Signal Recovery Using Gabor Frames

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Motivation

Let's send two **messages** using two different approaches:

"Hi"
↓
0100100001101001
↓
■100■0000110■001
No way to restore the original signal unless we know more :(

"Hi"
↓
0100100001101001
↓
DFT:
$[\,\widehat{f}_0, \widehat{f}_1, \widehat{f}_2, \widehat{f}_3, \widehat{f}_4, \widehat{f}_5, \widehat{f}_6, \widehat{f}_7, \widehat{f}_8, \widehat{f}_9, \widehat{f}_{10}, \widehat{f}_{11}, \widehat{f}_{12}, \widehat{f}_{13}, \widehat{f}_{14}, \widehat{f}_{15}\,]$
↓
*R*eceived:
$[\,\blacksquare, \widehat{f}_1, \widehat{f}_2, \widehat{f}_3, \blacksquare, \blacksquare, \widehat{f}_6, \widehat{f}_7, \widehat{f}_8, \widehat{f}_9, \widehat{f}_{10}, \widehat{f}_{11}, \widehat{f}_{12}, \blacksquare, \widehat{f}_{14}, \widehat{f}_{15}\,]$

We receive only part of a signal/frequencies - the rest is missing.

Questions we want to answer:

- Is it possible to reconstruct the full message?
- What are sufficient conditions for reconstruction?

Background

- We will call an arbitrary function $f : \mathbb{Z}_N^d \to \mathbb{C}$ a **signal**.
- We will call an arbitrary function's Fourier transform $\widehat{f} : \mathbb{Z}_N^d \to \mathbb{C}$ a **frequency**.

Definition: Let $f : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$. The **normalized DFT**, $\hat{f} : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$, of f is given by

$$\hat{f}(m,n) := \frac{1}{\sqrt{NT}} \sum_{x\in\mathbb{Z}_N} \sum_{y\in\mathbb{Z}_T} f(x,y) \exp\left(-2\pi i \left(\frac{xm}{N} + \frac{yn}{T}\right)\right)$$

.

Classical Recovery Condition [2]

Let $f : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$, and suppose we transmit the frequencies $\hat{f}$, but the values of $\hat{f}$ are missing in $M \subset \mathbb{Z}_N \times \mathbb{Z}_T$. If f is supported in $E \subset \mathbb{Z}_N \times \mathbb{Z}_T$ and

$$|E||M| < \frac{NT}{2},$$

then f can be recovered exactly using Logan's method [1], which consists of finding $f = \arg\min_g \|g\|_{L^1(\mathbb{Z}_N \times \mathbb{Z}_T)}$ subject to $\hat{g}(m) = \hat{f}(m)$ for all $m \notin M$.

Improved Recovery Condition Using the Gabor Transform

We are interested in applying the Gabor transform, an object from continuous harmonic analysis, to our discrete setting.

Definition: Let $f : \mathbb{R} \to \mathbb{C}$ be (Lebesgue) integrable. The **Gabor transform** of f is defined by

$$G_f(\omega, \tau) = \int_{-\infty}^{\infty} f(t) g(t-\tau) e^{-i\omega t} dt,$$

where g is a window function, commonly taken to be $g(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{t^2}{2\sigma^2}}$.

The window function isolates the function within a short time span, as it can be easier to reconstruct f with reasonable accuracy based on knowledge of its approximate frequencies in short periods of time than based on an imperfect transmission of the Fourier transform of the entire function.

In the discrete setting, rather than taking the window function to be the normal distribution, we can take the window function to restrict f to particular rows or columns in its domain.

Definition: Given a function $f : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$, we define its **row-wise Gabor Transform** $Gf : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$ by

$$Gf(m,a) := N^{-1/2} \sum_{t\in\mathbb{Z}_N} f(t,a) e^{-\frac{2\pi i m\cdot t}{N}},$$

i.e., $Gf(m,a) := \widehat{f(-,a)}(m)$.

Theorem (SMALL 2025)

Suppose $f : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$, where $T : \mathbb{N} \to \mathbb{N}$ such that $T(N) = o\left(\sqrt{N}e^N\right)^a$, and define

$$E_{\max} := \max_{a\in\mathbb{Z}_T} |\mathrm{supp}_t(f(t,a))|.$$

Suppose we transmit $Gf(m,a)$ for all $(m,a) \in \mathbb{Z}_N \times \mathbb{Z}_T$ and that the distribution of lost frequencies is binomial with fixed probability $0 < \theta < \frac{1}{2E_{\max}}$. Let M be the set of missing frequencies and define

$$M_{\max} := \max_{a\in\mathbb{Z}_T} |M \cap \{Gf(t,a) : t \in \mathbb{Z}_N\}|,$$
$$M_{\min} := \min_{a\in\mathbb{Z}_T} |M \cap \{Gf(t,a) : t \in \mathbb{Z}_N\}|.$$

As $N \to \infty$, $\mathbb{P}\left(M_{\max} < \frac{N}{2E_{\max}}\right) \to 1$, which implies that the probability of unique recovery converges to 1.

Furthermore, for $\theta > \frac{1}{2E_{\max}}$, as $N \to \infty$, $\mathbb{P}\left(M_{\min} < \frac{N}{2E_{\max}}\right) \to 0$.

[a] We use standard asymptotic notation: $f(N) = O(g(N))$ means $|f(N)| \le C|g(N)|$ for some constant $C > 0$ and sufficiently large N, while $f(N) = o(g(N))$ means $\lim_{N\to\infty} f(N)/g(N) = 0$.

Discussion

Suppose $f : \mathbb{Z}_N \times \mathbb{Z}_T \to \mathbb{C}$ has support $E \subset \mathbb{Z}_N \times \mathbb{Z}_T$, and we transmit $\hat{f}$, its Fourier transform in $\mathbb{Z}_N \times \mathbb{Z}_T$. Suppose also that the values of $\hat{f}$ are missing in M, and that each element of M is lost independently with probability θ. Then, the expectation of lost frequencies is $NT\theta$. So, according to the classical recovery condition

$$|E||M| < \frac{NT}{2},$$

$|E|$ can be as large as

$$\left\lceil \frac{NT}{2NT\theta} \right\rceil - 1 = \left\lceil \frac{1}{2\theta} \right\rceil - 1$$

while guaranteeing unique recovery with high probability for N sufficiently large. In our theorem, we only require

$$E_{\max} < \frac{1}{2\theta},$$

which means that $|E|$ can be as large as

$$T\left(\left\lceil \frac{1}{2\theta} \right\rceil - 1\right)$$

while nearly guaranteeing unique recovery for N large enough, with the restriction on the growth of T being $T = o\left(\sqrt{N}e^N\right)$.

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