expected Value



Answer is $2 * 6 - \frac{36}{11} = \frac{96}{11}$ (or about 8.7 rolls until you get both sixes).

Is this answer reasonable? Are you surprised by it? What tests can you do to see if it makes sense? What lower or upper bounds can you find?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

The Double Sixes Game: Expected Value

Answer is $2 * 6 - \frac{36}{11} = \frac{96}{11}$ (or about 8.7 rolls until you get both sixes).

Is this answer reasonable? Are you surprised by it? What tests can you do to see if it makes sense?

In the six game (roll one die, stop when you get a 6) we saw the expected number of rolls is 6; as we now need TWO 6s, reasonable that it takes LONGER, and 6 is a LOWER BOUND.

If we played the six game twice (roll the first die until we get a 6, then start rolling the second die till we get a 6) expect to need 12 rolls. Thus 12 should be an UPPER BOUND. (Actually, can improve to 11 as an upper bound….)

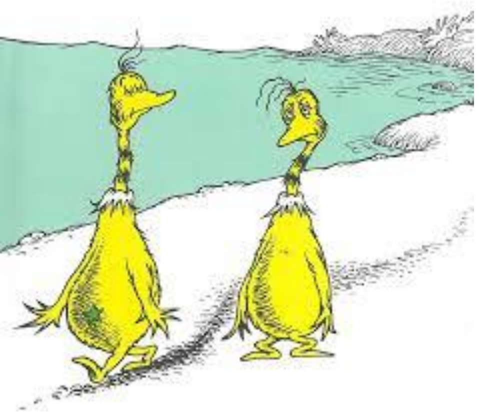
Review: Big Takeaways

The Darth Vader Theorem: If the probability of a success is p, then the expected number of trials until a success is 1/p. Furthermore:
$S(p) = p(1 + 2 * (1-p) + 3 * (1-p)^2 + 4(1-p)^3 + \cdots) = 1/\text{p}.$

The Law of Complementary Events: If the probability something happens is p, then the probability it does not happen is 1-p.

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen: Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).

The Power of Algebra: Sometimes have to do a bit of algebraic manipulations to make what you have look like something you know.



Part VI: Bridge Hands: Long Suits

JOURNAL OF NUMBER THEORY

Teachers As Scholars
henry.bolter@gmail.com

These math outreach lectures are supported in part by the Journal of Number Theory and the Teachers as Scholars program; it is a pleasure to thank them for their support.

$$nCr = \binom{n}{r}$$

Thus $5C2 = \binom{5}{2} = 10$

N
♠A9743
♥K8763
♦A6
♣7

W
♠None
♥QJ952
♦109
♣KQ10982

E
♠Q82
♥104
♦QJ85432
♣J

S
♠KJ1065
♥A
♦K7
♣A6543

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

Review: nCr

Recall the combinatorial function nCr: it is the number of ways to choose r objects from n, when order DOES NOT matter.

We use C for "combinations".

We often write $\binom{n}{r}$ for nCr, and it equals $\frac{n!}{r!(n-r)!}$.

For example, 5C2 = 10. If we have 5 people {A,B,C,D,E} there are 5 ways to choose the first and then 4 ways to choose the second, and that gives us 5*4 = 20; however, this is ORDERED, this is 5P2. We remove the order by dividing by 2!, the number of ways to order two objects.

Review: Distinct Deals in Bridge

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

- How many ways are there to deal the cards?

  Answer: $52C13 \ * 39C13 \ * 26C13 \ * 13C13$

  Equals 53,644,737,765,488,792,839,237,440,000 (about $10^{28.7295}$)

Do you think in all of human history there have ever been two deals the same?

Number of seconds since the universe began:

$60 * 60 * 24 * 366 \ * 14{,}000{,}000{,}000$ or about $10^{17.6}$.

About 108,000,000,000 people have been born, if each deals a hand a second since the dawn of time get up to about $10^{28.6795}$.

Long Suits in Bridge

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

What is the probability you are dealt at least 7 cards in a suit?

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Long Suits in Bridge

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

What is the probability you are dealt at least 7 cards in a suit?

Break a hard problem into a lot of easier problems.

Note cannot have two 7 card suits (this is why we start with 7 and not 6!).

It is Prob(exactly one 7 card suit) + … + Prob(exactly one 13 card suit).

What are these probabilities? What is Prob(exactly one 7 card suit)?

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Long Suits in Bridge

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

What is the probability you are dealt at least 7 cards in a suit?

Break a hard problem into a lot of easier problems.

Note cannot have two 7 card suits (this is why we start with 7 and not 6!).

It is Prob(exactly one 7 card suit) + … + Prob(exactly one 13 card suit).

What are these probabilities?

Prob(exactly one 7 card suit) = $4C1 \ * 13C7 \ * 39C6$

Why? 4C1 ways to choose the suit, 13C7 ways to choose 7 cards in that suit, 39C6 ways to fill out the hand.

Long Suits in Bridge

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get. NOTE: $nCr \ = \ \binom{n}{r}$

What is the probability you are dealt at least 7 cards in a suit?

It is Prob(exactly one 7 card suit) + … + Prob(exactly one 13 card suit).

$$4C1 * 13C7 * 39C6 \ + \ 4C1 * 13C8 * 39C5 + \cdots + 4C1 * 13C13 * 39C0$$

Can write compactly as $\sum_{k=7}^{13} \binom{4}{1}\binom{13}{k}\binom{39}{13-k}$ = 25,604,567,408.

There are $52C13$ = 635,013,559,600 hands.

Probability at least 7 in a suit is $\frac{25{,}604{,}567{,}408}{635{,}013{,}559{,}600}$ or about .04 (thus 4%).

Low probability, but happens enough that need to be prepared for it!

Long Suits in Bridge

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

Probability at least 7 in a suit is $\frac{25{,}604{,}567{,}408}{635{,}013{,}559{,}600}$ or about .04 (thus 4%).

Similarly probability at least 8 is $\frac{3{,}209{,}923{,}136}{635{,}013{,}559{,}600}$ or about .005 (thus .5%).

Not surprisingly, almost all hands with at least 7 in a suit have exactly 7 in a suit.

It is $\frac{25{,}604{,}567{,}408 \; - \; 3{,}209{,}923{,}136}{25{,}604{,}567{,}408}$ or about 87.5%

Long Suits in Bridge

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

Probability at least 7 in a suit is $\frac{25,604,567,408}{635,013,559,600}$ or about .04 (thus 4%).

Similarly probability at least 8 is $\frac{3,209,923,136}{635,013,559,600}$ or about .005 (thus .5%).

What is the probability you have exactly 6 in a suit? What is the complication?

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Long Suits in Bridge

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

Probability at least 7 in a suit is $\frac{25{,}604{,}567{,}408}{635{,}013{,}559{,}600}$ or about .04 (thus 4%).

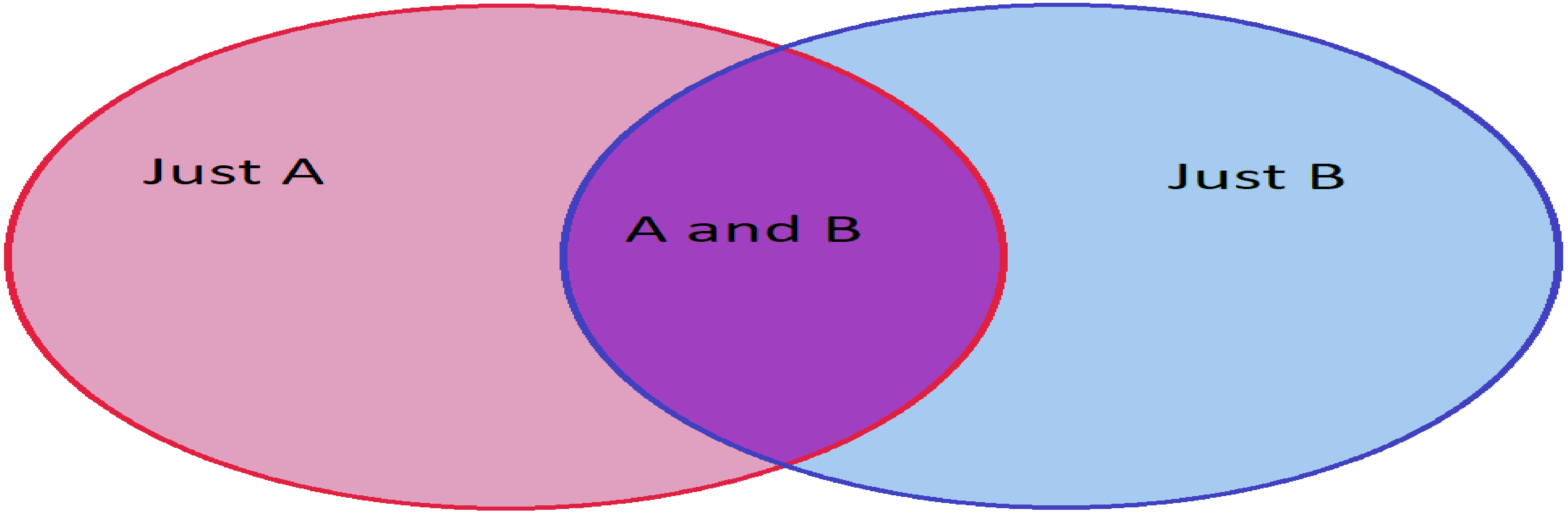
Similarly probability at least 8 is $\frac{3{,}209{,}923{,}136}{635{,}013{,}559{,}600}$ or about .005 (thus .5%).

What is the probability you have exactly 6 in a suit? What is the complication? The challenge is you could have TWO 6 card suits.

Recall a Great Probability Result

We recall a WONDERFUL idea in probability:

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen: Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).



Probability have a suit with EXACTLY 6 cards

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen: Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).

How many ways are there for a player to have exactly 6 cards in a specific suit?

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Probability have a suit with EXACTLY 6 cards

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen: Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).

How many ways are there for a player to have exactly 6 cards in a specific suit?

$$4C1 \ * 13C6 \ * 39C7.$$

There are 4C1 ways to choose the specific suit that they have 6 cards in, 13C6 ways to choose the 6 cards in the suit, and then 39C7 ways to choose the remaining cards.

The problem is while unlikely, it is POSSIBLE that they have another 6 (or even 7!) card suit in the remaining 39C7 cards…. How do we fix?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

Probability have a suit with EXACTLY 6 cards

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen: Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).

How many ways are there for a player to have exactly 6 cards in a specific suit?

$$4C1 \ * 13C6 \ * 39C7.$$

How many ways are there to have two suits with exactly 6 cards?

$$???.$$

How many ways are there to have one suit with 7 and one with 6 cards?

$$???.$$

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

Probability have a suit with EXACTLY 6 cards

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen: Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).

How many ways are there for a player to have exactly 6 cards in a specific suit?

$$4C1 \ * 13C6 \ * 39C7.$$

How many ways are there to have two suits with exactly 6 cards?

$$4C2 \ * 13C6 \ * 13C6 \ * 26C1.$$

How many ways are there to have one suit with 7 and one with 6 cards?

$$4C1 \ * 13C7 \ * 3C1 \ * 13C6.$$

How likely do you think the last two probabilities are relative to the first? Is this REALLY something we need to worry about?

Probability have a suit with EXACTLY 6 cards

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen: Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).

How many ways are there for a player to have exactly 6 cards in a specific suit?

$$4C1 \ * 13C6 \ * 39C7 \ = 105{,}574{,}751{,}568.$$

How many ways are there to have two suits with exactly 6 cards?

$$4C2 \ * 13C6 \ * 13C6 \ * 26C1 \ = 459{,}366{,}336.$$

How many ways are there to have one suit with 7 and one with 6 cards?

$$4C1 \ * 13C7 \ * 3C1 \ * 13C6 \ = 11{,}778{,}624.$$

NO!!! While 459 million is a big number relative to most of our bank accounts, it is pretty small relative to 105 billion!

Probability have a suit with EXACTLY 6 cards

The Law of Double Counting: The probability A or B happens is the sum of the probability each happens minus the probability they both happen: Prob(A or B) = Prob(A) + Prob(B) – Prob(A and B).

How many ways are there for a player to have exactly 6 cards in a specific suit?

$$4C1 \ * 13C6 \ * 39C7 \ = 105{,}574{,}751{,}568.$$

How many ways are there to have two suits with exactly 6 cards?

$$4C2 \ * 13C6 \ * 13C6 \ * 26C1 \ = 459{,}366{,}336.$$

How many ways are there to have one suit with 7 and one with 6 cards?

$$4C1 \ * 13C7 \ * 3C1 \ * 13C6 \ = 11{,}778{,}624.$$

There are $52C13 = \ 635{,}013{,}559{,}600$ possible hands.

Thus the probability have EXACTLY one 6 card suit is

$$\frac{105{,}574{,}751{,}568 - 459{,}366{,}336 - 11{,}778{,}624}{635{,}013{,}559{,}600} = \frac{938{,}425{,}059}{5{,}669{,}763{,}925},$$ about 16.6%!

Probability have a suit with EXACTLY 6 cards

The probability have EXACTLY one 6 card suit is

$$\frac{105{,}574{,}751{,}568 - 459{,}366{,}336 - 11{,}778{,}624}{635{,}013{,}559{,}600} = \frac{938{,}425{,}059}{5{,}669{,}763{,}925},$$ about 16.6%!

The probability have EXACTLY TWO 6 card suits is

$$\frac{459{,}366{,}336}{635{,}013{,}559{,}600} = \frac{28{,}710{,}396}{39{,}688{,}347{,}475},$$ about .07%!

The probability have EXACTLY 7 in one suit and EXACTLY 6 in another is

$$\frac{11{,}778{,}624}{635{,}013{,}559{,}600} = \frac{736{,}164}{39{,}688{,}347{,}475},$$ about .00185%!

If play 100 games, first happens 17 times on average, we do expect to see often!

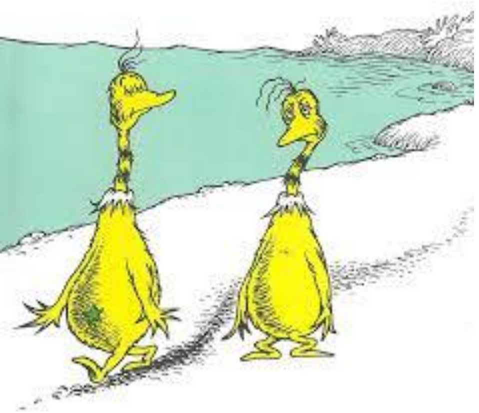
If play 1000 games, second happens .72 times on average, so don't expect to see!

If play 10,000 games, third happens .19 times on average, so don't expect to see!

This helps us figure out what bidding conventions we need – what is worth having!

Recap

- Often more than one way to compute an answer.
- Break a complicated probability into a sum of simpler probabilities; important that the cases are disjoint and cover all the possibilities.
- Tremendous power in using nCr to compute the number of combinations.
- Frequently there are smaller effects / lower order terms that you TECHNICALLY need to have the right answer, but they change things by a negligible amount….
- If you can compute something two ways, do so – a great way to check your work!
- If you can write a computer program to test your work that is OUTSTANDING!



Part VII: Bridge Hands with bad splits, Aces Up

JOURNAL OF NUMBER THEORY
Teachers As Scholars
henry.bolter@gmail.com
These math outreach lectures are supported in part by the Journal of Number Theory and the Teachers as Scholars program; it is a pleasure to thank them for their support.

N
♠AQ10975
♥9
♦J4
♣AQ94

W
♠K63
♥K8632
♦109876
♣None

E
♠842
♥Q107
♦K2
♣J10763

S
♠J
♥AJ54
♦AQ53
♣K852

West	North	East	South
			1♦
Pass	1♠	Pass	2♣
Pass	2♥[1]	Pass	3♥[2]
Pass	4♣	Pass	4♦[3]
Pass	4♠[3]	Pass	6♣
Pass	Pass	Pass	

(1) Fourth suit forcing, asking for more information
(2) Natural, showing four hearts (3) A cue bid with clubs agreed as trumps.

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

Review: nCr

Recall the combinatorial function nCr: it is the number of ways to choose r objects from n, when order DOES NOT matter.

We use C for “combinations”.

We often write $\binom{n}{r}$ for nCr, and it equals $\frac{n!}{r!(n-r)!}$.

For example, 5C2 = 10. If we have 5 people {A,B,C,D,E} there are 5 ways to choose the first and then 4 ways to choose the second, and that gives us 5*4 = 20; however, this is ORDERED, this is 5P2. We remove the order by dividing by 2!, the number of ways to order two objects.

Trump Splits II: The Bad 5-0 Split

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

What if you and your partner have 8 trump; what are the odds the remaining 5 are all in the same hand?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

Trump Splits II: The Bad 5-0 Split

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

What if you and your partner have 8 trump; what are the odds the remaining 5 are all in the same hand?

One solution: There are $2C1 * 5C5 * 21C8 * 13C13 =$ 406,980.

(there are 2 ways to choose which player gets the 5 trump, then give all 5 to that player, then give that player any 8 of the remaining 21 cards, then give all the remaining cards to the final player)

The number of ways to assign the remaining 26 cards is $26C13 * 13C13 =$ 104,006,000.

Thus probability is 406,980 / 104,006,000 = 9/230 or about .039 (or 3.9%).

Trump Splits II: The Bad 5-0 Split

In a hand of bridge, each of the four players is dealt 13 cards. It does not matter what order you get the cards, only which cards you get.

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Could we say the answer is 2 * $(1/2)^5$ as there are two players who could get all 5, and each card has a 50-50 chance?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

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Could we say the answer is $2 * (1/2)^5$ as there are two players who could get all 5, and each card has a 50-50 chance? Note this equals 1/16 or 6.25%. Why is this wrong?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

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Thus probability is 406,980 / 104,006,000 = 9/230 or about .039 (or 3.9%).

Could we say the answer is $2 * (1/2)^5$ as there are two players who could get all 5, and each card has a 50-50 chance? Note this equals 1/16 or 6.25%. Why is this wrong? As we hand out the cards, subsequent ones no longer have a 50-50 chance. Answer is 2 (13/26)(12/25)(11/24)(10/23)(9/22) = 9/230.

Trump Splits II: The Bad 5-0 Split

What if you and your partner have 8 trump; what are the odds the remaining 5 are all in the same hand?

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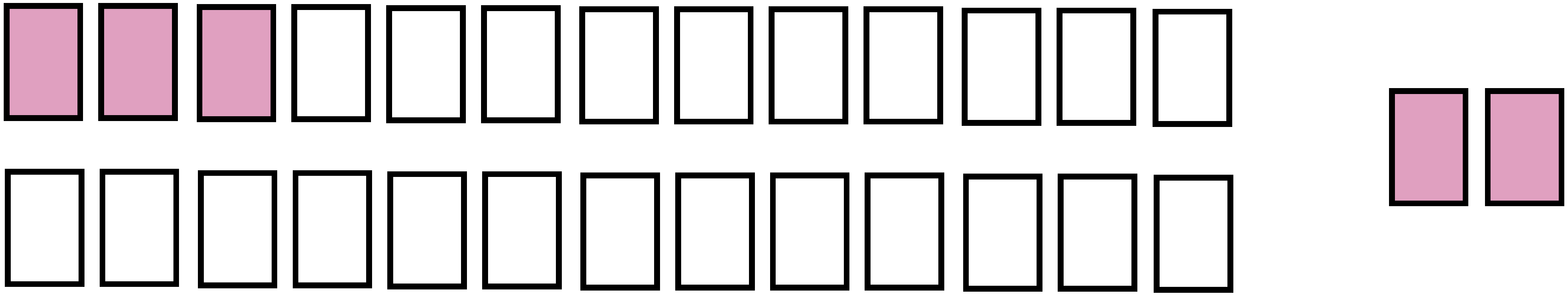


The first three trump are in the top hand. There are two trump left. As each space is equally likely, the probability the next goes in the top hand is 10 / 23 (as 10 of the 23 spaces are in the top hand).

Aces Up

One of my favorite solitaire games is **aces up**. It goes as follows: shuffle the deck and then deal four cards face up. You now have four piles. If the top card on a pile has the same suit but has a lower number than the top card of another pile, that card can be moved into the discard pile. For example, if there are four piles and the top cards are $4\diamondsuit$, $3\heartsuit$, $6\spadesuit$ and $2\diamondsuit$, then we can move the $2\diamondsuit$ into the discard pile. If there was a card below the $2\diamondsuit$, that's now the top card of the pile; if there was no card below it, we now have a free pile and we can move *any* top card onto that pile. The goal of the game is to end with just the four aces showing (as obviously we can't do better than that!).

What is the probability the last four cards are in different suits, and thus no matter how well you play, you will LOSE?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

Aces Up

What is the probability the last four cards are in different suits?

There are many ways to compute this.

- We could look at each card, one at a time, and each is in a different suit than the previous.
- We can choose one card from each suit.

Which do you prefer? Why? Will they give the same answer (they better!).

Try to solve it. Note there are 52P4 = 6,497,400 ways to choose 4 cards when order DOES matters, and 52C4 = 270,725 ways to choose 4 cards when order DOES NOT matter.

Aces Up

What is the probability the last four cards are in different suits?

There are many ways to compute this.

- We could look at each card, one at a time, and each is in a different suit than the previous. $\frac{52}{52} * \frac{??}{??} * \frac{??}{??} * \frac{??}{??} = \frac{??}{??}$
- We can choose one card from each suit.

Which do you prefer? Why? Will they give the same answer (they better!).

Try to solve it. Note there are 52P4 = 6,497,400 ways to choose 4 cards when order DOES matters, and 52C4 = 270,725 ways to choose 4 cards when order DOES NOT matter.

Aces Up

What is the probability the last four cards are in different suits?

There are many ways to compute this.

- We could look at each card, one at a time, and each is in a different suit than the previous. $\frac{52}{52} * \frac{39}{51} * \frac{??}{??} * \frac{??}{??} = \frac{??}{??}$
- We can choose one card from each suit.

Which do you prefer? Why? Will they give the same answer (they better!).

Try to solve it. Note there are 52P4 = 6,497,400 ways to choose 4 cards when order DOES matters, and 52C4 = 270,725 ways to choose 4 cards when order DOES NOT matter.

Aces Up

What is the probability the last four cards are in different suits?

There are many ways to compute this.

- We could look at each card, one at a time, and each is in a different suit than the previous. $\frac{52}{52} * \frac{39}{51} * \frac{26}{50} * \frac{13}{49} = \frac{2197}{20825}$ or about 10.5%
- We can choose one card from each suit. $\frac{\binom{??}{??} * \binom{??}{??} * \binom{??}{??} * \binom{??}{??}}{\binom{??}{??}}$

Which do you prefer? Why? Will they give the same answer (they better!).

Try to solve it. Note there are 52P4 = 6,497,400 ways to choose 4 cards when order DOES matters, and 52C4 = 270,725 ways to choose 4 cards when order DOES NOT matter.

Aces Up

What is the probability the last four cards are in different suits?

There are many ways to compute this.

- We could look at each card, one at a time, and each is in a different suit than the previous. $\frac{52}{52} * \frac{39}{51} * \frac{26}{50} * \frac{13}{49} = \frac{2197}{20825}$ or about 10.5%
- We can choose one card from each suit.. $\frac{\binom{13}{1}*\binom{13}{1}*\binom{13}{1}*\binom{13}{1}}{\binom{52}{4}} = \frac{2197}{20825}$.

Always good to calculate something two different ways!

Also, if you can, test with a computer program!!!

Aces Up

What is the probability the last four cards are in different suits?

```
acesup[numdo_] := Module[{},
  fail = 0;
  deck = {};
  For[i = 1, i ≤ 13, i++,
   For[j = 0, j ≤ 3, j++,
    deck = AppendTo[deck, 10^j];
   ]; (* end of j loop *)
  ]; (* end of i loop *)
  (* nice trick, deck has 13 cards in a "suit", all cards in a suit the same *)
  (* if sum of four cards in hand is 1111 then have four suits, else do not! *)
  For[n = 1, n ≤ numdo, n++,
   {
    hand = RandomSample[deck, 4];
    If[Sum[hand[[h]], {h, 1, 4}] == 1111, fail = fail + 1];
   }]; (* end of n loop *)
  Print["We failed ", fail, " out of ", numdo " times, or ", 100.0*fail/numdo, "%."];
 ]
```

Aces Up

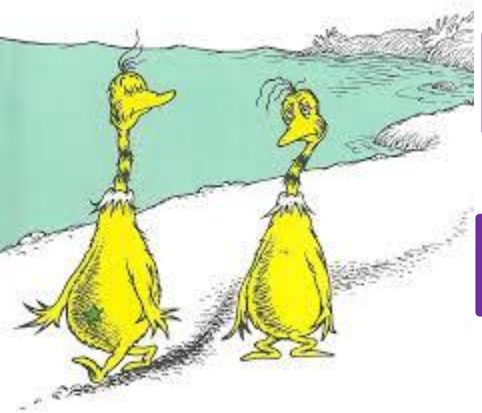
What is the probability the last four cards are in different suits?


```
acesup[100]
acesup[1000]
acesup[10000]
acesup[100000]
acesup[1000000]
acesup[10000000]
Print["Theoerical probability of failure is ", 100.0*2197/20825, "%."]
```

```
We failed 16 out of 100 times, or 16.%.
We failed 114 out of 1000 times, or 11.4%.
We failed 1058 out of 10000 times, or 10.58%.
We failed 10695 out of 100000 times, or 10.695%.
We failed 105253 out of 1000000 times, or 10.5253%.
We failed 1052450 out of 10000000 times, or 10.5245%.
Theoerical probability of failure is 10.5498%.
```

Recap

- Often more than one way to compute an answer.
- Break a complicated probability into a sum of simpler probabilities; important that the cases are disjoint and cover all the possibilities.
- Tremendous power in using nCr to compute the number of combinations.
- Frequently there are smaller effects / lower order terms that you TECHNICALLY need to have the right answer, but they change things by a negligible amount….
- If you can compute something two ways, do so – a great way to check your work!
- If you can write a computer program to test your work that is OUTSTANDING!



Part VIII: Advanced Bridge and Poker Hands

These math outreach lectures are supported in part by the Journal of Number Theory and the Teachers as Scholars program; it is a pleasure to thank them for their support.

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

Steven J Miller – Williams College – sjm1@williams.edu

Review: nCr

Recall the combinatorial function nCr: it is the number of ways to choose r objects from n, when order DOES NOT matter.

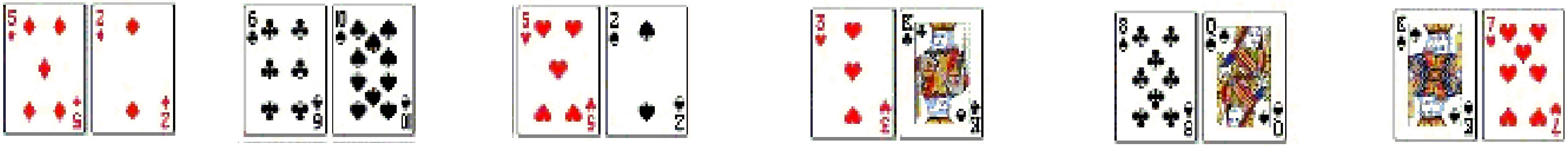
We use C for “combinations”.

We often write $\binom{n}{r}$ for nCr, and it equals $\frac{n!}{r!(n-r)!}$.

For example, 5C2 = 10. If we have 5 people {A,B,C,D,E} there are 5 ways to choose the first and then 4 ways to choose the second, and that gives us 5*4 = 20; however, this is ORDERED, this is 5P2. We remove the order by dividing by 2!, the number of ways to order two objects.

Modified Two Handed Bridge: Getting all Trump

Let’s say hearts are trump. Go through the deck looking at the cards two at a time. In each pair choose one card, and discard the other.



You end with 26 cards; one comes from cards 1 and 2, one comes from cards 3 and 4, …, one comes from cards 51 and 52.

If you always choose a heart if it is available, what is the probability you end up with all 13 hears?

What could prevent you from being able to have all 13 hearts?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

Modified Two Handed Bridge: Getting all Trump

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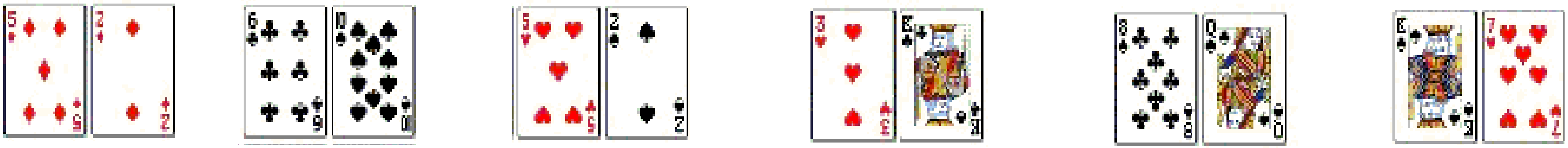
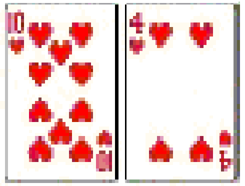


You end with 26 cards; one comes from cards 1 and 2, one comes from cards 3 and 4, …, one comes from cards 51 and 52.

If you always choose a heart if it is available, what is the probability you end up with all 13 hears?

What could prevent you from being able to have all 13 hearts?

You lose if you ever have two hearts in the same pair.



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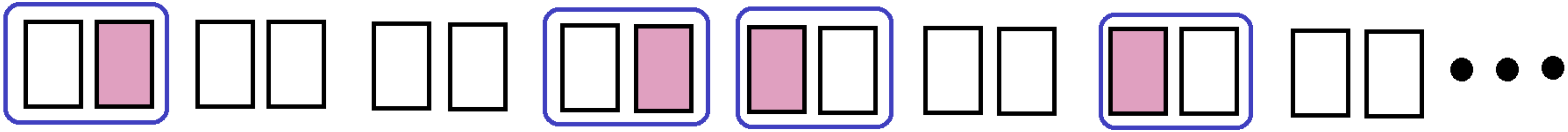
Let's say hearts are trump. Go through the deck looking at the cards two at a time. In each pair choose one card, and discard the other. You end with 26 cards; one comes from cards 1 and 2, …, one comes from cards 51 and 52. If you always choose a heart if it is available, what is the probability you end up with all 13 hears?



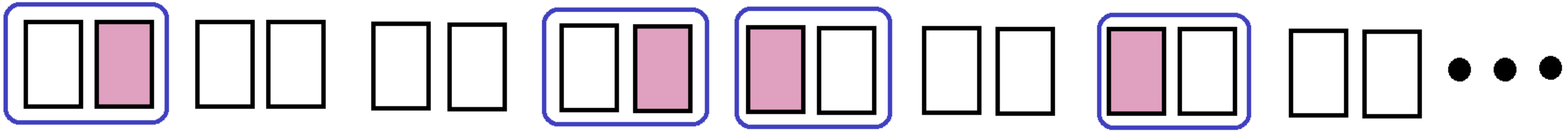
Here is one way to view it: You choose **???** of the 26 pairs to have exactly one heart, each pair you have **???** choices to place the heart, you then have **???** ways to place the 13 hearts in these spots, and you then have **???** ways to place the remaining 39 cards. You then divide by the number of ways to place the 52 cards, which is **???**. Notice using order.

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

Modified Two Handed Bridge: Getting all Trump

Let’s say hearts are trump. Go through the deck looking at the cards two at a time. In each pair choose one card, and discard the other. You end with 26 cards; one comes from cards 1 and 2, …, one comes from cards 51 and 52. If you always choose a heart if it is available, what is the probability you end up with all 13 hears?

H



heart, each pair you have **2** choices to place the heart, you then have **13!** ways to place the 13 hearts in these spots, and you then have **39!** ways to place the remaining 39 cards. You then divide by the number of ways to place the 52 cards, which is **52P52 = 52!**. Notice using order.

Answer: $26C13 * 213 * 13! * 39! \ / \ 52!$ = 77824/580027, approximately 13.4%.

Modified Two Handed Bridge: Getting all Trump

```
gettingalltrumpsinpairsoftwo[numdo_] := Module[{},
   success = 0;
   deck = {};
   For[i = 1, i ≤ 13, i++, deck = AppendTo[deck, 1]];
   For[i = 14, i ≤ 52, i++, deck = AppendTo[deck, 0]];
   (* creates the deck, the trump suit is all 1's,
   rest 0's *)
   For[n = 1, n ≤ numdo, n++,
    {
     hand = RandomSample[deck, 52];
     value = Sum[hand[[2*i]] * hand[[2*i - 1]],
       {i, 1, 26}];
     If[value == 0, success = success + 1];
     (* if value is 0 never have two trump in same pair,
     doable *)
    }]; (* end of n loop *)
   Print["Did ", numdo, " samples and succeeded ",
    success, " times, or ", 100.0 success / numdo, "%."];
  ];
```

```
gettingalltrumpsinpairsoftwo[1000]
gettingalltrumpsinpairsoftwo[10000]
gettingalltrumpsinpairsoftwo[100000]
gettingalltrumpsinpairsoftwo[1000000]
gettingalltrumpsinpairsoftwo[10000000]
Did 1000 samples and succeeded 138 times, or 13.8%.
Did 10000 samples and succeeded 1359 times, or 13.59%.
Did 100000 samples and succeeded 13507 times, or 13.507%.
Did 1000000 samples and succeeded 133508 times, or 13.3508%.
Did 10000000 samples and succeeded 1342922 times, or 13.4292%.
```

Answer: About 13.4173%.

Modified Two Handed Bridge: Getting all Trump

•

```
gettingalltrumpsinpairsoftwo[numdo_] := Module[{},
   success = 0;
   deck = {};
   For[i = 1, i ≤ 13, i++, deck = AppendTo[deck, 1]];
   For[i = 14, i ≤ 52, i++, deck = AppendTo[deck, 0]];
   (* creates the deck, the trump suit is all 1's,
   rest 0's *)
   For[n = 1, n ≤ numdo, n++,
    {
     hand = RandomSample[deck, 52];
     value = Sum[hand[[2*i]] * hand[[2*i - 1]],
       {i, 1, 26}];
     If[value == 0, success = success + 1];
     (* if value is 0 never have two trump in same pair,
     doable *)
    }]; (* end of n loop *)
   Print["Did ", numdo, " samples and succeeded ",
    success, " times, or ", 100.0 success / numdo, "%."];
  ];
```

Worth commenting on the code.

The deck is {1,…,1,0,…,0} where we have 13 cards are a 1, and 39 cards are a 0.

Why?

We only care if a card is trump or not trump – that is all we need to keep track of when we code!

Modified Two Handed Bridge: Getting all Trump

.

Other versions:

It is trivial to do if you look at the cards one at a time.

What if you look at the cards 4 at a time and you choose one each time? It is JUST barely possible – compute the probability you can do it.

Can you do it if you choose one out of every 6 cards?

Poker Problem: 5 Cards, at least two Aces, two Kings

Consider a 5 card poker hand. What is the probability have at least two Aces and at least two Kings?

How can you do this? What hands would work? What is the difficulty?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

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How can you do this? What hands would work? What is the difficulty?

AAAKK, AAKKK, AAKKX

Challenge is could have three of a kind and a pair.

Have to be careful not to double count.

What are the probabilities of each of the three configurations? How many hands?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

Poker Problem: 5 Cards, at least two Aces, two Kings

Consider a 5 card poker hand. What is the probability have at least two Aces and at least two Kings?

How can you do this? What hands would work? What is the difficulty?

- AAAKK: $4C3 \ * 4C2 \ = \ 4 \ * 6 \ = \ 24$
- AAKKK: $4C2 \ * 4C3 \ = \ 6 \ * 4 \ = \ 24$
- AAKKX: $4C2 \ * 4C2 \ * 44C1 \ = \ 6 \ * 6 \ * 44 \ = 1584$
- 5 card hands: $52C5 \ = 2{,}598{,}960$.

- Probability is $\frac{24+24+1584}{2598960} = \frac{1633}{2598960} = \ .0628\%.$

Poker Problem: 5 Cards, at least two Aces, two Kings

Consider a 5 card poker hand. What is the probability have at least two Aces and at least two Kings

- AAAKK: $4C3 \ * 4C2 \ = \ 4 \ * 6 \ = \ 24$
- AAKKK: $4C2 \ * 4C3 \ = \ 6 \ * 4 \ = \ 24$
- AAKKX: $4C2 \ * 4C2 \ * 44C1 \ = \ 6 \ * 6 \ * 44 \ = 1584$
- 5 card hands: $52C5 \ = 2{,}598{,}960.$

- Probability is $\frac{24+24+1584}{2598960} = \frac{1633}{2598960} = .0628328\%.$

Another Approach: Choose two Aces, choose two kings, choose another card.

$4C2 \ * 4C2 \ * 48C1 \ = 1728, \quad so \ \frac{1728}{2598960}$ or about .0664881%.

Which is right? Why?

STOP STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION. STOP

Poker Problem: 5 Cards, at least two Aces, two Kings

Consider a 5 card poker hand. What is the probability have at least two Aces and at least two Kings?

- AAAKK: $4C3 \ * 4C2 \ = \ 4 \ * 6 \ = \ 24$
- AAKKK: $4C2 \ * 4C3 \ = \ 6 \ * 4 \ = \ 24$
- AAKKX: $4C2 \ * 4C2 \ * 44C1 \ = \ 6 \ * 6 \ * 44 \ = 1584$
- 5 card hands: $52C5 \ = 2{,}598{,}960.$
- Probability is $\frac{24+24+1584}{2598960} \ = \ \frac{1633}{2598960} \ = \ .0628328\%.$

Another Approach: Choose two Aces, choose two kings, choose another card.

Another Approach: $4C2 \ * 4C2 \ * 48C1 \ = 1728, \quad so \ \frac{1728}{2598960}$ or about .0664881%.

The first – the second approach **triple** counts! The “third” ace could be in the 48C1.

Note (1728-1584) = 144, and 144/3 = 48 = 24+24, the AAAKK and AAKKK.

Poker Problem: 5 Cards, at least two Aces, two Kings

Consider a 5 card poker hand. What is the probability have at least two Aces and at least two Kings? Is it $.0628328\%$ *or* $.0664881\%$?

```
twoacestwokings[numdo_] := Module[{},
  success = 0;
  deck = {1, 1, 1, 1, 10, 10, 10, 10};
  For[i = 1, i ≤ 44, i++, deck = AppendTo[deck, 0]];
  For[n = 1, n ≤ numdo, n++,
   {
    hand = RandomSample[deck, 5];
    value = Sum[hand[[i]], {i, 1, 5}];
    If[value == 22 || value == 23 || value == 32,
     success = success + 1];
   }]; (* end of n loop *)
  Print["Did ", numdo, " hands and succeeded ",
   SetAccuracy[100. success / numdo, 5], "%."];
 ]
```

```
twoacestwokings[1000]
twoacestwokings[10000]
twoacestwokings[100000]
twoacestwokings[1000000]
twoacestwokings[10000000]
```

```
Did 1000 hands and succeeded 0.×10^-5%.
Did 10000 hands and succeeded 0.0800%.
Did 100000 hands and succeeded 0.0590%.
Did 1000000 hands and succeeded 0.0614%.
Did 10000000 hands and succeeded 0.0649%.
```

Poker Problem: 5 Cards, at least two Aces, two Kings

Consider a 5 card poker hand. What is the probability have at least two Aces and at least two Kings? Is it $.0628328\%$ *or* $.0664881\%$?

```
twoacestwokings[numdo_] := Module[{},
  success = 0;
  deck = {1, 1, 1, 1, 10, 10, 10, 10};
  For[i = 1, i ≤ 44, i++, deck = AppendTo[deck, 0]];
  For[n = 1, n ≤ numdo, n++,
   {
    hand = RandomSample[deck, 5];
    value = Sum[hand[[i]], {i, 1, 5}];
    If[value == 22 || value == 23 || value == 32,
     success = success + 1];
   }]; (* end of n loop *)
  Print["Did ", numdo, " hands and succeeded ",
   SetAccuracy[100. success / numdo, 5], "%."];
 ]
```

```
twoacestwokings[1000]
twoacestwokings[10000]
twoacestwokings[100000]
twoacestwokings[1000000]
twoacestwokings[10000000]

Did 1000 hands and succeeded 0.×10^-5%.
Did 10000 hands and succeeded 0.0800%.
Did 100000 hands and succeeded 0.0590%.
Did 1000000 hands and succeeded 0.0614%.
Did 10000000 hands and succeeded 0.0649%.
```

The problem is that the simulation is not good enough to determine which probability is correct, as they are so close. We need to go up to 100,000,000; it took about 20 seconds to do a million....

Poker Problem: 5 Cards, at least two Aces, two Kings

Consider a 5 card poker hand. What is the probability have at least two Aces and at least two Kings? Is it $\boxed{.0628328\%}$ *or* 0664881%?

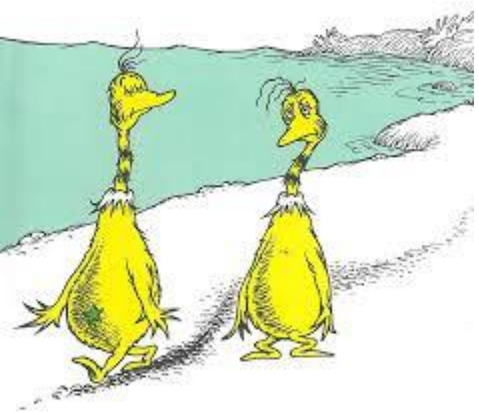
```
twoacestwokings[numdo_] := Module[{},
  success = 0;
  deck = {1, 1, 1, 1, 10, 10, 10, 10};
  For[i = 1, i ≤ 44, i++, deck = AppendTo[deck, 0]];
  For[n = 1, n ≤ numdo, n++,
   {
    If[Mod[n, numdo / 10] == 0,
     Print["We have done ", 100.0 n / numdo, "%."]];
    hand = RandomSample[deck, 5];
    value = Sum[hand[[i]], {i, 1, 5}];
    If[value == 22 || value == 23 || value == 32,
     success = success + 1];
   }]; (* end of n loop *)
  Print["Did ", numdo, " hands and succeeded ",
   SetAccuracy[100. success / numdo, 5], "%."];
 ]
```

```
twoacestwokings[1000]
twoacestwokings[10000]
twoacestwokings[100000]
twoacestwokings[1000000]
twoacestwokings[10000000]

Did 1000 hands and succeeded 0.×10^-5%.
Did 10000 hands and succeeded 0.0800%.
Did 100000 hands and succeeded 0.0590%.
Did 1000000 hands and succeeded 0.0614%.
Did 10000000 hands and succeeded 0.0649%.
```

Did 100,000,000 hands and succeeded approximately 0.0626%.

Took 21.46 seconds to do 1,000,000 and 2157.58 seconds for 100,000,000.



Part IX: Advanced Combinations

JOURNAL OF NUMBER THEORY

Teachers As Scholars
henry.bolter@gmail.com

These math outreach lectures are supported in part by the Journal of Number Theory and the Teachers as Scholars program; it is a pleasure to thank them for their support.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

Expanding $(x + y)^n$

$(x + y)^0 = \mathbf{1}$

$(x + y)^1 = \mathbf{1}\, x + \mathbf{1}\, y$

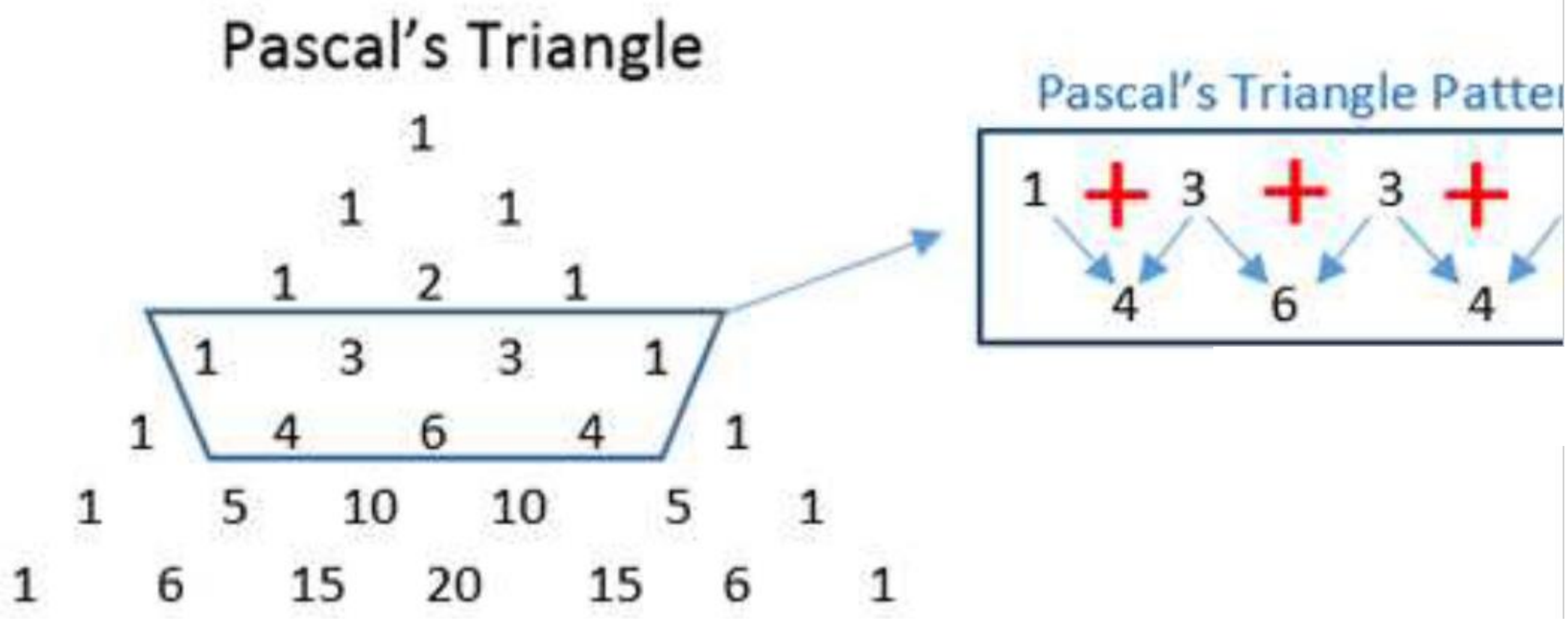
$(x + y)^2 = \mathbf{1}\, x^2 + \mathbf{2}\, x\, y + \mathbf{1}\, y^2.$

$(x + y)^3 = \mathbf{1}\, x^3 + \mathbf{3}\, x^2\, y + \mathbf{3}\, x\, y^2 + \mathbf{1}\, y^3.$

We can keep going and get more and more rows…..

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

Why is the Pascal Relation true? Each number is the sum of what is immediately above to the right and to the left.



FOIL

FOIL stands for FIRST, OUTSIDE, INSIDE and LAST.

It provides a framework to multiply (a+b) and (c+d).

We have:

$$(a + b) * (c + d) = \underbrace{a * c}_{\text{FIRST}} \quad + \quad \underbrace{a * d}_{\text{OUTSIDE}} \quad + \quad \underbrace{b * c}_{\text{INSIDE}} \; + \; \underbrace{b * d}_{\text{LAST}}.$$

FOIL

FOIL stands for FIRST, OUTSIDE, INSIDE and LAST.

We can repeatedly apply it, and its generalizations.....

We have:

$(x + y)^2 = (x + y) * (x + y) = x * x + x * y + y * x + y * y = x^2 + x\,y + y\,x + y^2 = x^2 + 2\,x\,y + y^2.$

So:

$$
\begin{aligned}
(x + y)^3 &= (x + y) * (x + y)^2 = (x + y) * (x^2 + 2\,x\,y + y^2) \\
&= x * (x^2 + 2\,x\,y + y^2) + y * (x^2 + 2\,x\,y + y^2) \\
&= (x^3 + 2\,x^2\,y + x\,y^2) + (x^2\,y + 2\,x\,y^2 + y^3) \\
&= x^3 + 3\,x^2\,y + 3\,x\,y^2 + y^3.
\end{aligned}
$$

Expanding $(x + y)^n$

$(x + y)^1 = \mathbf{1}\, x + \mathbf{1}\, y$

$(x + y)^2 = \mathbf{1}\, x^2 + \mathbf{2}\, x\, y + \mathbf{1}\, y^2.$

$(x + y)^3 = \mathbf{1}\, x^3 + \mathbf{3}\, x^2\, y + \mathbf{3}\, x\, y^2 + \mathbf{1}\, y^3.$

This is the start of Pascal’s Triangle…..

How should we define $(x + y)^0$? Well, we often say things to to zeroth power are 1, so we extend to….

Expanding $(x + y)^n$ and nCr

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Consider $(x + y)^4$

We have $(x + y) * (x + y) * (x + y) * (x + y)$

When we multiply out, for each factor we take an x or a y (but not both)

Thus we could have

- xxxx = x^4 (1 = 4C4 way),
- xxxy, xxyx, xyxx, yxxx, all of which give $x^3 y$ (4 = 4C3 ways),
- xxyy, xyxy yxxy, xyyx, yxyx, yyxx, all of which give $x^2 y^2$ (6 = 4C2 ways),
- yyyx, yyxy, yxyy, xyyy, all of which give $x y^3$ (4 = 4C1 ways),
- yyyy = y^4 (1 = 4C0 way).

So every term will be of the form $x^a y^b$ with a+b = 4 and a, b non-negative; so $x^a y^{4-a}$.

What is the connection with nCr? The coefficient of $x^a y^{4-a}$ is 4Ca.

Makes sense: we have 4 factors, choosing a of them to be x, and 4-a to be y.

Pascal's Triangle

The numbers in the n^{th} row of Pascal's Triangle are the coefficients we obtain in expanding $(x+y)^n$.

Equivalently, we have two diagonals of 1, and all other elements are the sum of the elements in the row above immediately to the left and immediately to the right.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Expanding $(x + y)^n$ and nCr

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Consider $(x + y)^4$

We have $(x + y) * (x + y) * (x + y) * (x + y)$

When we multiply out, for each factor we take an x or a y (but not both)

So every term will be of the form $x^a y^b$ with a+b = 4 and a, b non-negative; so $x^a y^{4-a}$.

What is the connection with nCr? The coefficient of $x^a y^{4-a}$ is 4Ca.

Makes sense: we have 4 factors, choosing a of them to be x, and 4-a to be y.

Thus $(x + y)^4$ = 4C4 x^4 + 4C3 x^3 y + 4C2 $x^2 y^2$ + 4C1 x y^3 + 4C0 y^4.

More generally have the

Binomial Theorem: $(x+y)^n = {}_nC_n\, x^n + {}_nC_{n-1}\, x^{n-1} y + \cdots + {}_nC_1\, x\, y^{n-1} + {}_nC_0\, y^n$

Sketch of the proof:

Assume we know one row, say
$(x+y)^5 = x^5 + 5\,x^4\,y + 10\,x^3\,y^2 + 10\,x^2\,y^3 + 5\,x\,y^4 + y^5$.

Then

$$
\begin{aligned}
(x+y)^6 &= (x+y)\,(x+y)^5 \\
&= x\,(x+y)^5 + y\,(x+y)^5 \\
&= x\,(x^5 + 5\,x^4\,y + 10\,x^3\,y^2 + 10\,x^2\,y^3 + 5\,x\,y^4 + y^5) + y\,(x^5 + 5\,x^4\,y + 10\,x^3\,y^2 + 10\,x^2\,y^3 + 5\,x\,y^4 + y^5) \\
&= (x^6 + 5\,x^5\,y + 10\,x^4\,y^2 + 10\,x^3\,y^3 + 5\,x^2\,y^4 + x\,y^5) + (x^5\,y + 5\,x^4\,y^2 + 10\,x^3\,y^3 + 10\,x^2\,y^4 + 5\,x\,y^5 + y^6) \\
&= x^6 + 5\,x^5\,y + 10\,x^4\,y^2 + 10\,x^3\,y^3 + 5\,x^2\,y^4 + x\,y^5 \\
&\quad + x^5\,y + 5\,x^4\,y^2 + 10\,x^3\,y^3 + 10\,x^2\,y^4 + 5\,x\,y^5 + y^6 \\
&= x^6 + (5+1)\,x^5\,y + (10+5)\,x^4\,y^2 + (10+10)\,x^3\,y^3 + (5+10)\,x^2\,y^4 + (1+5)\,x\,y^5 + y^6 \\
&= x^6 + 6\,x^5\,y + 15\,x^4\,y^2 + 20\,x^3\,y^3 + 15\,x^2\,y^4 + 6\,x\,y^5 + y^6
\end{aligned}
$$

Pascal's Identity: Often write nCk as $\binom{n}{k}$

Rather than doing algebra, we can tell a story involving nCr's….

Pascal's identity

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.$$

Proof: Assume *n* Red Sox fans, 1 Yankee fan, how many ways to choose a group of *k*?

$$\binom{n+1}{k} = \binom{1}{0}\binom{n}{k} + \binom{1}{1}\binom{n}{k-1}.$$

Why is this true? Note have n+1 fans and must choose k, order doesn't matter.

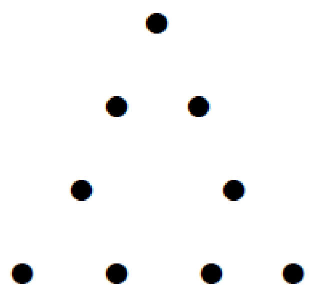
Left: There are $\binom{n}{k+1}$ ways to choose k+1 people from n.

Right: $\binom{1}{0}$ way not to take the Yankee fan, then $\binom{n}{k}$ ways to choose k Sox fans, and second term is $\binom{1}{1}$ way to choose the Yankee fan, and then $\binom{n}{k-1}$ ways to choose k-1 Sox fans (so again have k fans overall).

Pascal's Triangle

Modify Pascal's triangle: • if odd, blank if even.

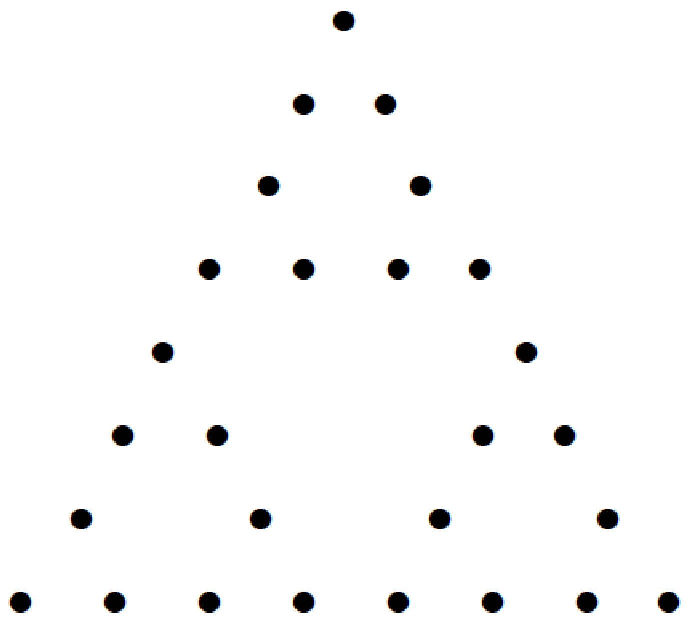
If we have just one row we would see •, if we have four rows we would see



Pascal’s Triangle

Modify Pascal’s triangle: • if odd, blank if even.

For eight rows we find



Pascal's Triangle

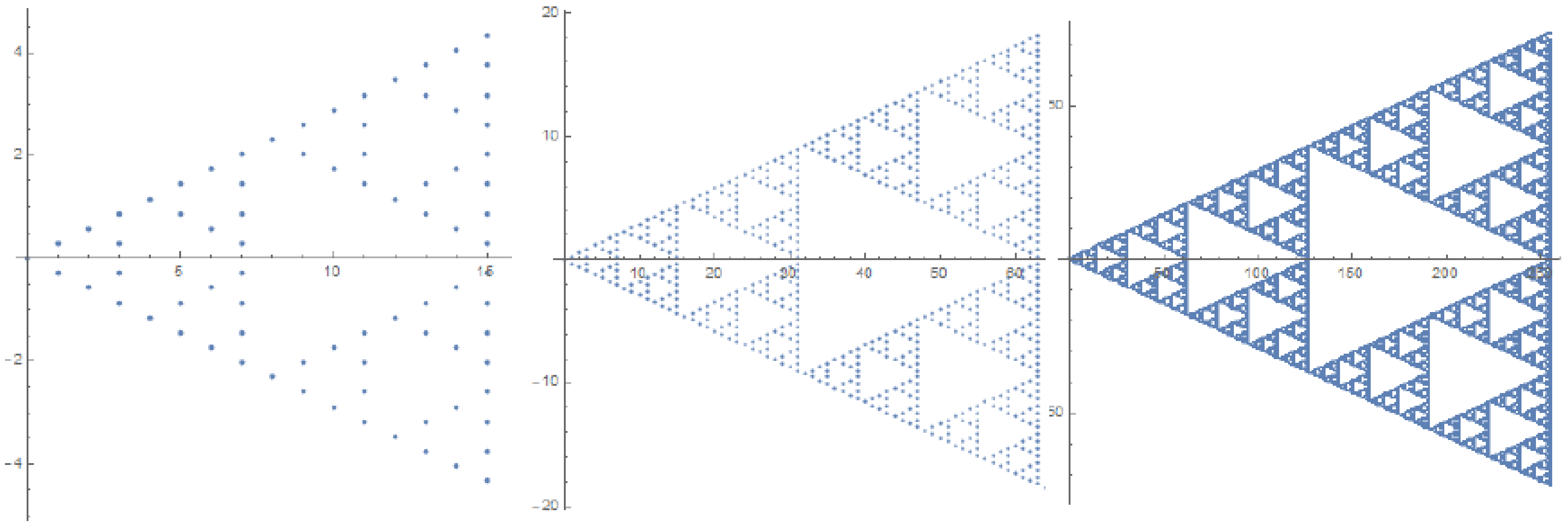


Figure: Plot of Pascal's triangle modulo 2 for 2^4, 2^8 and 2^{10} rows.

https://www.youtube.com/watch?v=tt4_4YajqRM (start 1:35)

Combinations: nCr or $_nC_r$

nCr or $_nC_r$ is the number of ways to choose r objects from n when order DOES NOT matter; the C stands for combinations.

Theorem: $nCr \ * r! = nPr = \frac{n!}{(n-r)!}$, thus $nCr = \frac{n!}{r!(n-r)!}$.

What value of r leads to the largest value of nCr?

STOP! PAUSE THE VIDEO NOW TO THINK ABOUT THE QUESTION.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

Combinations: nCr or ${}_nC_r$

nCr or ${}_nC_r$ is the number of ways to choose r objects from n when order DOES NOT matter; the C stands for combinations.

Theorem: $nCr\ * r! = nPr = \frac{n!}{(n-r)!}$, thus $nCr = \frac{n!}{r!(n-r)!}$.

What value of r leads to the largest value of nCr? It is r = n/2 (if n/2 is not an integer, it is the integer on either side).

To see this, note nCr looks like a polynomial of degree n^r if r is at most n/2, so we want the exponent to be as large as possible, and then use symmetry for r greater than n/2 to relate those values to r less than n/2. (This is NOT a proof, this is a heuristic – see how you go from ${}_nC_r$ to ${}_nC_{r+1}$.

Combinations: nCr or ${}_nC_r$

nCr or ${}_nC_r$ is the number of ways to choose r objects from n when order DOES NOT matter; the C stands for combinations.

Theorem: $nCr \ * r! = nPr = \frac{n!}{(n-r)!}$, thus $nCr = \frac{n!}{r!(n-r)!}$.

What value of r leads to the largest value of nCr? It is r = n/2 (if n/2 is not an integer, it is the integer on either side).

PROOF:

$$ {}_nC_{r+1} = \frac{n!}{(r+1)!(n-(r+1))!} = \frac{n!}{(r+1)r!(n-(r+1))!}\frac{(n-r)}{(n-r)} = \frac{n!}{r!(n-r)!}\frac{n-r}{r+1} = {}_nC_r\frac{n-r}{r+1} $$

As r < n/2, we have $\frac{n-r}{r+1} > 1$. Note $\frac{n-r}{r+1} > 1$ if n-r > r+1 or r < $\frac{n-1}{2}$.