

A Pair of Diophantine Equations Involving Fibonacci Numbers

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January 8, 2025 - Joint Mathematics Meetings

A Pair of Equations

For relatively prime $a, b \in \mathbb{N}$, consider

$$ax + by = \frac{(a-1)(b-1)}{2}$$

$$ax + by + 1 = \frac{(a-1)(b-1)}{2}$$

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Theorem (Beiter (1964), extended by Chu (2020))

Exactly one of the equations has a **nonnegative integral** solution (x, y) . The solution is unique.

A Pair of Equations

$$ax + by = \frac{(a-1)(b-1)}{2} \quad (1)$$

$$ax + by + 1 = \frac{(a-1)(b-1)}{2} \quad (2)$$

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Define $\Gamma : \{(a, b) \in \mathbb{N}^2 : \gcd(a, b) = 1\} \rightarrow \{1, 2\}$:

$$\Gamma(a, b) = \begin{cases} 1, & \text{if (1) has a solution;} \\ 2, & \text{if (2) has a solution.} \end{cases}$$

Fibonacci Numbers

$$F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5, F_6 = 8, \dots$$

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$$\gcd(F_n, F_{n+1}) = 1$$

$$a \boxed{x} + b \boxed{y} = \frac{(a-1)(b-1)}{2} \quad (1)$$

$$a \boxed{x} + b \boxed{y} + 1 = \frac{(a-1)(b-1)}{2} \quad (2)$$

Fibonacci Numbers

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$$\gcd(F_n, F_{n+1}) = 1$$

$$F_n \boxed{x} + F_{n+1} \boxed{y} = \frac{(F_n - 1)(F_{n+1} - 1)}{2} \quad (1)$$

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Fibonacci Numbers

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$$\Gamma(F_n, F_{n+1}) = ?$$

Previous work

Chu 2020:

| | | | | | | |
|------------------------|---|---|---|---|---|---|
| n | 3 | 4 | 5 | 6 | 7 | 8 |
| $\Gamma(F_n, F_{n+1})$ | 2 | 2 | 2 | 1 | 1 | 1 |

Previous work

Chu 2020:

| | | | | | | |
|------------------------|---|---|---|---|---|---|
| n | 3 | 4 | 5 | 6 | 7 | 8 |
| $\Gamma(F_n, F_{n+1})$ | 2 | 2 | 2 | 1 | 1 | 1 |

$$F_{6k} \cdot \frac{F_{6k-1} - 1}{2} + F_{6k+1} \cdot \frac{F_{6k-1} - 1}{2} = \frac{(F_{6k} - 1)(F_{6k+1} - 1)}{2}$$

$$1 + F_{6k+3} \cdot \frac{F_{6k+2} - 1}{2} + F_{6k+4} \cdot \frac{F_{6k+2} - 1}{2} = \frac{(F_{6k+3} - 1)(F_{6k+4} - 1)}{2}$$

Previous work

Chu 2020:

| | | | | | | |
|------------------------|---|---|---|---|---|---|
| n | 3 | 4 | 5 | 6 | 7 | 8 |
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R. K. Davala 2023: Identities for other famous sequences

Previous work

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R. K. Davala 2023: Identities for other famous sequences

Polymath Jr. 2023: Characterize (a, b) for $\Gamma(a, b) = 1$ or 2

Our goals

Chu 2020:

$$F_{6k} \cdot \frac{F_{6k-1} - 1}{2} + F_{6k+1} \cdot \frac{F_{6k-1} - 1}{2} = \frac{(F_{6k} - 1)(F_{6k+1} - 1)}{2}$$

$$1 + F_{6k+3} \cdot \frac{F_{6k+2} - 1}{2} + F_{6k+4} \cdot \frac{F_{6k+2} - 1}{2} = \frac{(F_{6k+3} - 1)(F_{6k+4} - 1)}{2}$$

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$$\Gamma(F_n^2, F_{n+1}^2) = ?$$

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$$F_{6k} \cdot \frac{F_{6k-1} - 1}{2} + F_{6k+1} \cdot \frac{F_{6k-1} - 1}{2} = \frac{(F_{6k} - 1)(F_{6k+1} - 1)}{2}$$

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$$\Gamma(F_n^2, F_{n+1}^2) = ?$$

Are there similar identities for Fibonacci squared?

Fibonacci Squared - The Data

$$ax + by = \frac{(a-1)(b-1)}{2} \quad (1)$$

$$ax + by + 1 = \frac{(a-1)(b-1)}{2} \quad (2)$$

| n | F_n^2 | F_{n+1}^2 | x | y | $\Gamma(F_n^2, F_{n+1}^2)$ |
|-----|---------|-------------|-----|-----|----------------------------|
| 2 | 1 | 4 | 0 | 0 | (1) |
| 3 | 4 | 9 | 3 | 0 | (1) |
| 4 | 9 | 25 | 5 | 2 | (2) |
| 5 | 25 | 64 | 20 | 4 | (1) |
| 6 | 64 | 169 | 51 | 12 | (1) |
| 7 | 169 | 441 | 83 | 52 | (2) |

Table: Fibonacci squared

Fibonacci Squared - The Data

| n | F_n^2 | F_{n+1}^2 | x | y | $\Gamma(F_n^2, F_{n+1}^2)$ | $F_n^2 - x - y$ |
|----------|----------|-------------|----------|----------|----------------------------|-----------------|
| 2 | 1 | 4 | 0 | 0 | (1) | 1 |
| 3 | 4 | 9 | 3 | 0 | (1) | 1 |
| 4 | 9 | 25 | 5 | 2 | (2) | 2 |
| 5 | 25 | 64 | 20 | 4 | (1) | 1 |
| 6 | 64 | 169 | 51 | 12 | (1) | 1 |
| 7 | 169 | 441 | 83 | 52 | (2) | 34 |
| 8 | 441 | 1156 | 356 | 84 | (1) | 1 |
| 9 | 1156 | 3025 | 935 | 220 | (1) | 1 |
| 10 | 3025 | 7921 | 1513 | 934 | (2) | 578 |
| \vdots | \vdots | \vdots | \vdots | \vdots | \vdots | \vdots |
| 13 | 54289 | 142129 | 27143 | 16776 | (2) | 10370 |

Table: Fibonacci Squared Pattern

Fibonacci Squared - The Data

| n | F_n^2 | F_{n+1}^2 | x | y | $\Gamma(F_n^2, F_{n+1}^2)$ | $F_n^2 - x - y$ |
|-----|--------------|-------------|--------------|-------|----------------------------|-----------------|
| 2 | 1 | 4 | 0 | 0 | (1) | 1 |
| 3 | 4 | 9 | 3 | 0 | (1) | 1 |
| 4 | 9 | 25 | 5 | 2 | (2) | 2 |
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| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
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Table: Fibonacci Squared Pattern

Identities for Fibonacci Squared

Theorem

For $n \geq 7$ with $n \equiv 1 \pmod{6}$, the following holds

$$F_n^2 \cdot \frac{F_n^2 - 3}{2} + F_{n+1}^2 \cdot \frac{F_n^2 - F_{n-1}^2 - 1}{2} + 1 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}.$$

Identities for Fibonacci Squared

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Theorem

For $n \geq 1$ with $n \equiv 4 \pmod{6}$, the following holds

$$F_n^2 \cdot \frac{F_n^2 + 1}{2} + F_{n+1}^2 \cdot \frac{F_n^2 - F_{n-1}^2 - 1}{2} + 1 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}.$$

Identities for Fibonacci Squared

Theorem

For $n \geq 1$ with $n \equiv 0, 2, 3, 5 \pmod{6}$, the following holds

$$F_n^2 \cdot \left(F_n^2 - \frac{F_{n-1}^2 + 1}{2} \right) + F_{n+1}^2 \cdot \frac{F_{n-1}^2 - 1}{2} = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}.$$

Proof

Theorem

For $n \geq 1$ with $n \equiv 0, 2, 3, 5 \pmod{6}$, the following identity holds

$$F_n^2 \cdot \left(F_n^2 - \frac{F_{n-1}^2 + 1}{2} \right) + F_{n+1}^2 \cdot \frac{F_{n-1}^2 - 1}{2} = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}$$

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Proof:

$$F_n = F_{n+1} - F_{n-1}$$

Proof

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Proof:

$$F_n = F_{n+1} - F_{n-1} \implies F_n^2 - F_{n-1}^2 - F_{n+1}^2 = -2F_{n-1}F_{n+1}$$

Proof

Theorem

For $n \geq 1$ with $n \equiv 0, 2, 3, 5 \pmod{6}$, the following identity holds

$$F_n^2 \cdot \left(F_n^2 - \frac{F_{n-1}^2 + 1}{2} \right) + F_{n+1}^2 \cdot \frac{F_{n-1}^2 - 1}{2} = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}$$

Proof:

$$F_n = F_{n+1} - F_{n-1} \implies F_n^2 - F_{n-1}^2 - F_{n+1}^2 = -2F_{n-1}F_{n+1}$$

$$(F_{n-1}F_{n+1} - F_n^2)^2 = ((-1)^n)^2 \quad (\text{Cassini's Identity})$$

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Proof:

$$F_n = F_{n+1} - F_{n-1} \implies F_n^2 - F_{n-1}^2 - F_{n+1}^2 = -2F_{n-1}F_{n+1}$$

$$(F_{n-1}F_{n+1} - F_n^2)^2 = ((-1)^n)^2 \quad (\text{Cassini's Identity})$$

$$\implies F_n^4 - 2F_n^2F_{n-1}F_{n+1} + F_{n-1}^2F_{n+1}^2 = 1$$

Proof

$$F_n^2 - F_{n-1}^2 - F_{n+1}^2 = -2F_{n-1}F_{n+1}$$

$$F_n^4 - 2F_n^2F_{n-1}F_{n+1} + F_{n-1}^2F_{n+1}^2 = 1$$

$$\implies 2F_n^4 - F_n^2F_{n-1}^2 - F_n^2F_{n+1}^2 + F_{n-1}^2F_{n+1}^2 = 1$$

Proof

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$$\implies 2F_n^4 - F_n^2F_{n-1}^2 - F_n^2F_{n+1}^2 + F_{n-1}^2F_{n+1}^2 = 1$$

$$(2F_n^2 - F_{n-1}^2 - 1) \cdot F_n^2 + (F_{n-1}^2 - 1) \cdot F_{n+1}^2 = (F_n^2 - 1)(F_{n+1}^2 - 1)$$

Proof

$$F_n^2 - F_{n-1}^2 - F_{n+1}^2 = -2F_{n-1}F_{n+1}$$

$$F_n^4 - 2F_n^2F_{n-1}F_{n+1} + F_{n-1}^2F_{n+1}^2 = 1$$

$$\implies 2F_n^4 - F_n^2F_{n-1}^2 - F_n^2F_{n+1}^2 + F_{n-1}^2F_{n+1}^2 = 1$$

$$(2F_n^2 - F_{n-1}^2 - 1) \cdot F_n^2 + (F_{n-1}^2 - 1) \cdot F_{n+1}^2 = (F_n^2 - 1)(F_{n+1}^2 - 1)$$

$$\left(F_n^2 - \frac{F_{n-1}^2 + 1}{2}\right) \cdot F_n^2 + \left(\frac{F_{n-1}^2 - 1}{2}\right) \cdot F_{n+1}^2 = \frac{(F_n^2 - 1)(F_{n+1}^2 - 1)}{2}$$

Fibonacci Cubed - The Data

What about Fibonacci cubed?

Fibonacci Cubed - The Data

What about Fibonacci cubed?

| n | F_n^3 | F_{n+1}^3 | x_n | y_n | $\Gamma(F_n^3, F_{n+1}^3)$ |
|-----|---------|-------------|-------|-------|----------------------------|
| 2 | 1 | 8 | 0 | 0 | (1) |
| 3 | 8 | 27 | 8 | 1 | (1) |
| 4 | 27 | 125 | 18 | 9 | (2) |
| 5 | 125 | 512 | 106 | 36 | (1) |
| 6 | 512 | 2197 | 405 | 161 | (2) |
| 7 | 2197 | 9261 | 1791 | 673 | (1) |

Table: Fibonacci cubed

Fibonacci Cubed - The Data

What about Fibonacci cubed?

| n | F_n^3 | F_{n+1}^3 | x_n | y_n | $\Gamma(F_n^3, F_{n+1}^3)$ |
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Table: Fibonacci cubed

$$x_4 = 3^3 - x_3 - 1; \quad y_4 = y_3 + 2^3$$

$$x_5 = 5^3 - x_4 - 1; \quad y_5 = y_4 + 3^3$$

$$x_6 = 8^3 - x_5 - 1; \quad y_6 = y_5 + 5^3$$

Fibonacci Cubed - The Data

$$\begin{cases} x_n &= F_n^3 - x_{n-1} - 1 \\ y_n &= y_{n-1} + F_{n-1}^3. \end{cases}$$

Fibonacci Cubed - The Data

$$\begin{cases} x_n &= F_n^3 - x_{n-1} - 1 \\ y_n &= y_{n-1} + F_{n-1}^3. \end{cases}$$

Theorem

Let $n \in \mathbb{N}$ with $n \geq 2$. The following hold

$$\begin{aligned} \left[\sum_{k=1}^{2n-1} (-1)^{k-1} F_k^3 \right] F_{2n-1}^3 + \left(\sum_{k=2}^{2n-2} F_k^3 \right) F_{2n}^3 &= \frac{(F_{2n-1}^3 - 1)(F_{2n}^3 - 1)}{2}, \\ 1 + \left[\sum_{k=1}^{2n} (-1)^k F_k^3 - 1 \right] F_{2n}^3 + \left(\sum_{k=2}^{2n-1} F_k^3 \right) F_{2n+1}^3 &= \frac{(F_{2n}^3 - 1)(F_{2n+1}^3 - 1)}{2}. \end{aligned}$$

Proof Sketch

Frontczak's formulas (2018) to replace sums of Fibonacci cubes:

$$\begin{aligned}
 & \left(\frac{F_{6n}}{4} - \frac{F_{6n-3}}{4} - F_{2n}^3 + F_{2n-1}^3 - \frac{1}{2} \right) F_{2n-1}^3 \\
 & + \left(\frac{F_{6n-3}}{4} + \frac{F_{6n-6}}{4} - F_{2n-1}^3 - F_{2n-2}^3 - \frac{1}{2} \right) F_{2n}^3 \\
 & = \frac{(F_{2n-1}^3 - 1)(F_{2n}^3 - 1)}{2}
 \end{aligned}$$

Proof Sketch

Frontczak's formulas (2018) to replace sums of Fibonacci cubes:

$$\begin{aligned} & \left(\frac{F_{6n}}{4} - \frac{F_{6n-3}}{4} - F_{2n}^3 + F_{2n-1}^3 - \frac{1}{2} \right) F_{2n-1}^3 \\ & + \left(\frac{F_{6n-3}}{4} + \frac{F_{6n-6}}{4} - F_{2n-1}^3 - F_{2n-2}^3 - \frac{1}{2} \right) F_{2n}^3 \\ & = \frac{(F_{2n-1}^3 - 1)(F_{2n}^3 - 1)}{2} \end{aligned}$$

Equivalently, we need to prove

$$\begin{aligned} (F_{6n} - F_{6n-3})F_{2n-1}^3 + 4F_{2n-1}^6 + (F_{6n-3} + F_{6n-6})F_{2n}^3 \\ - 4F_{2n-2}^3 - 10F_{2n-1}^3F_{2n}^3 = 2 \end{aligned}$$

Proof Sketch

$$F_{3n} = 5F_n^3 + 3(-1)^n F_n$$

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$$\implies (F_{6n} - F_{6n-3}) F_{2n-1}^3 = (5F_{2n}^3 + 3F_{2n} - 5F_{2n-1}^3 + 3F_{2n-1}) F_{2n-1}^3$$

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$$F_{2n-2} F_{2n} = (-1)^{2n-1} + F_{2n-1}^2 = -1 + F_{2n-1}^2 \quad (\text{Cassini's Identity})$$

Proof Sketch

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$$F_{2n-2} F_{2n} = (-1)^{2n-1} + F_{2n-1}^2 = -1 + F_{2n-1}^2 \quad (\text{Cassini's Identity})$$

$$\implies (F_{6n} - F_{6n-3}) F_{2n-1}^3 + 4F_{2n-1}^6 + (F_{6n-3} + F_{6n-6}) F_{2n}^3 - 4F_{2n-2}^3 - 10F_{2n-1}^3 F_{2n}^3 = 2$$

Another Pair of Diophantine Equations

Theorem

Given $a, b \in \mathbb{N}$ with $(a, b) = 1$, consider the two following equations.

$$xa + yb = \frac{(a-1)(b-1)}{2},$$
$$1 + xa + yb = \frac{(a-1)(b-1)}{2}.$$

Exactly one of the equations above has a **nonnegative integral** solution (x, y) , and the solution is unique.

Another Pair of Diophantine Equations

$(a, b) = (7, 8)$ gives $(x, y) = (3, 0)$:

$$3 \cdot 7 + 0 \cdot 8 = \frac{(7-1)(8-1)}{2}.$$

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$(a, b) = (7, 8)$ gives $(x, y) = (3, 0)$:

$$3 \cdot 7 + 0 \cdot 8 = \frac{(7-1)(8-1)}{2}.$$

Can we find a pair of equations that give only **positive** solutions?

Another Pair of Diophantine Equations

Theorem

Given $a, b \in \mathbb{N}$ with $(a, b) = 1$, consider the two following equations

$$xa + yb = \frac{(a-1)(b-1)}{2} + (a+b),$$
$$1 + xa + yb = \frac{(a-1)(b-1)}{2} + (a+b).$$

Exactly one of the equations above has a **positive integral** solution (x, y) , and the solution is unique.

Another Pair of Diophantine Equations

$$(a, b) = (7, 8):$$

$$\text{Old: } 3 \cdot 7 + 0 \cdot 8 = \frac{(7-1)(8-1)}{2}$$

$$\text{New: } 4 \cdot 7 + 1 \cdot 8 = \frac{(7-1)(8-1)}{2} + (7+8)$$

$$(3, 0) \rightarrow (3+1, 0+1)$$

Another Pair of Diophantine Equations

Theorem

Given $a, b \in \mathbb{N}$ with $(a, b) = 1$, $b \geq 2$, and $(a + 1)b \equiv 0 \pmod{2}$, consider the two following equations

$$xa + yb = \frac{(a + 1)b}{2} + 1, \quad (3)$$

$$xa + yb = \frac{(a + 1)b}{2} - 1. \quad (4)$$

Exactly one of the two equations has a **positive integral** solution (x, y) , and the solution is unique.

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$$xa + yb = \frac{(a + 1)b}{2} + 1, \quad (3)$$

$$xa + yb = \frac{(a + 1)b}{2} - 1. \quad (4)$$

Exactly one of the two equations has a **positive integral** solution (x, y) , and the solution is unique.

Define $\Gamma' : \{(a, b) : (a, b) = 1, b \geq 2, 2 \text{ divides } (a + 1)b\} \rightarrow \{1, 2\}$:

$$\Gamma'(a, b) = \begin{cases} 1, & \text{if (3) has a solution;} \\ 2, & \text{if (4) has a solution.} \end{cases}$$

Another Pair of Diophantine Equations

Asymmetric: possible that $\Gamma'(a, b) \neq \Gamma'(b, a)$.

$(a, b) = (3, 5)$ and $(a, b) = (5, 3)$

$$2 \cdot 3 + 1 \cdot 5 = \frac{(3+1)5}{2} + 1$$

$$1 \cdot 5 + 1 \cdot 3 = \frac{(5+1)3}{2} - 1$$

What's next

Examine higher powers of Fibonacci numbers

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Explore $\Gamma(F_n^i, F_{n+1}^j)$ for $i \neq j$

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Characterize when $\Gamma'(a, b) \neq \Gamma'(b, a)$