From Alligators to All In - Generalized Predator-Prey Problems and Guided Online Research

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AMS Contributed Paper Session on Mathematical Biology

https://web.williams.edu/Mathematics/sjmiller/public_html/math/talks/talks.html

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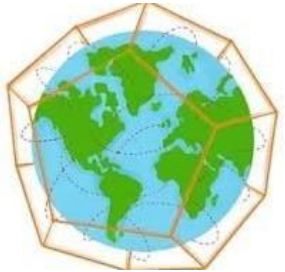
Outline

1) Project Context: Polymath Jr. 2025

2) Biological Background and Motivation

3) Model Formulation: ODE System

Polymath Junior Program

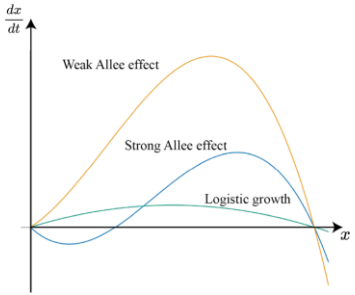


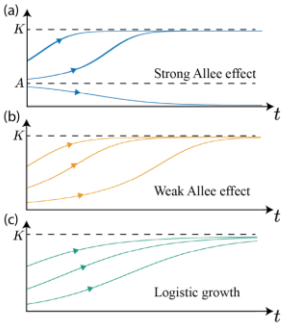
- Provide research opportunities to undergraduates.
- Online, runs in the spirit of the Polymath Project.
- Projects run by researcher with experience in undergraduate mentoring.
- Most 15-25 students, a main mentor, grad students / postdocs assisting.

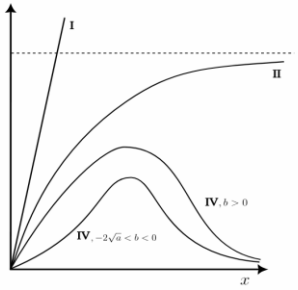
https://geometrynyc.wixsite.com/polymathreu

During World War I, fishing in regions such as the Adriatic Sea declined dramatically, leading to unusual fluctuations in fish populations. Vito Volterra became intrigued by this phenomenon.

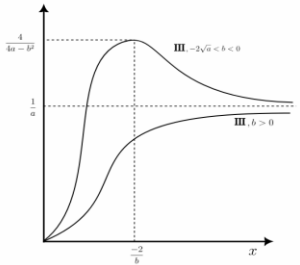
Extensions







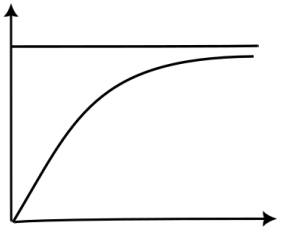
(a) Holling Type I,II, and IV



(b) Holling Type III

$$\frac{dx}{dt} = G_1(x) - \alpha_1 F(x)y - H_1(x)$$

$$\frac{dy}{dt} = -G_2(y) + \alpha_2 F(x)y - H_2(y)$$



Background (Biology)

1800s to mid-1900s: Intense hunting for leather and meat.

1950s to 1960s: Populations crash; poaching persists; weak enforcement.

1967: Listed as endangered in the United States.

1973: Endangered Species Act enabled.

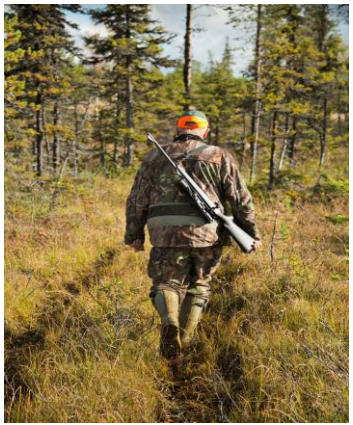
1970s: States close hunting, protect nests, begin science-based management.

Background (Biology)

Late 1970s to 1980s: Rebound via protection, anti-poaching, wetland conservation.

1987: Delisted federally; managed as "threatened due to similarity of appearance."

FWC lottery permits; two alligators per permit.



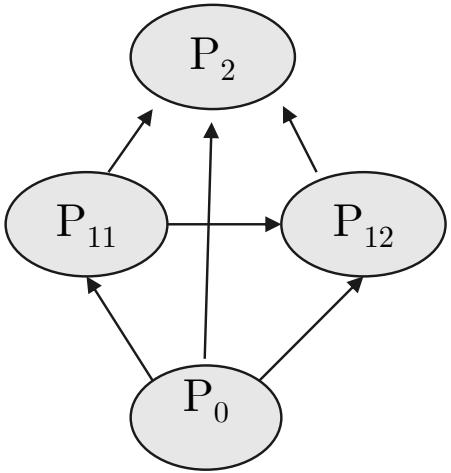
Burmese Pythons

- Intentional releases by owners who could no longer keep large snakes.
- Accidental escapes from pet stores, breeders, and private collections.
- Facility damage during Hurricane Andrew (1992) likely released additional snakes into the Everglades.

Ecological implications

- Sharp declines of midsized mammals in parts of the Everglades (raccoons, opossums, marsh rabbits)
- Pressure on threatened species that nest or forage on the ground
- Competition with native apex predators for prey, altering food webs

Model Illustrated:



P_0 - Racoon/prey population

P_{11} - Juvenile Alligator population

P_{12} - Adult Alligator population

P_2 - Burmese Python population

Initial Model

$$\begin{cases} \frac{dP_0}{dt} &= f_1(P_0) - m_{20} f_2(P_0) P_{12} - m_{30} f_3(P_0) P_2 \\ \frac{dP_{11}}{dt} &= c_{20} f_2(P_0) P_{12} - m_{31} f_4(P_{11}) P_3 - d_1 P_{11} \\ \frac{dP_{12}}{dt} &= d_1 P_{11} - d_2 P_{12} - H_2(P_{12}) \\ \frac{dP_2}{dt} &= c_{30} f_3(P_0) P_2 + c_{31} f_4(P_{11}) P_2 - d_3 P_2 - H_3(P_2) \end{cases}$$

$f_m(P_n) = P_n$

$H_2(P_2) = h_2$

$H_3(P_3) = h_3 P_3$

$$\begin{cases} \frac{dP_0}{dt} &= r P_0 - m_{20} P_0 P_{12} - m_{30} P_0 P_2 \\ \frac{dP_{11}}{dt} &= c_{20} P_0 P_{12} - m_{31} P_{11} P_3 - d_{11} P_{11} \\ \frac{dP_{12}}{dt} &= d_{12} P_{11} - d_2 P_{12} - h_2 \\ \frac{dP_2}{dt} &= c_{30} P_0 P_2 + c_{31} P_2 P_{11} - d_3 P_2 - h_3 P_2 \end{cases}$$

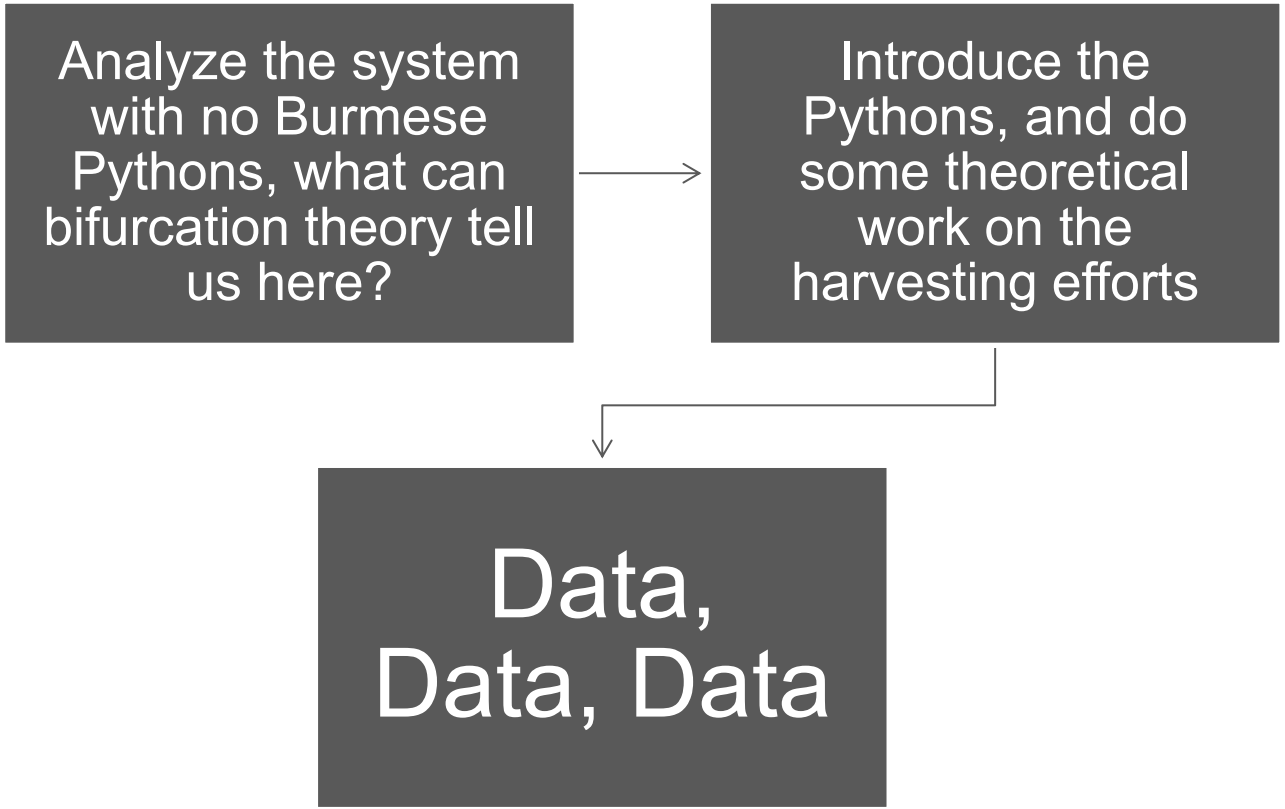
$h_2 \Rightarrow$ *Saturated*

$$\begin{cases} \frac{dP_0}{dt} = r P_0 \left(1 - \frac{P_0}{K}\right) - m_{10} P_0 P_1 - m_{20} P_0 P_2, \\ \frac{dP_1}{dt} = c_{10} P_0 P_1 - d_1 P_1 - \frac{h_1 P_1}{a + P_1} - m_{31} P_1 P_2, \\ \frac{dP_2}{dt} = c_{20} P_0 P_2 + c_{21} P_1 P_2 - d_2 P_2 - h_2 P_2. \end{cases}$$

Equilibria: 7 Equilibrium points

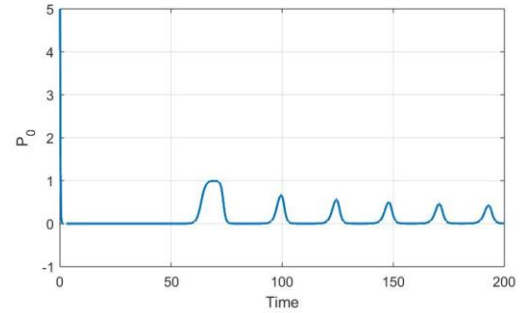
- 2 Equilibria on the P_0 axis.
- 2 equilibria on the $P_0 - P_1$ plane suggesting that the prey can coexist with the alligators.
- 2 non-zero equilibria where all 3 species coexist.
- 1 equilibrium in the $P_0 - P_2$ plane suggesting that python can coexist with the general prey.

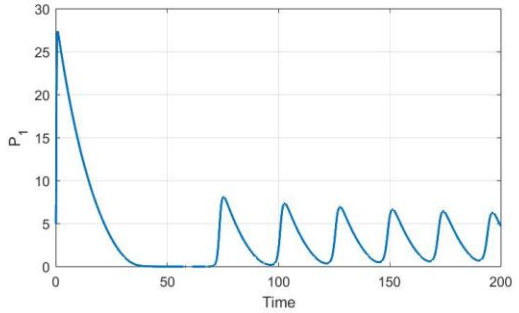
Ultimate goal: Policy making

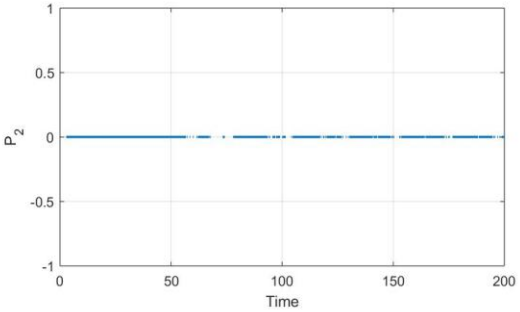


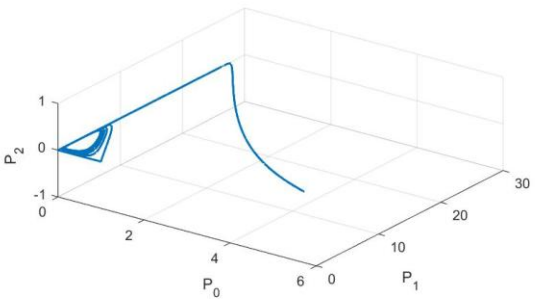
The case with no Burmese Pythons

- If no Burmese Pythons are present our 3d model is reduced to a 2d system: The system hereby would exhibit 2 non-zero equilibria.









Bogdanov Takens Bifurcation:

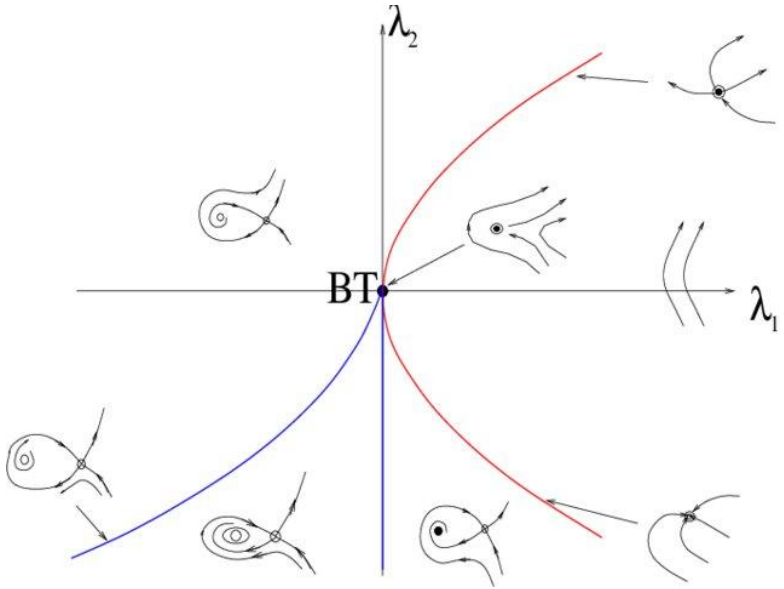
$$\dot{x} = f(x, \mu), \quad x \in \mathbb{R}^2, \ \mu = (\mu_1, \mu_2)$$

A Bogdanov Takens Bifurcation occurs when the system has an equilibrium point x∗ such that the Jacobian $Df(x^*, \mu^*)$ has a double zero eigenvalue:

$$\mathrm{tr}(Df) = 0, \quad \det(Df) = 0, \quad Df \neq 0,$$

And $Df(x^*, \mu^*)$ has a single Jordan block (geometric multiplicity = 1). The system can then be transformed to the normal form:

$$\begin{cases} \dot{u} = v, \\ \dot{v} = \beta_1 + \beta_2 u + u^2 + s\, uv. \end{cases}$$



Steps followed:

1. **Shift the equilibrium:** Find the equilibrium $P^* = (P_0^*, P_1^*, P_2^*)$ and shift variables:

$$u = P_0 - P_0^*, \quad v = P_1 - P_1^*, \quad w = P_2 - P_2^*$$

so that the system becomes $\dot{z} = F(z)$ with $F(0) = 0$.

2. **Linearize and identify eigen-directions:** Compute the Jacobian $J = DF(0)$. At the BT point: $\text{spec}(J) = \{0, 0, \lambda_s\}$ with $\lambda_s < 0$. The eigenvectors corresponding to zero eigenvalues define the center subspace.

3. **Center manifold reduction (3D → 2D):** There exists a smooth invariant manifold $w = h(u, v)$ with $h(0) = 0$, $Dh(0) = 0$. Substituting $w = h(u, v)$ reduces the system to:

$$\dot{u} = \tilde{F}_1(u, v), \quad \dot{v} = \tilde{F}_2(u, v)$$

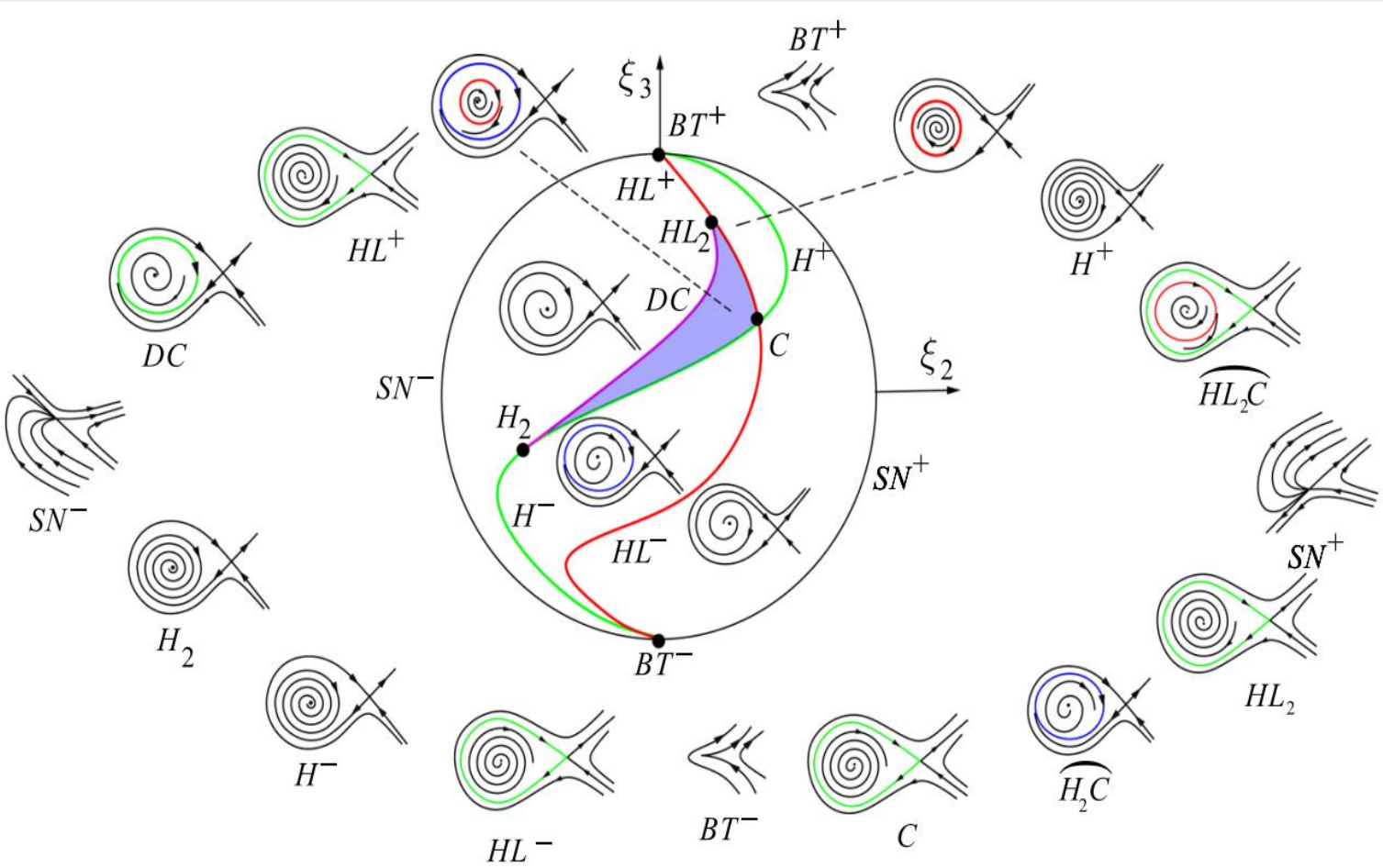
4. **Reduced dynamics:** Expand $\tilde{F}_1, \tilde{F}_2$ in a Taylor series near the origin and retain quadratic terms:

$$\dot{u} = v + \mathcal{O}(2), \quad \dot{v} = au^2 + buv + \cdots$$

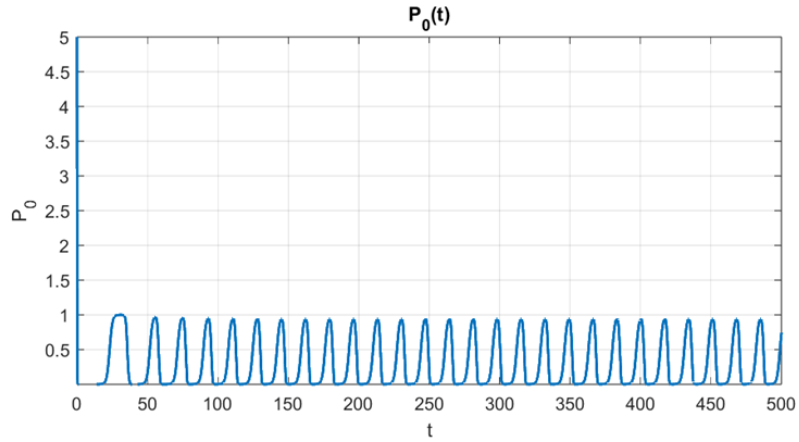
5. **Transform to BT normal form:** Apply smooth coordinate and parameter changes to obtain:

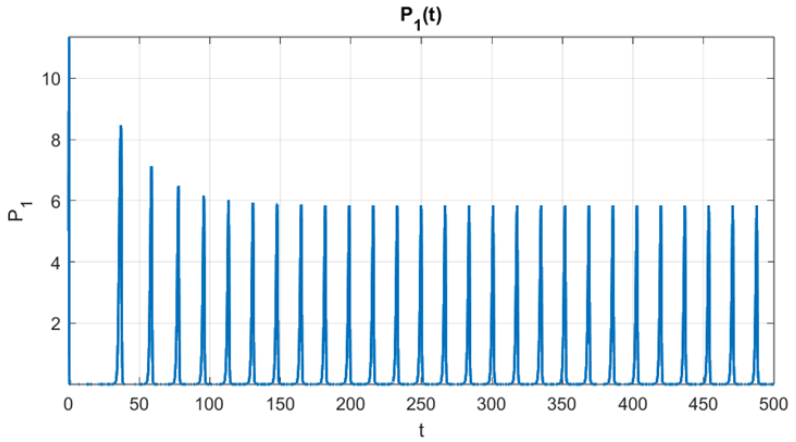
$$\boxed{\dot{x} = y, \quad \dot{y} = \beta_1 + \beta_2 x + ax^2 + bxy + \mathcal{O}(3)}$$

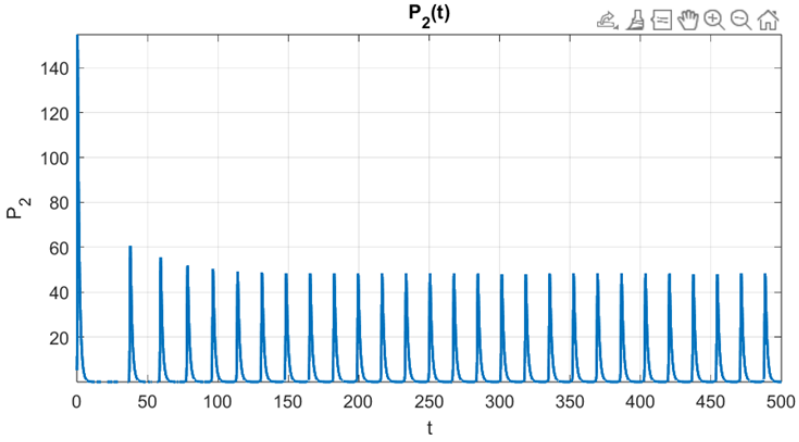
High Codimension Bogdanov Takens Bifurcation:

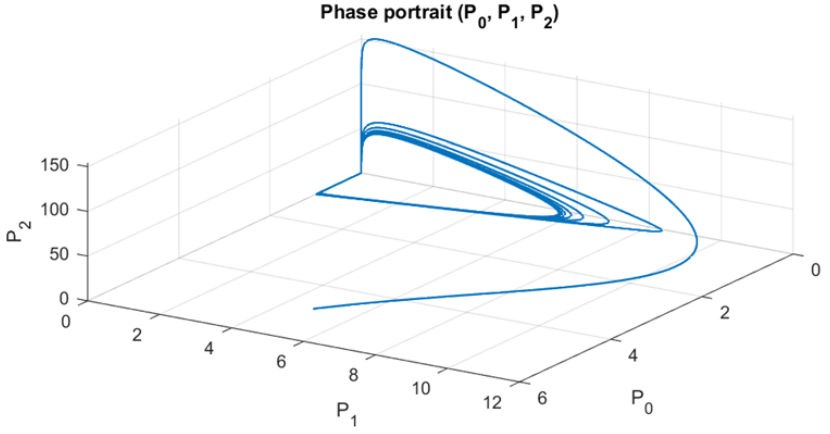


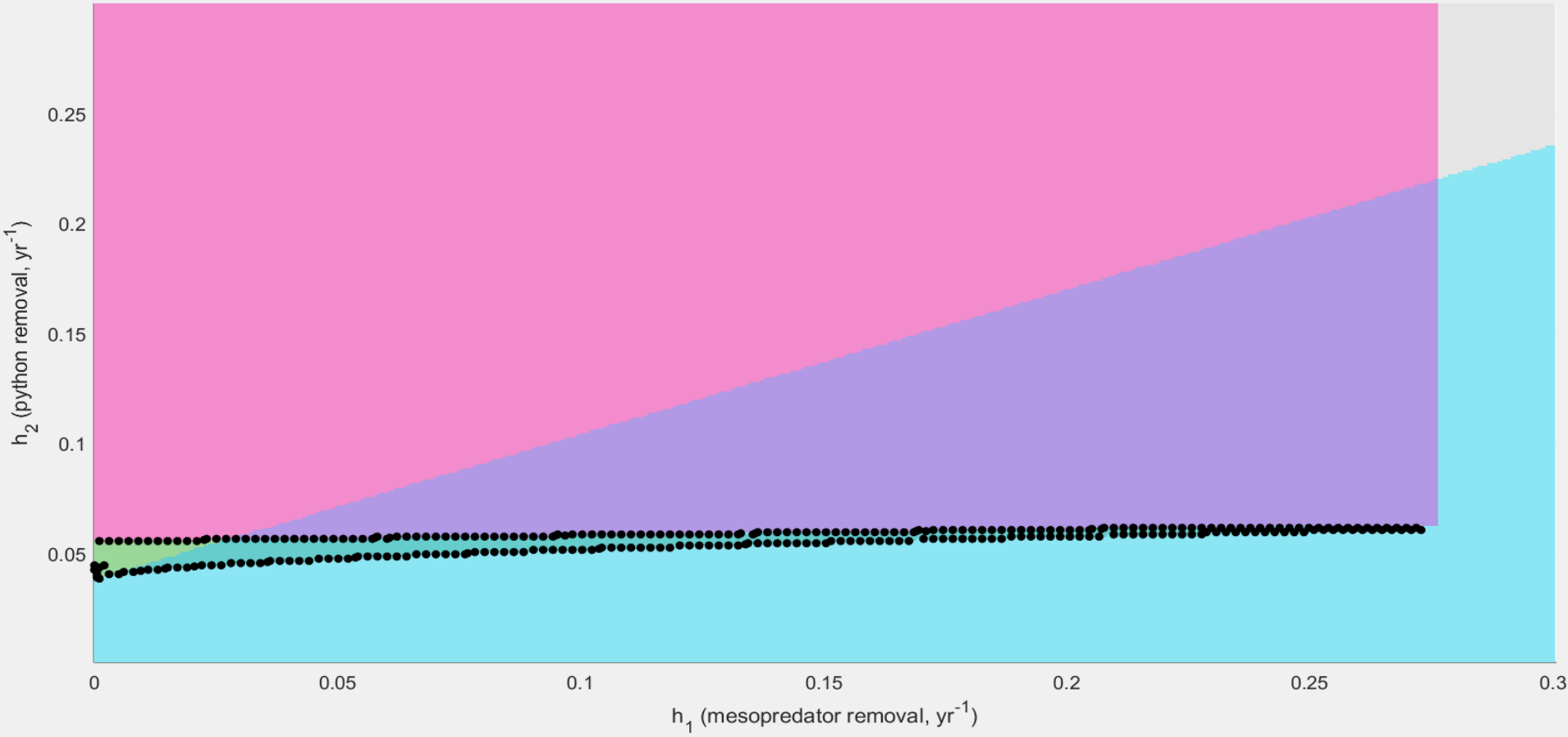
Case 2: Burmese Pythons are present










0.25
0.2
0.15
0.1
0.05
h2 (python removal, yr-1)
0
0.05
0.1
0.15
0.2
0.25
0.3
h1 (mesopredator removal, yr-1)
Both Alligators and Pythons go extinct
Alligators and Prey can only coexist
Pythons and Prey can only coexist
Pythons and Alligators can coexist with Prey, but not with one another
Pythons and Alligators can only coexist at the same time
Alligators and Prey can coexist without Pythons, and all three can coexist simultaneously
Pythons and Prey can coexist without Alligators, and all three can coexist simultaneously
Pythons and Prey can coexist, Alligators and prey can coexist, all three can coexist simultaneously
coexistence stability boundary

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Thank you!

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