Fixed Points in the Fibonacci Sequence

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Introduction

The **Fibonacci sequence:** let $F_0 = 0$, $F_1 = 1$ and for $n > 1$, we have $F_n = F_{n-1} + F_{n-2}$.

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \ldots$$
$$F_0\ \ F_1\ \ F_2\ \ F_3\ \ F_4\ \ F_5\ \ F_6\ \ F_7\ \ F_8\ \ F_9\ \ F_{10}\ \ F_{11}\ \ F_{12}$$

The **order of appearance** $z(n)$ for a natural n is the smallest positive integer ℓ such that $n \mid F_\ell$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$z(n)$	1	3	4	6	5	12	8	6	12	15	10	12

Table 1: The order of appearance $z(n)$

A **fixed point** in the Fibonacci sequence occurs when $z(n) = n$.

Theorem

$z(n) = n$ if and only if $n = 5^k$ or $n = 12 \cdot 5^k$ for some $k \geq 0$.

Fixed Point Order

Definition

The **fixed point order** $z^k(n)$ for a natural n is the smallest positive integer k such that $z^k(n)$ is a fixed point. If n is a fixed point, the fixed point order is 0.

$n \setminus k$	1	2	3	4
1	**1**			
2	3	4	6	**12**
3	4	6	**12**	
4	6	**12**		
5	**5**			
6	**12**			
7	8	6	**12**	
8	6	**12**		
9	**12**			
10	15	20	30	**60**

Table 2: Iterations of z on n, bold values are fixed points

Families of Integers k-Iterations from a Fixed Point

We study the sequence of k-iterations of z on n required to reach a fixed point. This result tells us if there exists an integer n that takes exactly k iterations of z to reach a fixed point, then there are infinitely many integers that take exactly k iterations of z to reach a fixed point.

Theorem (FMV)

For all positive integers k, there exist infinitely many n with fixed point order k.

Example:
When $n = 6 \cdot 5^k$ with $k \geq 0$, we know $z^k(n)$ has fixed point order 1. These include

$$n = 6,\ 30,\ 150,\ 750,\ 3750,\ 18750,\ \ldots$$

To see why this is a fixed point, consider

$$z\left(6 \cdot 5^k\right) = \operatorname{lcm}\left(z(2), z(3), z\left(5^k\right)\right) = 3 \cdot 4 \cdot 5^k = 12 \cdot 5^k.$$

Thus, after one iteration of z on $n = 6 \cdot 5^k$, we have arrived at a fixed point.

When $z^k(n)$ is expressed as the product of a constant relatively prime to 5 and a power of 5 , then $z^k(n \cdot 5^a)$ can be written as the product of that same constant and another power of 5.

Lemma 1 (FMV): Suppose $z^k(5^a \cdot n) = c_k 5^{a_i}$, where $\gcd(c_k, 5) = 1$. For all non-negative integers a, the coefficient c_k remains constant.

For powers of 10, the k^{th} iteration of z is given by Lemma 2. This allows us to find integers that require exactly k iterations of z to reach a fixed point for any positive integer k

Lemma 2 (FMV): For all integers k and m with $k \geq 0, m \geq 4$ and $2k + 2 \leq m$, we have

$$z^k\left(10^m\right) = 3 \cdot 5^m \cdot 2^{m-2k}.$$

Finite Fixed Point Order

We are interested in knowing how many iterations of z on n are required in order to reach a fixed point and what fixed point a given number is sent to.

Theorem (FMV)

All positive integers n have finite fixed point order.

This result has been proven by showing $z^k(n) = 2^a 3^b 5^c$ for positive integers a, b, c and by using a relationship between the Pisano period of n and $z(n)$. We prove this using a minimal counterexample argument.

Properties of $z(n)$

We use the following property for $n = ab$, where a and b are relatively prime

$$z^k(n) = \operatorname{lcm}\left(z^k(a), z^k(b)\right).$$

This lets us consider different proof strategies

- Case 1: n is prime
- Case 2: n is a power of a prime
- Case 3: n has at least two distinct prime factors

Future Directions

1. Where are fixed points located when initial conditions are varied?
2. How does $z^k(n)$ behave for related sequences?
3. For a given integer, can the fixed point order be bounded as a function of n?

k	1	2	3	4	5	6	7	8	9	10
n	1	4	3	2	11	89	1069	2137	4273	59833
FP	1	12	12	12	60	60	60	60	60	60

Table 3: First n that takes k iterations to reach a fixed point, calculated in Mathematica

References

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