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President, Fibonacci Association

What do you mean? Mirror, mirror on the wall, who's the most irrational number of all?

From Zombies to Fibonaccis: An Introduction to the Theory of Games

Steven J. Miller, Williams College

http://www.williams.edu/Mathematics/sjmiller/public_html

New Jersey Math Camp: Summer 2018

$\sqrt{2}$

$\sqrt{2}$ Is Irrational

Standard Proof: Assume $\sqrt{2} = a/b$.

WLOG, assume b is the smallest denominator among all fractions that equal $\sqrt{2}$.

$2b^2 = a^2$ thus $a = 2m$ is even.

Then $2b^2 = 4m^2$ so $b^2 = 2m^2$ so $b = 2n$ is even.

Thus $\sqrt{2} = a/b = 2m/2n = m/n$, contradicts minimality of n.

(Could also do by contradiction from a, b relatively prime.)

Tennenbaum's Proof

Assume $\sqrt{2} = a/b$ with b minimal.

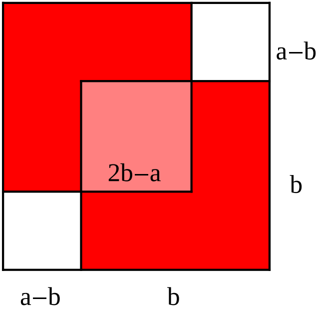


Figure: $2b^2 = a^2$ so $(2b-a)^2 = 2(a-b)^2$ and $\sqrt{2} = \frac{2b-a}{a-b}$.

As $0 < a - b < b$ (if not, $a - b \geq b$ so $a \geq 2b$ and $\sqrt{2} = a/b \geq 2$), contradicts minimality of b.

Challenge

WHAT OTHER NUMBERS HAVE GEOMETRIC IRRATIONALITY PROOFS?

More Irrationals

$\sqrt{3}$

Assume $\sqrt{3} = a/b$ with b minimal.

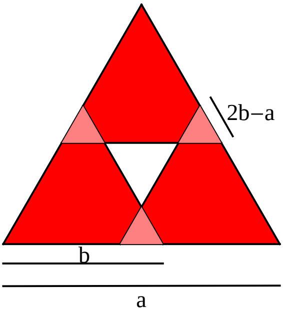


Figure: Geometric proof of the irrationality of $\sqrt{3}$. The white equilateral triangle in the middle has sides of length $2a - 3b$.

Have $3(2b-a)^2 = (2a-3b)^2$ so $\sqrt{3} = (2a-3b)/(2b-a)$, note $2b - a < b$ (else $b \geq a$), violates minimality.

$\sqrt{5}$

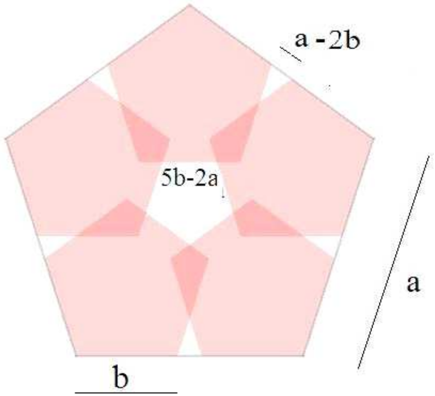
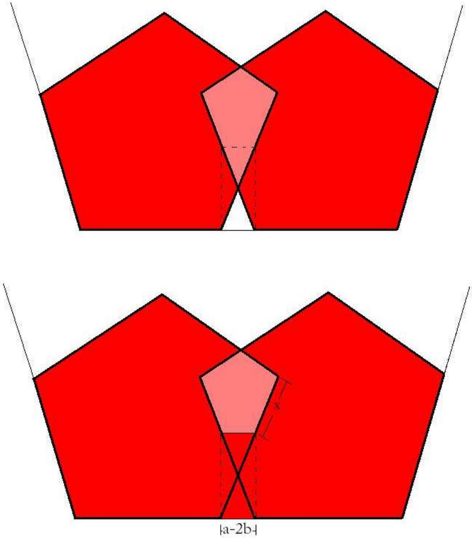


Figure: Geometric proof of the irrationality of $\sqrt{5}$.

$\sqrt{5}$



$\sqrt{5}$

A straightforward analysis shows that the five doubly covered pentagons are all regular, with side length $a - 2b$, and the middle pentagon is also regular, with side length $b - 2(a - 2b) = 5b - 2a$.

We now have a smaller solution, with the five doubly counted regular pentagons having the same area as the omitted pentagon in the middle. Specifically, we have $5(a - 2b)^2 = (5b - 2a)^2$; as $a = b\sqrt{5}$ and $2 < \sqrt{5} < 3$, note that $a - 2b < b$ and thus we have our contradiction.

$\sqrt{6}$

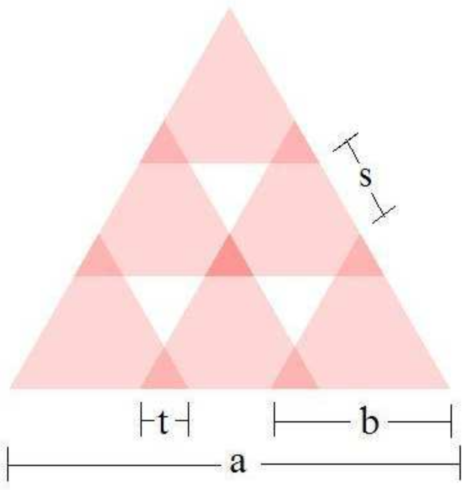


Figure: Geometric proof of the irrationality of $\sqrt{6}$.

Closing Thoughts

Could try to do $\sqrt{10}$ but eventually must break down. Note $3, 6, 10$ are triangular numbers ($T_n = n(n+1)/2$).

$T_8 = 36$ and thus $\sqrt{T_8}$ is an integer!

Can you get a cube-root?

What other numbers?

From $\mathbb{C}$ to Shining $\mathbb{C}$: $\mathbb{C}$omplex Dynamics from $\mathbb{C}$ombinatorics to $\mathbb{C}$oastlines

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Introduction to Applications of Calculus:
Hampshire College 8/8/2022

Introduction

Turbulent '60s: Goal is to (begin to) understand papers

- Edward N. Lorenz, *Deterministic nonperiodic flow*, Journal of Atmospheric Sciences **20** (1963), 130–141. http://journals.ametsoc.org/doi/pdf/10.1175/1520-0469%281963%29020%3C0130%3ADNF%3E2.0.CO%3B2.

- Benoit Mandelbrot, *How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension*, Science, New Series, Vol. 156, No. 3775 (May 5, 1967), pp. 636–638. https://classes.soe.ucsc.edu/ams214/Winter09/foundingpapers/Mandelbrot1967.pdf and http://www.jstor.org/stable/1721427?origin=JSTOR-pdf&seq=1#page_scan_tab_contents.

Lorenz Paper

From the conclusion: *All solutions, and in particular the period solutions, are found to be unstable. .... When our results concerning the instability of nonperiodic flow are applied to the atmosphere, which is ostensibly nonperiodic, they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly. In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long range forecasting would seem to be non-existent.*

Mandelbrot Paper

From the abstract: *Geographical curves are so involved in their detail that their lengths are often infinite or, rather, undefinable. However, many are statistically "self-similar," meaning that each portion can be considered a reduced-scale image of the whole. In that case, the degree of complication can be described by a quantity D that has many properties of a "dimension," though it is fractional; that is, it exceeds the value unity associated with the ordinary, rectifiable, curves.*

Examples of country dimensions from the paper: Britain 1.25, Germany (land frontier in 1899) 1.15, Spain-Portugal (land boundary) 1.14, Australia 1.13, South Africa (coastline) 1.02.

Link

What is the link between the two papers?

Link

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Extreme sensitivity to initial conditions.

Dimension

What is dimension?

Define dimension....

What is dimension?

Define dimension....

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}^2$ are pairs of real numbers, and so on.

Dilating a set by r means multiply each point by r; thus a unit circle centered at the origin becomes a circle of radius r when we dilate by r.

Hausdorff Dimension

Let

$$S \subset \mathbb{R}^n := \{(x_1, \dots, x_n) : x_i \in \mathbb{R}\}$$

be a set. If dilating S by a factor of r yields c copies of S, then the dimension d of S satisfies $r^d = c$.

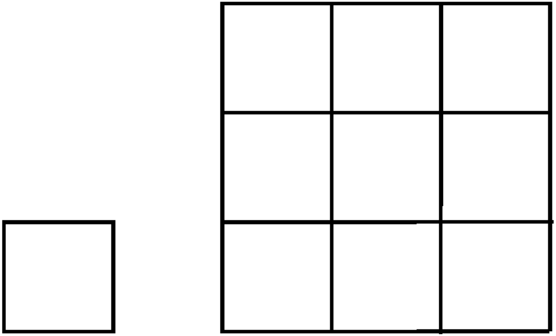
Example: Remember $r^d = c$ where d dimension, r dilation, c copies

What is the easiest example?

Example: Remember $r^d = c$ where d dimension, r dilation, c copies

Segment of length 1. We take $r = 3$ and get $c = 3$ copies; thus $d = 1$ as $3^1 = 3$.

Example: Remember $r^d = c$ where d dimension, r dilation, c copies



Increasing the sides of a square by a factor of $r = 3$ increases the area by a factor of $9 = 3^2$; the dimension is 2 as $3^{2} = 9$.

Cantor Set: $r^d = c$ where d dimension, r dilation, c copies

- Let $C_0 = [0, 1]$, the unit interval.
- Given C_n, let C_{n+1} be the set formed by removing the middle third of each interval in C_n.

$C_1 = \{0, 1/3\} \cup \{2/3, 1\}$ and
$C_2 \ = \ \{0, 1/9\} \cup \{2/9, 3/9\} \cup \{2/3, 7/9\} \cup \{8/9, 1\}$.

Figure: The zeroth iteration of the construction of the Cantor set. Image from Sarang (Wikimedia Commons). Thoughts on dimension?

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Figure: The first iteration of the construction of the Cantor set. Image from Sarang (Wikimedia Commons). Thoughts on dimension?

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Figure: The first three iterations of the construction of the Cantor set. Image from Sarang (Wikimedia Commons). Thoughts on dimension?

Cantor Set: $r^d = c$ where d dimension, r dilation, c copies

- Let $C_0 = [0, 1]$, the unit interval.
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Figure: The first four iterations of the construction of the Cantor set. Image from Sarang (Wikimedia Commons). Thoughts on dimension?

Cantor Set: $r^d = c$ where d dimension, r dilation, c copies

- Let $C_0 = [0, 1]$, the unit interval.
- Given C_n, let C_{n+1} be the set formed by removing the middle third of each interval in C_n.

$C_1 = \{0, 1/3\} \cup \{2/3, 1\}$ and
$C_2 = \{0, 1/9\} \cup \{2/9, 3/9\} \cup \{2/3, 7/9\} \cup \{8/9, 1\}$.

Figure: The first five iterations of the construction of the Cantor set. Image from Sarang (Wikimedia Commons). Thoughts on dimension?

Cantor Set: $r^d = c$ where d dimension, r dilation, c copies

- Let $C_0 = [0, 1]$, the unit interval.
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Figure: The first six iterations of the construction of the Cantor set. Image from Sarang (Wikimedia Commons). Thoughts on dimension?

Cantor Set: $r^d = c$ where d dimension, r dilation, c copies

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Figure: The first six iterations of the construction of the Cantor set. Image from Sarang (Wikimedia Commons). Thoughts on dimension?

Dilate by $r = 3$ and get $c = 2$ copies, thus dimension d satisfies $3^d = 2$, or $d = \log_3 2 \approx 0.63093$; note *not* an integer, but....

Pascal's Triangle

Pascal's triangle: k^{th} entry in the n^{th} row is $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

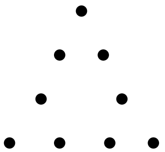
Pascal's Triangle Modulo 2

Modify Pascal's triangle: • if $\binom{n}{k}$ is odd, blank if even.

Pascal's Triangle Modulo 2

Modify Pascal's triangle: • if $\binom{n}{k}$ is odd, blank if even.

If we have just one row we would see •, if we have four rows we would see

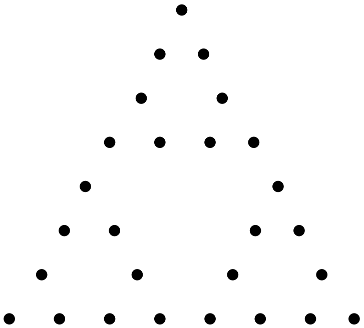


Note: Often write $a \bmod b$ for the remainder of a divided by b; thus $15 \bmod 12$ is 3.

Pascal’s Triangle Modulo 2

Modify Pascal’s triangle: • if $\binom{n}{k}$ is odd, blank if even.

For eight rows we find



Pascal mod 2 Plots

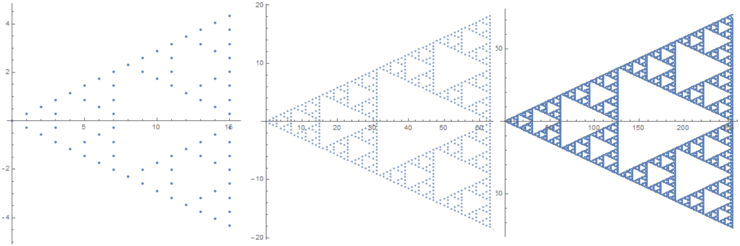


Figure: Plot of Pascal's triangle modulo 2 for 2^4, 2^8 and 2^{10} rows.

https://www.youtube.com/watch?v=tt4_4YajqRM
(start 1:35)
Fixed: https://youtu.be/_vkGakVt1RA?t=264 (start 4:24)

Sierpinski Triangle: Remember $r^d = c$ where d dimension, r dilation, c copies

Figure: The construction process leading to the Sierpinski triangle; first four stages. Image from Wereon (Wikimedia Commons).

What's its dimension?

Sierpinski Triangle: Remember $r^d = c$ where d dimension, r dilation, c copies

Figure: The construction process leading to the Sierpinski triangle; first four stages. Image from Wereon (Wikimedia Commons).

What's its dimension?

If double get three copies; so in $r^d = c$ have $r = 2, c = 3$ and thus $d = \log_2 3 \approx 1.58496$ (note exceeds 1, less than 2).

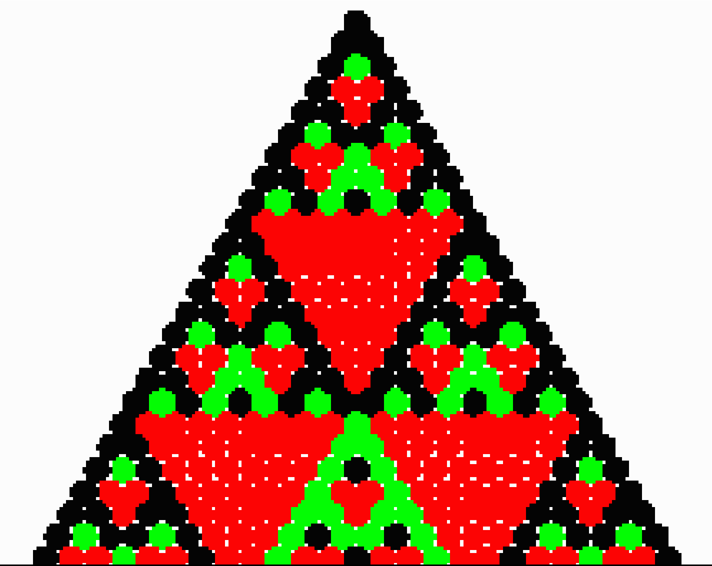
More Pascal

Question: What would be a good way to generalize what we've done?

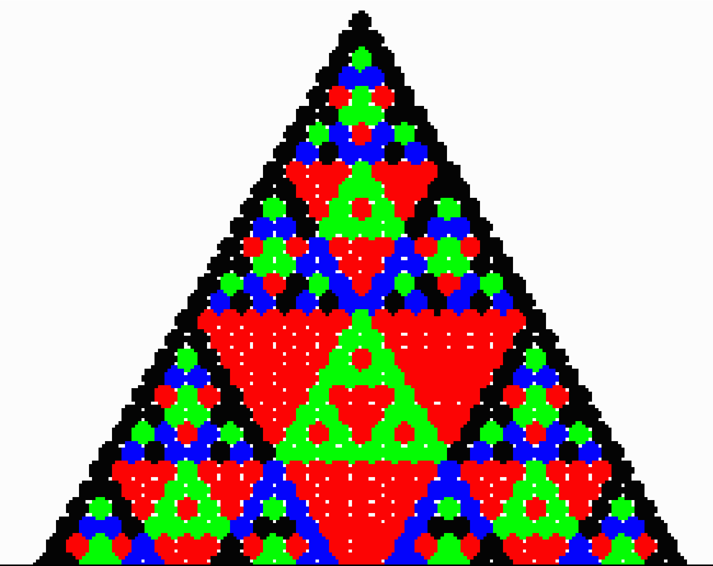
Some links....

- https://www.youtube.com/watch?v=wcxmdiuYjhk
- https://www.youtube.com/watch?v=b2GEQPZQxk0
- https://www.youtube.com/watch?v=XMriWTvPXHI
- https://www.youtube.com/watch?v=QBTiqiIiRpQ

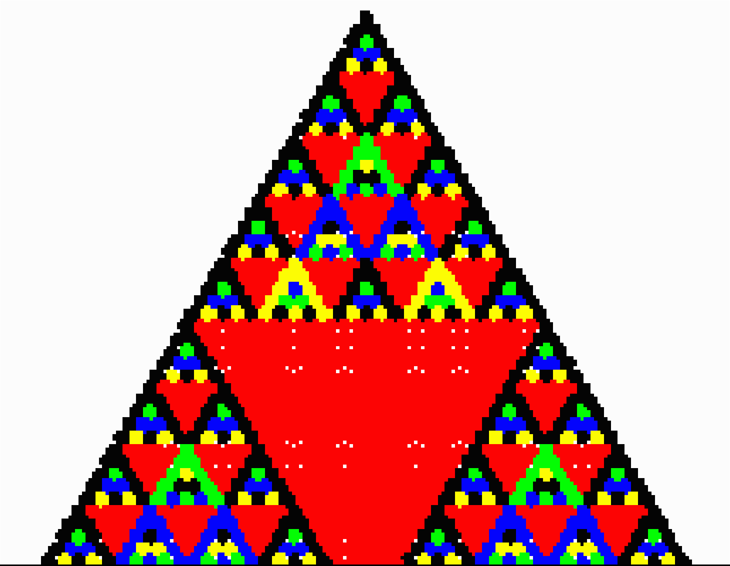
Generalization: Pascal mod 3



Generalization: Pascal mod 4



Generalization: Pascal mod 5



Research Problems

Always ask new questions, try to extend.

Guided 600+ students, two years ago asked in class: can any $r \in \mathbb{R}$ be a fractal dimension?

Coastline

Coastline Dimension

Coastline paradox: measured length of a coastline changes with the scale of measurement.

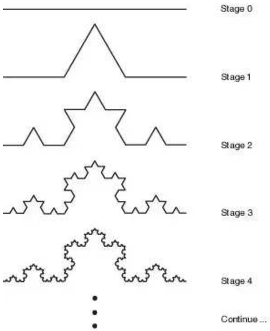
Led to $L(G) = CG^{1-d}$ where C is a constant, G is the scale of measurement, d the dimension.

British Coastline

$L(G) = CG^{1-d}$ where C is a constant, G is the scale of measurement, d the dimension.

Figure: *How Long is the Coastline of the Law* (D. Katz, posted 10/18/10).

Koch Snowflake



Koch snowflake (showing 1 of 3 sides)

Draw an equilateral triangle in the middle, remove bottom.

Repeat on each line segment. Lather, rinse, repeat....

Length at stage n+1 is 4/3 length at stage n; length goes to infinity.

Exercise to show area is bounded.

Dimension: As r^d = c, since r=3 yields c=4, d = log 4 / log 3.

Thus dimension is approximately 1.26186.

Chaos

Finding roots

Much of math is about solving equations.

Finding roots

Much of math is about solving equations.

Example: polynomials:

- $ax + b = 0$, root $x = -b/a$.
- $ax^2 + bx + c = 0$, roots $(-b \pm \sqrt{b^2 - 4ac})/2a$.
- Cubic, quartic: formulas exist in terms of coefficients; not for quintic and higher.

In general cannot find exact solution, how to estimate?

Cubic: For fun, here's the solution to $ax^3 + bx^2 + cx + d = 0$

```
Solve[a x^3 + b x^2 + c x + d == 0, x]
```

$$\Big\{\Big\{x \to -\frac{b}{3a} - \frac{2^{1/3}\left(-b^2+3ac\right)}{3a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} + \frac{\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{3\times 2^{1/3}a}\Big\},$$

$$\Big\{x \to -\frac{b}{3a} + \frac{\left(1+i\sqrt{3}\right)\left(-b^2+3ac\right)}{3\times 2^{2/3}a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} - \frac{\left(1-i\sqrt{3}\right)\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{6\times 2^{1/3}a}\Big\},$$

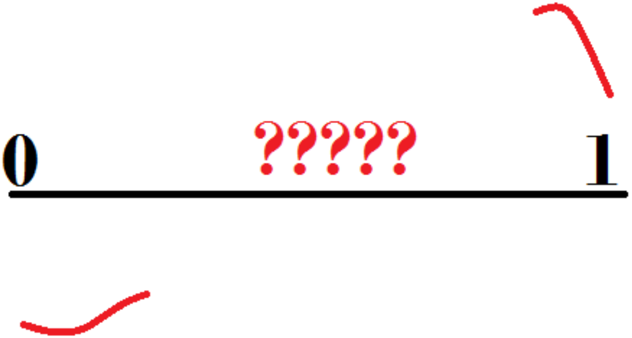
$$\Big\{x \to -\frac{b}{3a} + \frac{\left(1-i\sqrt{3}\right)\left(-b^2+3ac\right)}{3\times 2^{2/3}a\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}} - \frac{\left(1+i\sqrt{3}\right)\left(-2b^3+9abc-27a^2d+\sqrt{4\left(-b^2+3ac\right)^3+\left(-2b^3+9abc-27a^2d\right)^2}\right)^{1/3}}{6\times 2^{1/3}a}\Big\}\Big\}$$

One of four solutions to quartic $ax^4 + bx^3 + cx^2 + dx + e = 0$

```
Solve[a x^4 + b x^3 + c x^2 + d x + e == 0, x]
```

$$
\begin{aligned}
\Bigg\{\Bigg\{x \to & -\frac{b}{4a} - \frac{1}{2}\sqrt{\left(\frac{b^2}{4a^2} - \frac{2c}{3a} + \right.} \\
& \left(2^{1/3}\left(c^2 - 3bd + 12ae\right)\right) \Big/ \left(3a\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\right) + \\
& \frac{1}{3 \times 2^{1/3} a}\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\Bigg) - \frac{1}{2}\sqrt{\left(\frac{b^2}{2a^2} - \frac{4c}{3a} - \right.} \\
& \left(2^{1/3}\left(c^2 - 3bd + 12ae\right)\right) \Big/ \left(3a\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\right) - \\
& \frac{1}{3 \times 2^{1/3} a}\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3} - \\
& \left(-\frac{b^3}{a^3} + \frac{4bc}{a^2} - \frac{8d}{a}\right) \Bigg/ \left(4\sqrt{\left(\frac{b^2}{4a^2} - \frac{2c}{3a} + \left(2^{1/3}\left(c^2 - 3bd + 12ae\right)\right) \Big/ \right.}\right. \\
& \quad \left(3a\left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\right) + \frac{1}{3 \times 2^{1/3} a} \\
& \quad \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace + \sqrt{-4\left(c^2 - 3bd + 12ae\right)^3 + \left(2c^3 - 9bcd + 27ad^2 + 27b^2e - 72ace\right)^2}\right)^{1/3}\Bigg)\Bigg)\Bigg)\Bigg\},
\end{aligned}
$$

Divide and Conquer: Partial plot of continuous function $f(x)$



Divide and Conquer

Divide and Conquer

Assume f is continuous, $f(a) < 0 < f(b)$. Then f has a root between a and b. To find, look at the sign of f at the midpoint $f\left(\frac{a+b}{2}\right)$; if sign positive look in $[a, \frac{a+b}{2}]$ and if negative look in $[\frac{a+b}{2}, b]$. Lather, rinse, repeat.

Divide and Conquer

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Example:

- $f(0) = -1, f(1) = 3$, look at $f(.5)$.
- $f(.5) = 2$, so look at $f(.25)$.
- $f(.25) = -.4$, so look at $f(.375)$.

Divide and Conquer (continued)

How fast? Every 10 iterations uncertainty decreases by a factor of $2^{10} = 1024 \approx 1000$.

Thus 10 iterations essentially give three decimal digits.

		f(x) = x^2 - 3, sqrt(3)		1.732051		
n	left	right	f(left)	f(right)	left error	right error
1	1	2	-2	1	0.732051	-0.26795
2	1.5	2	-0.75	1	0.232051	-0.26795
3	1.5	1.75	-0.75	0.0625	0.232051	-0.01795
4	1.625	1.75	-0.35938	0.0625	0.107051	-0.01795
5	1.6875	1.75	-0.15234	0.0625	0.044551	-0.01795
6	1.71875	1.75	-0.0459	0.0625	0.013301	-0.01795
7	1.71875	1.734375	-0.0459	0.008057	0.013301	-0.00232
8	1.726563	1.734375	-0.01898	0.008057	0.005488	-0.00232
9	1.730469	1.734375	-0.00548	0.008057	0.001582	-0.00232
10	1.730469	1.732422	-0.00548	0.001286	0.001582	-0.00037

Figure: Approximating $\sqrt{3} \approx 1.7320508075688772935274463415$.

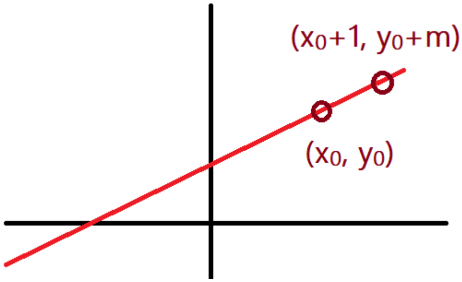
Equation of a Line

Lots of ways to write: Point-Slope: given $P = (x_0, y_0)$ and m,

$$y - y_0 = m(x - x_0)$$

or

$$y = m(x - x_0) + y_0.$$



Tangent Line

One of most important uses of calculus; approximate a curve by a straight line.

Locally good: for small changes in time, speed approximately constant.

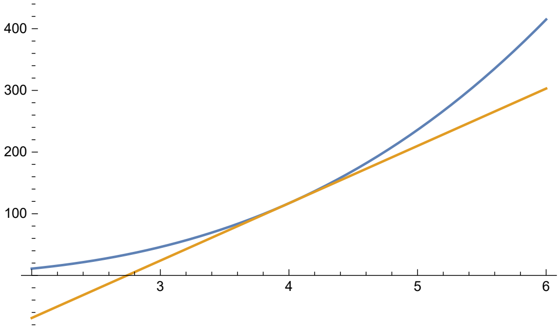
New location $f(x)$ is approximately $f(x_0) + f'(x_0)(x - x_0)$ (where start plus speed at x_0 times elapsed time).

Get the tangent line by Point-Slope: $P = (x_0, f(x_0))$ and slope $m = f'(x_0)$.

Tangent Line

New location $f(x)$ is approximately $f(x_0) + f'(x_0)(x - x_0)$ (where start plus speed at x_0 times elapsed time).

Get the tangent line by Point-Slope: $P = (x_0, f(x_0))$ and slope $m = f'(x_0)$.
$f(x) = 2x^3 - 3x + 1$, $f'(x) = 6x^2 - 3$, $x_0 = 4$.



Tangent Line

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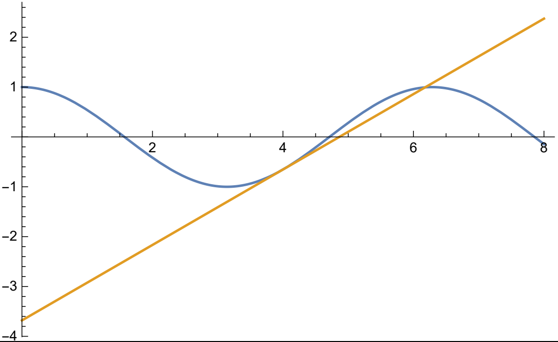
Get the tangent line by Point-Slope: $P = (x_0, f(x_0))$ and slope $m = f'(x_0)$.
$f(x) = \cos(x)$, $f'(x) = -\sin(x)$, $x_0 = 4$.

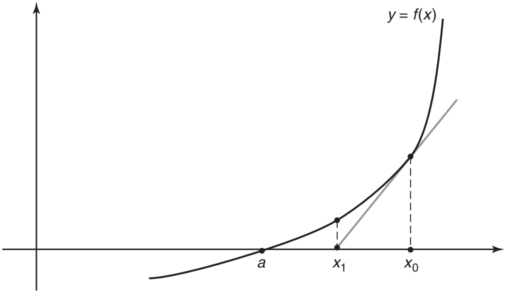


Newton's Method

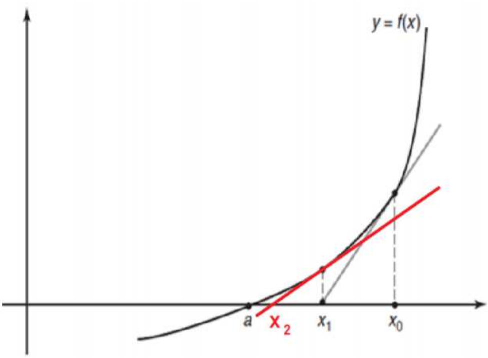
Newton's Method

Assume f is continuous and differentiable. We generate a sequence hopefully converging to the root of $f(x) = 0$ as follows. Given x_n, look at the tangent line to the curve $y = f(x)$ at x_n; it has slope $f'(x_n)$ and goes through $(x_n, f(x_n))$ and gives line $y - f(x_n) = f'(x_n)(x - x_n)$. This hits the x-axis at $y = 0, x = x_{n+1}$, and yields $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

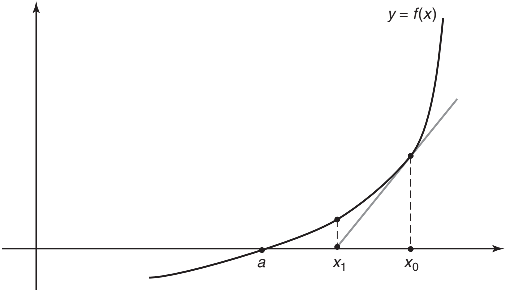
Newton's Method



Newton's Method



Newton's Method



For example, $f(x) = x^2 - 3$ after algebra get
$x_{n+1} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right)$.

Doing the algebra: Approximating roots of $f(x) = 0$

Have n^{th} approx x_n to the root of $f(x) = 0$, want next, x_{n+1}.
Tangent line $y = f(x)$ at point $(x_n, f(x_n))$ with slope $m = f'(x_n)$:

$$y = f(x_n) + f'(x_n)(x - x_n).$$

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$$y = f(x_n) + f'(x_n)(x - x_n).$$

Tangent line hits x-axis when $y = 0$, call that x_{n+1}, so

$$0 = f(x_n) + f'(x_n)(x_{n+1} - x_n).$$

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If $f(x) = x^2 - 3$: $f'(x) = 2x$, $f(x_n) = 2x_n^2 - 3$, $f'(x_n) = 2x_n$:

$$-\frac{f(x_n)}{f'(x_n)} = x_{n+1} - x_n \quad \text{or} \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad \text{thus}$$

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$$x_{n+1} = x_n - \frac{x_n^2 - 3}{2x_n} = \frac{2x_n^2 - x_n^2 + 3}{x_n} = \frac{1}{2}\left(x_n + \frac{3}{x_n}\right).$$

Rational Approximations: $\sqrt{3} = 1.7320508076\ldots$

$$
\begin{aligned}
x_{n+1} &= \frac{1}{2}\left(x_n + \frac{3}{x_n}\right) \\
x_0 &= 2
\end{aligned}
$$

Rational Approximations: $\sqrt{3} = 1.7320508076\ldots$

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x_3 &= \frac{1}{2}\left(\frac{97}{56} + \frac{3}{97/56}\right) = \frac{18817}{10864} \approx 1.7320508100.
\end{aligned}
$$

Newton's Method

n	x[n]	1.0 x[n]	Sqrt[3] - x[n]
0	2	2.0000000000000000000000000000000000000000	-0.2679491924311227064725536584941276330 57
1	$\frac{7}{4}$	1.7500000000000000000000000000000000000000	-0.0179491924311227064725536584941276330 57
2	$\frac{97}{56}$	1.7321428571428572062984585500089451670 65	-0.0000920495739798493296965156369847759 14
3	$\frac{18817}{10864}$	1.7320508100147276042690691610914655029774	$-2.4458502469732900355191644519 08 \times 10^{-9}$

Sqrt[3] = 1.7320508075688772935274463415058723669428

x[5] = 1.7320508075688772935274463415058723678037

x[4] = 1.7320508075688772952543539460721719142351

$$\sqrt{3} = 1.732050807568877293527446341505872366\mathbf{6}9428$$

$$x_5 = 1.732050807568877293527446341505872367\mathbf{7}8037$$

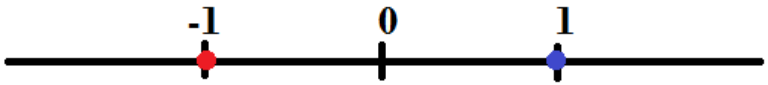
$$x_5 = \frac{1002978273411373057}{579069776145402304}.$$

Newton Method: $x^2 - 3 = 0$

Consider $x^2 - 1 = (x - 1)(x + 1) = 0$.

Roots are 1, -1; if we start at a point x_0 do we approach a root? If so which?

Recall $x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right)$.



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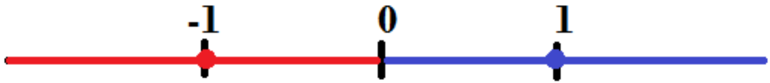

Newton Fractal: $x^3 - 1 = 0$: https://www.youtube.com/watch?v=ZsFixqGFNRc

What are the roots to $x^3 - 1 = 0$?

Here comes Complex Numbers!
$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}, i = \sqrt{-1}\}$.

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Here comes Complex Numbers!
$\mathbb{C} = \{x + iy : x, y \in \mathbb{R}, i = \sqrt{-1}\}$.

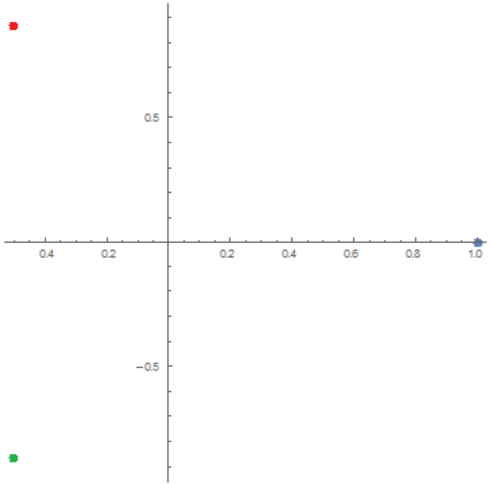
$$\begin{aligned} x^3 - 1 &= (x-1)(x^2 + x + 1) \\ &= (x-1) \cdot \left(x - \frac{-1 + \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2}\right) \cdot \left(x - \frac{-1 - \sqrt{1^2 - 4 \cdot 1 \cdot 1}}{2}\right) \\ &= (x-1) \cdot \left(x - \frac{-1 + \sqrt{-3}}{2}\right) \cdot \left(x - \frac{-1 - \sqrt{-3}}{2}\right) \\ &= (x-1) \cdot \left(x - \frac{-1 + i\sqrt{3}}{2}\right) \cdot \left(x - \frac{-1 - i\sqrt{3}}{2}\right). \end{aligned}$$

Roots are $1, \; -1/2 + i\sqrt{3}/2, \; -1/2 - i\sqrt{3}/2$.

Newton Fractal: $x^3 - 1 = 0$:

https://www.youtube.com/watch?v=ZsFixqGFNRc

If start at (x, y), what root do you iterate to?



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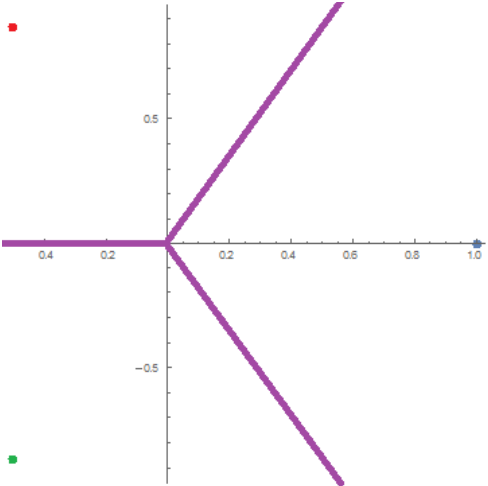
If start at (x, y), what root do you iterate to? Guess



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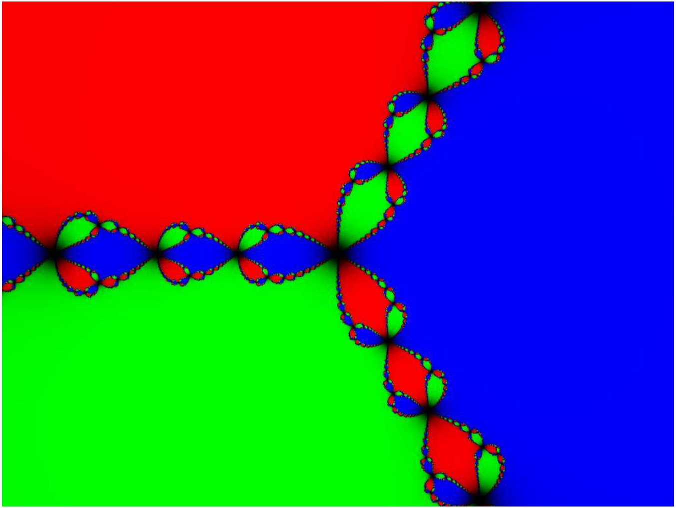

Mandelbrot Set:

https://www.youtube.com/watch?v=0jGaio87u3A

Definition: Set of all complex numbers $c = x + iy$ ($i = \sqrt{-1}$) such that if $f_c(u) = u^2 + c$ then the sequence

$$z_1 = f_c(0), \quad z_2 = f_c(z_1) = f_c(f_c(0)), \quad \cdots, \quad z_{n+1} = f_c(z_n)$$

$$z_1 = c, \quad z_2 = c^2 + c, \quad z_3 = (c^2 + c)^2 + c, \quad \cdots$$

remains bounded as $n \to \infty$.

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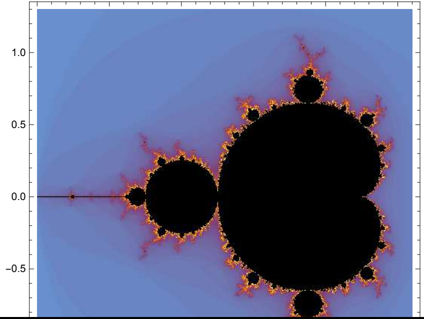
MandelbrotSetPlot[]



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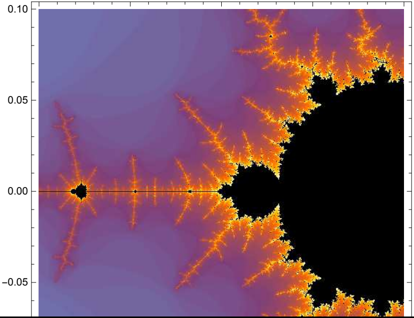
MandelbrotSetPlot[-1.5 - .1 I, -1.3 + .1 I]



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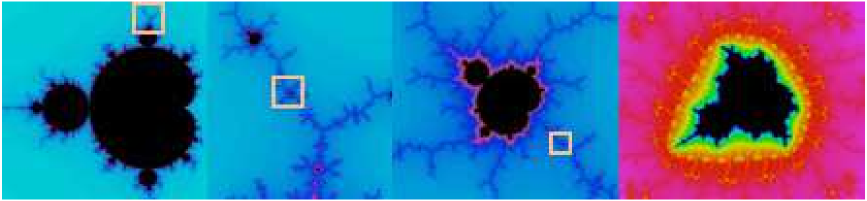
Zooming in....



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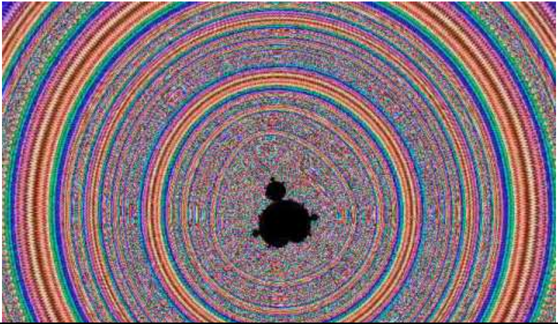
Extreme zoom!



Mandelbrot Links: Especially http://www.hpdz.net/index.htm

- https://www.youtube.com/watch?v=0jGaio87u3A
- https://www.youtube.com/watch?v=9j2yV1nLCEI
- https://www.youtube.com/watch?v=ZsFixqGFNRc
- https://www.youtube.com/watch?v=PD2XgQOyCCk
- https://www.youtube.com/watch?v=vfteiiTfE0c

Consequences

Why do we care?

Consequences

Why do we care?

If such small changes can lead to such wildly different behavior, what happens when we try to solve the equations governing our world?

Lorenz equations and chaos (from Wikipedia)

Lorenz equations:

In 1963, Edward Lorenz developed a simplified mathematical model for atmospheric convection.[1] The model is a system of three ordinary differential equations now known as the Lorenz equations:

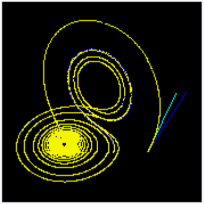
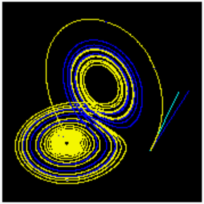
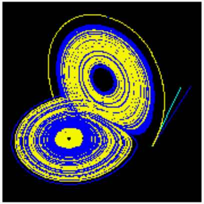
$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = x(\rho - z) - y \\ \dot{z} = xy - \beta z \end{cases}$$

The equations relate the properties of a two-dimensional fluid layer uniformly warmed from below and cooled from above. In particular, the equations describe the rate of change of three quantities with respect to time: x is proportional to the rate of convection, y to the horizontal temperature variation, and z to the vertical temperature variation.[2] The constants σ, ρ, and β are system parameters proportional to the Prandtl number, Rayleigh number, and certain physical dimensions of the layer itself.[3]

The Lorenz equations also arise in simplified models for lasers,[4] dynamos,[5] thermosyphons,[6] brushless DC motors,[7] electric circuits,[8] chemical reactions[9] and forward osmosis.[10]

Lorenz equations and chaos (from Wikipedia)

Sensitive dependence on the initial condition

Time t=1 (Enlarge)	**Time t=2** (Enlarge)	**Time t=3** (Enlarge)
		

These figures — made using ρ=28, σ = 10 and β = 8/3 — show three time segments of the 3-D evolution of 2 trajectories (one in blue, the other in yellow) in the Lorenz attractor starting at two initial points that differ only by 10^{-5} in the x-coordinate. Initially, the two trajectories seem coincident (only the yellow one can be seen, as it is drawn over the blue one) but, after some time, the divergence is obvious.

Take-aways

Takeaways

Math is applicable!

Similar behavior in very different systems.

Extreme sensitivity challenges.