# Black Hole Zeckendorf Games

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General Audience

# Background

**Zeckendorf Decompositions**

Any positive integer can be written as an unique sum of non-adjacent Fibonacci numbers.

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## The Zeckendorf Game

Baird-Smith, Epstein, Flint, and Miller converted the process of reaching an integer's Zeckendorf decomposition into a 2-player game where the last player to move wins.

# Orginal Zeckendorf Game Moves

- Merging

| $F_1$ | $F_2$ | $F_3$ | $F_4$ |
|---|---|---|---|
| 2 | 0 | 0 | 0 |

$\Longrightarrow$

| $F_1$ | $F_2$ | $F_3$ | $F_4$ |
|---|---|---|---|
| 0 | 1 | 0 | 0 |

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- Adding

| $F_1$ | $F_2$ | $F_3$ | $F_4$ |
|---|---|---|---|
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- Splitting for $F_2$

| $F_1$ | $F_2$ | $F_3$ | $F_4$ |
|---|---|---|---|
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| $F_1$ | $F_2$ | $F_3$ | $F_4$ |
|---|---|---|---|
| 0 | 0 | 1 | 0 |

- Splitting for $F_2$

| $F_1$ | $F_2$ | $F_3$ | $F_4$ |
|---|---|---|---|
| 0 | 2 | 0 | 0 |

$\Longrightarrow$

| $F_1$ | $F_2$ | $F_3$ | $F_4$ |
|---|---|---|---|
| 1 | 0 | 1 | 0 |

- Splitting for $F_i$ when $i \geq 3$

| $F_1$ | $F_2$ | $F_3$ | $F_4$ |
|---|---|---|---|
| 0 | 0 | 2 | 0 |

$\Longrightarrow$

| $F_1$ | $F_2$ | $F_3$ | $F_4$ |
|---|---|---|---|
| 1 | 0 | 0 | 1 |

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$$\mathbf{(25, \ldots)}$$
$$|$$
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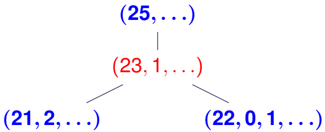

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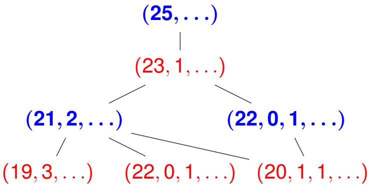

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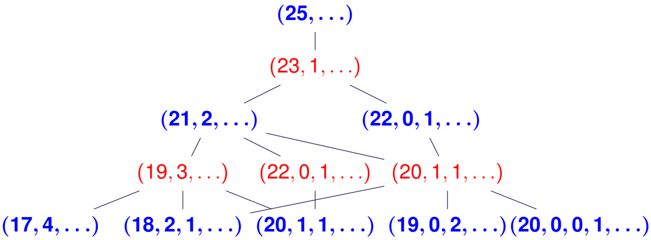

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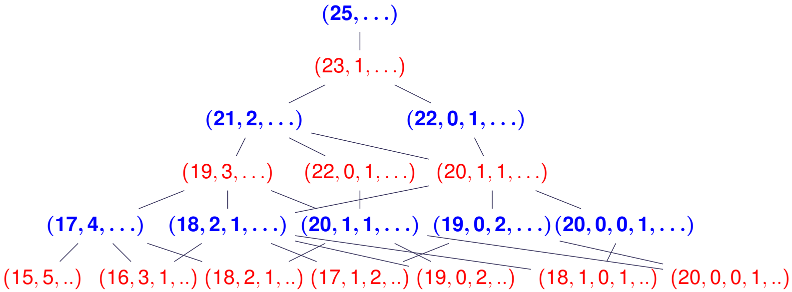
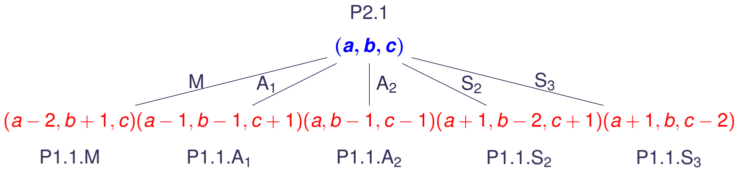


# Black Hole Zeckendorf Game

### $F_m$ Black Hole Variation

Any pieces placed in a column $F_i$ for $i \geq m$ are permanently removed from gameplay.

For the $F_4$ case, this allows for the following moves:



# Empty Board Black Hole Zeckendorf Game

Following this we define an empty board phase of the game. Players take turns placing one piece in the outermost columns of the board, namely $F_1$ and $F_{m-1}$, until the weighted sum equals $n$. Placing one piece in the $F_i$ column removes $F_i$ pieces from the pile of $n$. The last player to place a piece assumes the role of Player 2 in the decomposition phase.

# Results

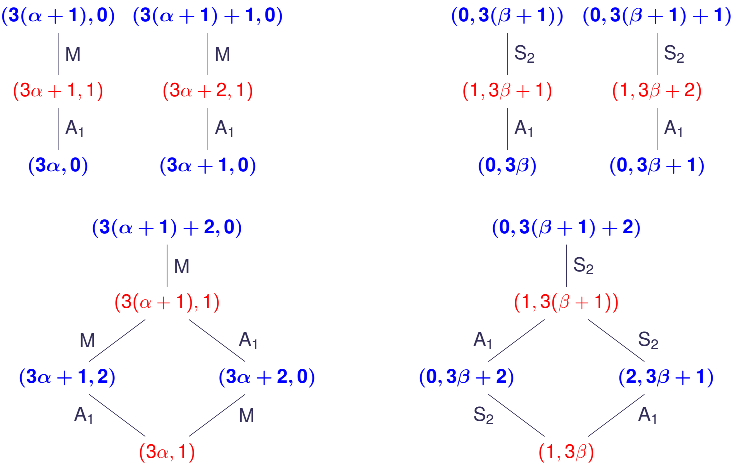
**Theorems: 4.4-4.8**

The winner of all possible games for a Black Hole on $F_3$ can be determined based on the value of $a$ and $b$ modulo 3. We find that Player 2 wins $(a, b)$ for all $a \equiv b \equiv 0$, $a \equiv 0, b \equiv 1$ or $a \equiv 1, b \equiv 0$. Player 1 wins for any other setup. For the EBBHZG, Player 1 wins $n \equiv 1, 2, 3, 6, 8 \pmod 9$ and Player 2 wins $n \equiv 0, 4, 5, 7 \pmod 9$.

# Single Column Winners

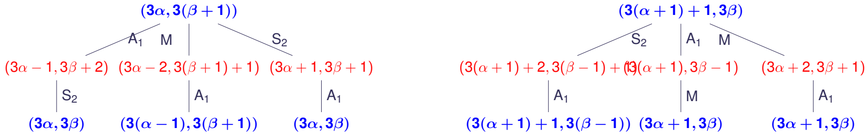
$$
\begin{array}{cccc}
\mathbf{(3(\alpha+1),0)} & \mathbf{(3(\alpha+1)+1,0)} & \mathbf{(0,3(\beta+1))} & \mathbf{(0,3(\beta+1)+1)} \\
\Big|\ \mathrm{M} & \Big|\ \mathrm{M} & \Big|\ \mathrm{S}_2 & \Big|\ \mathrm{S}_2 \\
(3\alpha+1,1) & (3\alpha+2,1) & (1,3\beta+1) & (1,3\beta+2) \\
\Big|\ \mathrm{A}_1 & \Big|\ \mathrm{A}_1 & \Big|\ \mathrm{A}_1 & \Big|\ \mathrm{A}_1 \\
\mathbf{(3\alpha,0)} & \mathbf{(3\alpha+1,0)} & \mathbf{(0,3\beta)} & \mathbf{(0,3\beta+1)}
\end{array}
$$

# Single Column Winners



# General Winning Setups

Player 2 wins for $a, b \equiv 0 \pmod 3$ or $a$ or $b \equiv 1 \pmod 3$, with the other equivalent to $0 \pmod 3$.




From any other initial setup, Player 1 can immediately place one of the setups above.

# Empty Board Phase

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|---|---|
| 1 | 0 |

| $F_1$ | $F_2$ |
|---|---|
| 1 | 1 |

# (a,0,0) Setup

**Theorem 5.1**

Let $(a, 0, 0)$ be an initial setup for an $F_4$ Black Hole Zeckendorf game. For any $n \neq 2 \in \mathbb{Z}^{\geq 0}$, Player 2 has a constructive solution.

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$(\boldsymbol{a}, \mathbf{0}, \mathbf{0})$

M

$(a-2, 1, 0)$

$A_1$

$(\boldsymbol{a}-\mathbf{3}, \mathbf{0}, \mathbf{1})$

M

$(a-5, 1, 1)$

$A_2$

$(\boldsymbol{a}-\mathbf{5}, \mathbf{0}, \mathbf{0})$

# (0,0,c) Setup

**Theorem 5.3**

Let $(0, 0, c)$ be an initial setup for an $F_3$ Black Hole Zeckendorf game. For any $c \neq 0, 1, 5 \in \mathbb{Z}^{\geq 0}$, Player 1 has a constructive winning strategy.

**Corollary 5.4**

Player 2 wins $(1, 0, c)$ for all $c \neq 3$.

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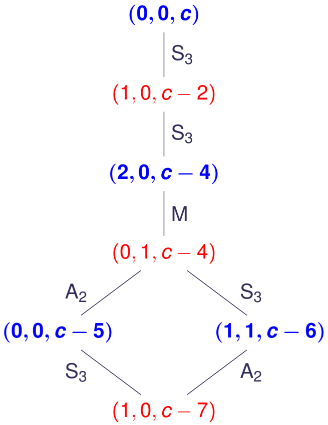


# (a,0,c) Setup

Consider a setup $(a, 0, c)$ as some $(3\alpha + k_1, 0, 4\gamma + k_3)$ with $k_1 \in \{0, 1, 2\}$ and $k_3 \in \{0, 1, 2, 3\}$. Player 2 wins are depicted in bold blue, and Player 1 wins are depicted in red.

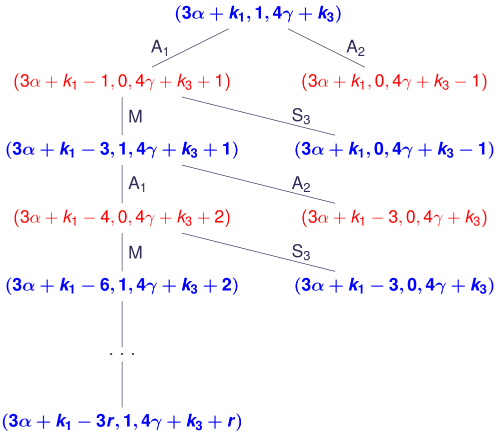
| | $a \equiv 0 \pmod 3$ | $a \equiv 1 \pmod 3$ | $a \equiv 2 \pmod 3$ |
|---|---|---|---|
| $c \equiv 0 \pmod 4$ | $\boldsymbol{\alpha \geq \gamma}$ <br> $\alpha \leq \gamma - 1$ | $\boldsymbol{\forall \alpha, \gamma}$ | $\boldsymbol{\alpha \geq \gamma + 1}$ <br> $\alpha \leq \gamma$ |
| $c \equiv 1 \pmod 4$ | $\boldsymbol{\alpha \geq \gamma - 1}$ <br> $\alpha \leq \gamma - 2$ | $\boldsymbol{\forall \alpha, \gamma}$ | $\boldsymbol{\alpha \geq \gamma}$ <br> $\alpha \leq \gamma - 1$ |
| $c \equiv 2 \pmod 4$ | $\forall \alpha, \gamma$ | $\alpha \geq \gamma + 1$ <br> $\boldsymbol{\alpha \leq \gamma}$ | $\forall \alpha, \gamma$ |
| $c \equiv 3 \pmod 4$ | $\forall \alpha, \gamma$ | $\alpha \geq \gamma$ <br> $\boldsymbol{\alpha \leq \gamma - 1}$ | $\forall \alpha, \gamma$ |

# A Non-Constructive Proof

**Lemma 5.5**

For all $\alpha, \gamma$ such that $k_1 \in \{1, 2\}$ and $k_3 \in \{0, 1, 2, 3\}$, Player 1 has a winning strategy for $(3\alpha + k_1, 1, 4\gamma + k_3)$

We use similar strategies to prove that some other key positions do in fact win.



# Empty Board Game on $F_4$

**Theorem 5.17**

Player 2 wins an Empty Board $F_4$ Black Hole Zeckendorf game for $n \equiv 0, 2, 4, 6, 9, 11, 13 \pmod{16}$ when $n \neq 2, 32$, and also wins $n = 17, 47$.
Player 1 wins for $n \equiv 1, 3, 5, 7, 8, 10, 12, 14, 15 \pmod{16}$ when $n \neq 17, 47$, and also wins $n = 2, 32$.

- We use similar move mirroring strategies as before.

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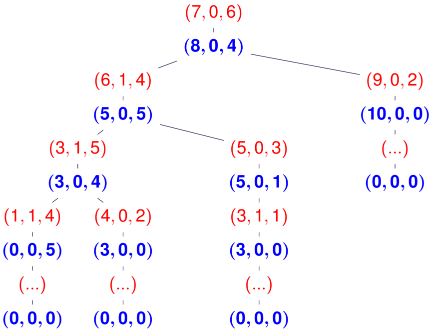
- We use similar move mirroring strategies as before.
- For large enough $n$, $\alpha \geq \gamma$ is always true when move mirroring is used.

# A return to $n = 25$

Player 2 can use move mirroring to force the Player 1 to set the board as $(7, 0, 6)$, so Player 2 moves first in the decomposition phase.



# Future Work

- Redefining the Empty Board Game to allow placement on all columns

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- Expanding to black holes on higher columns.

# Acknowledgements

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# Thank you!

Any questions?

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