Eigenvalue Statistics for Toeplitz and Circulant Ensembles

Murat Koloğlu[1], Gene Kopp[2], Steven J. Miller[1], and Karen Shen[3]

[1]Williams College [2]University of Michigan [3]Stanford University

http://www.williams.edu/Mathematics/sjmiller/

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Goals

- See how the structure of the ensembles affects limiting behavior.
- Discuss the tools and techniques needed to prove the results.

Real Symmetric Toeplitz Matrices

Chris Hammond and Steven J. Miller

Toeplitz Ensembles

Toeplitz matrix is of the form

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_{-1} & b_0 & b_1 & \cdots & b_{N-2} \\ b_{-2} & b_{-1} & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1-N} & b_{2-N} & b_{3-N} & \cdots & b_0 \end{pmatrix}$$

- Will consider Real Symmetric Toeplitz matrices.
- Main diagonal zero, $N-1$ independent parameters.
- Normalize Eigenvalues by $\sqrt{N}$.

Eigenvalue Density Measure

$$\mu_{A,N}(x)dx \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta\left(x-\frac{\lambda_i(A)}{\sqrt{N}}\right)dx.$$

The k^{th} moment of $\mu_{A,N}(x)$ is

$$M_k(A,N) \;=\; \frac{1}{N^{\frac{k}{2}+1}}\sum_{i=1}^{N}\lambda_i^k(A) \;=\; \frac{\mathrm{Trace}(A^k)}{N^{\frac{k}{2}+1}}.$$

Let

$$M_k \;=\; \lim_{N\to\infty}\mathbb{E}_A\left[M_k(A,N)\right];$$

have $M_2 = 1$ and $M_{2k+1} = 0$.

Even Moments

$$M_{2k}(N) = \frac{1}{N^{k+1}} \sum_{1 \le i_1, \cdots, i_{2k} \le N} \mathbb{E}(b_{|i_1-i_2|} b_{|i_2-i_3|} \cdots b_{|i_{2k}-i_1|}).$$

Main Term: b_j's matched in pairs, say

$$b_{|i_m - i_{m+1}|} = b_{|i_n - i_{n+1}|}, \quad x_m = |i_m - i_{m+1}| = |i_n - i_{n+1}|.$$

Two possibilities:

$$i_m - i_{m+1} = i_n - i_{n+1} \quad \text{or} \quad i_m - i_{m+1} = -(i_n - i_{n+1}).$$

$(2k-1)!!$ ways to pair, 2^k choices of sign.

Main Term: All Signs Negative (else lower order contribution)

$$M_{2k}(N) = \frac{1}{N^{k+1}} \sum_{1 \le i_1, \cdots, i_{2k} \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} \cdots b_{|i_{2k} - i_1|}).$$

Let $x_1, \dots, x_k$ be the values of the $|i_j - i_{j+1}|$'s, $\epsilon_1, \dots, \epsilon_k$ the choices of sign. Define $\widetilde{x}_1 = i_1 - i_2$, $\widetilde{x}_2 = i_2 - i_3, \dots$.

$$\begin{aligned} i_2 &= i_1 - \widetilde{x}_1 \\ i_3 &= i_1 - \widetilde{x}_1 - \widetilde{x}_2 \\ &\vdots \\ i_1 &= i_1 - \widetilde{x}_1 - \cdots - \widetilde{x}_{2k} \\ \widetilde{x}_1 + \cdots + \widetilde{x}_{2k} &= \sum_{j=1}^{k} (1 + \epsilon_j)\eta_j x_j \ = \ 0, \quad \eta_j \ = \ \pm 1. \end{aligned}$$

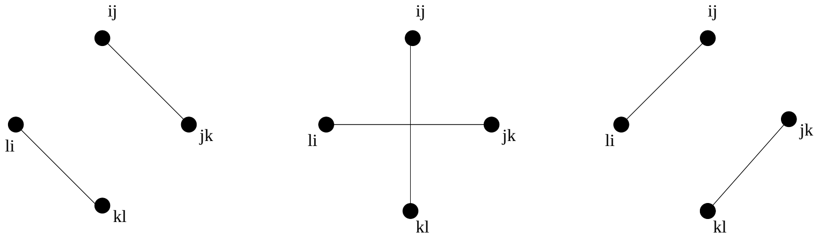
Even Moments: Summary

Main Term: paired, all signs negative.

$$M_{2k}(N) \ \le \ (2k-1)!! + O_k\left(\frac{1}{N}\right).$$

Bounded by Gaussian.

The Fourth Moment



$$M_4(N) = \frac{1}{N^3} \sum_{1 \le i_1, i_2, i_3, i_4 \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} b_{|i_3 - i_4|} b_{|i_4 - i_1|})$$

Let $x_j = |i_j - i_{j+1}|$.

The Fourth Moment

Case One: $x_1 = x_2, x_3 = x_4$:

$$i_1 - i_2 \; = \; -(i_2 - i_3) \quad \text{and} \quad i_3 - i_4 \; = \; -(i_4 - i_1).$$

Implies

$$i_1 \; = \; i_3, \quad i_2 \text{ and } i_4 \text{ arbitrary.}$$

Left with $\mathbb{E}[b_{x_1}^2 b_{x_3}^2]$:

$$N^3 - N \text{ times get } 1, \; N \text{ times get } p_4 \; = \; \mathbb{E}[b_{x_1}^4].$$

Contributes 1 in the limit.

The Fourth Moment

$$M_4(N) = \frac{1}{N^3} \sum_{1 \le i_1, i_2, i_3, i_4 \le N} \mathbb{E}(b_{|i_1-i_2|} b_{|i_2-i_3|} b_{|i_3-i_4|} b_{|i_4-i_1|})$$

Case Two: Diophantine Obstruction: $x_1 = x_3$ and $x_2 = x_4$.

$$i_1 - i_2 \;=\; -(i_3 - i_4) \quad \text{and} \quad i_2 - i_3 \;=\; -(i_4 - i_1).$$

This yields

$$i_1 \;=\; i_2 + i_4 - i_3, \quad i_1, i_2, i_3, i_4 \in \{1, \dots, N\}.$$

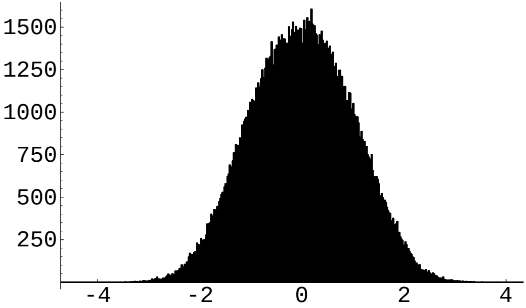
If $i_2, i_4 \ge \frac{2N}{3}$ and $i_3 < \frac{N}{3}$, $i_1 > N$: at most $(1 - \frac{1}{27})N^3$ valid choices.

The Fourth Moment

Theorem: Fourth Moment: Let p_4 be the fourth moment of p. Then

$$M_4(N) \;=\; 2\frac{2}{3} + O_{p_4}\left(\frac{1}{N}\right).$$

500 Toeplitz Matrices, 400×400.

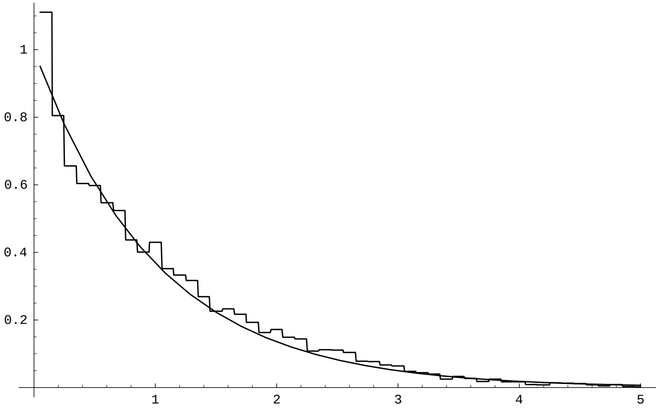


Main Result

Theorem: HM '05

For real symmetric Toeplitz matrices, the limiting spectral measure converges in probability to a unique measure of unbounded support which is not the Gaussian. If p is even have strong convergence).

Poissonian Behavior?



Not rescaled. Looking at middle 11 spacings, 1000 Toeplitz matrices (1000×1000), entries iidrv from the standard normal.

Real Symmetric Palindromic Toeplitz Matrices

Adam Massey, Steven J. Miller, Jon Sinsheimer

Real Symmetric Palindromic Toeplitz matrices

$$\begin{pmatrix} b_0 & b_1 & b_2 & b_3 & \cdots & b_3 & b_2 & b_1 & b_0 \\ b_1 & b_0 & b_1 & b_2 & \cdots & b_4 & b_3 & b_2 & b_1 \\ b_2 & b_1 & b_0 & b_1 & \cdots & b_5 & b_4 & b_3 & b_2 \\ b_3 & b_2 & b_1 & b_0 & \cdots & b_6 & b_5 & b_4 & b_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ b_3 & b_4 & b_5 & b_6 & \cdots & b_0 & b_1 & b_2 & b_3 \\ b_2 & b_3 & b_4 & b_5 & \cdots & b_1 & b_0 & b_1 & b_2 \\ b_1 & b_2 & b_3 & b_4 & \cdots & b_2 & b_1 & b_0 & b_1 \\ b_0 & b_1 & b_2 & b_3 & \cdots & b_3 & b_2 & b_1 & b_0 \end{pmatrix}$$

- Extra symmetry fixes Diophantine Obstructions.
- Always have eigenvalue at 0.

Results

Theorem: MMS '07

For real symmetric palindromic matrices, converge in probability to the Gaussian (if p is even have strong convergence).

Results

Theorem: MMS '07

Let $X_0, \dots, X_{N-1}$ be iidrv (with $X_j = X_{N-j}$) from a distribution p with mean 0, variance 1, and finite higher moments. For $\omega = (x_0, x_1, \dots)$ set $X_\ell(\omega) = x_\ell$, and

$$S_N^{(k)}(\omega) \;=\; \frac{1}{\sqrt{N}} \sum_{\ell=0}^{N-1} X_\ell(\omega) \cos(2\pi k \ell / N).$$

Then as $n \to \infty$

$$\mathrm{Prob}\left(\left\{\omega \in \Omega : \sup_{x \in \mathbb{R}} \left| \frac{1}{N} \sum_{k=0}^{N-1} I_{S_N^{(k)}(\omega) \le x} - \Phi(x) \right| \to 0\right\}\right) \;=\; 1;$$

I the indicator fn, Φ CDF of standard normal.

Real Symmetric Highly Palindromic Toeplitz Matrices

Steven Jackson, Victor Luo, Steven J. Miller, Vincent Pham, Nicholas George Triantafillou

Notation: Real Symmetric Highly Palindromic Toeplitz matrices

For fixed n, we consider $N \times N$ real symmetric Toeplitz matrices in which the first row is 2^n copies of a palindrome, entries are iidrv from a p with mean 0, variance 1 and finite higher moments.

For instance, a doubly palindromic Toeplitz matrix is of the form:

$$A_N = \begin{pmatrix} b_0 & b_1 & \cdots & b_1 & b_0 & b_0 & b_1 & \cdots & b_1 & b_0 \\ b_1 & b_0 & \cdots & b_2 & b_1 & b_0 & b_0 & \cdots & b_2 & b_1 \\ b_2 & b_1 & \cdots & b_3 & b_2 & b_1 & b_0 & \cdots & b_3 & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_2 & b_3 & \cdots & b_0 & b_1 & b_2 & b_3 & \cdots & b_1 & b_2 \\ b_1 & b_2 & \cdots & b_0 & b_0 & b_1 & b_2 & \cdots & b_0 & b_1 \\ b_0 & b_1 & \cdots & b_1 & b_0 & b_0 & b_1 & \cdots & b_1 & b_0 \end{pmatrix}.$$

Main Results

Theorem: JMP '12

Let n be a fixed positive integer, N a multiple of 2^n, consider the ensemble of real symmetric $N \times N$ palindromic Toeplitz matrices whose first row is 2^n copies of a fixed palindrome (independent entries iidrv from p with mean 0, variance 1 and finite higher moments).

1. As $N \to \infty$ the measures μ_{n,A_N} converge in probability to a limiting spectral measure which is even and has unbounded support.
2. If p is even, then converges almost surely.
3. The limiting measure has fatter tails than the Gaussian (or any previously seen distribution).

Work in Progress (with Victor Luo and Nicholas Triantafillou)

- Highly Palindromic Real Symmetric: all matchings contribute equally for fourth moment, conjectured equally in general.

- Highly Palindromic Hermitian: matchings do not contribute equally: fourth moment non-adjacent case is $\frac{1}{3}(2^n + 2^{-n})$, while the adjacent case is $\frac{1}{2}(2^n + 2^{-n})$.

Block Circulant Matrices

Murat Koloğlu, Gene Kopp and Steven J. Miller

The Ensemble of m-Block Circulant Matrices

Study symmetric matrices periodic with period m on wrapped diagonals, i.e., symmetric block circulant matrices.

8-by-8 real symmetric 2-block circulant matrix:

$$\left(\begin{array}{cc|cc|cc|cc} c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\ c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\ \hline c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\ c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\ \hline c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\ d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\ \hline c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\ d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0 \end{array}\right).$$

Choose distinct entries i.i.d.r.v.

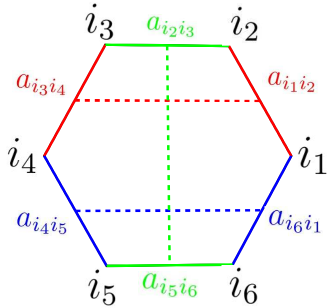
Oriented Matchings and Dualization

Compute moments of eigenvalue distribution (as m stays fixed and $N \to \infty$) using the combinatorics of pairings. Rewrite:

$$\begin{aligned} M_n(N) &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{1 \le i_1, \dots, i_n \le N} \mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}) \\ &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)}. \end{aligned}$$

where the sum is over oriented matchings on the edges $\{(1,2),(2,3),\dots,(n,1)\}$ of a regular n-gon.

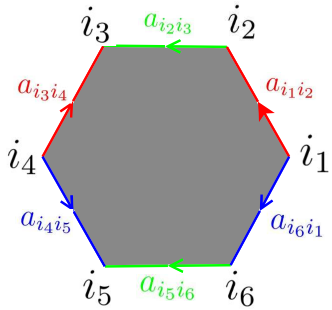
Oriented Matchings and Dualization



$$\left(\begin{array}{cc|cc|cc|cc}
c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\
c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\
\hline
c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\
c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\
\hline
c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\
d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\
\hline
c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0
\end{array}\right)$$

Figure: A matching in the expansion for $M_n(N) = M_6(8)$.

Oriented Matchings and Dualization

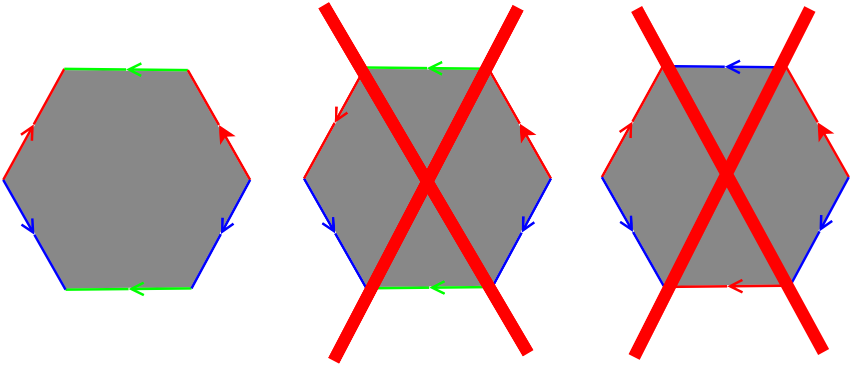


$$
\left(\begin{array}{cc|cc|cc|cc}
c_0 & \color{red}{c_1} & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\
c_1 & d_0 & d_1 & \color{green}{d_2} & d_3 & d_4 & c_3 & d_2 \\
\hline
c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & \color{blue}{d_3} \\
c_3 & d_2 & \color{red}{c_1} & d_0 & d_1 & d_2 & d_3 & d_4 \\
\hline
c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\
\color{blue}{d_3} & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\
\hline
c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
d_1 & d_2 & d_3 & d_4 & c_3 & \color{green}{d_2} & c_1 & d_0
\end{array}\right)
$$

Figure: An oriented matching in the expansion for $M_n(N) = M_6(8)$.

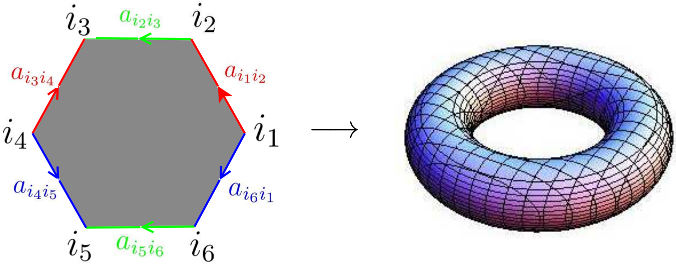
Contributing Terms

As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.



Only Topology Matters

Think of pairings as topological identifications; the contributing ones give rise to orientable surfaces.



Contribution from such a pairing is m^{-2g}, where g is the genus (number of holes) of the surface. Proof: combinatorial argument involving Euler characteristic.

Computing the Even Moments

Theorem: Even Moment Formula

$$M_{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k\left(\frac{1}{N}\right),$$

with $\varepsilon_g(k)$ the number of pairings of the edges of a $(2k)$-gon giving rise to a genus g surface.

J. Harer and D. Zagier (1986) gave generating functions for the $\varepsilon_g(k)$.

Harer and Zagier

$$\sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) r^{k+1-2g} = (2k-1)!!\, c(k,r)$$

where

$$1 + 2\sum_{k=0}^{\infty} c(k,r) x^{k+1} = \left(\frac{1+x}{1-x}\right)^r.$$

Thus, we write

$$M_{2k} = m^{-(k+1)} (2k-1)!!\, c(k,m).$$

A multiplicative convolution and Cauchy's residue formula yield the characteristic function of the distribution.

$$\phi(t) = \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k}}{(2k)!}$$

A multiplicative convolution and Cauchy's residue formula yield the characteristic function of the distribution.

$$\phi(t) \;=\; \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k}}{(2k)!} \;=\; \frac{1}{m} \sum_{k=0}^{\infty} \frac{(-t^2/2m)^k}{k!} c(k,m)$$

A multiplicative convolution and Cauchy's residue formula yield the characteristic function of the distribution.

$$\begin{aligned}\phi(t) &= \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k}}{(2k)!} = \frac{1}{m} \sum_{k=0}^{\infty} \frac{(-t^2/2m)^k}{k!} c(k,m) \\ &= \frac{1}{2\pi i m} \oint_{|z|=2} \frac{1}{2z^{-1}} \left(\left(\frac{1+z^{-1}}{1-z^{-1}} \right)^m - 1 \right) e^{-t^2 z/2m} \frac{dz}{z}\end{aligned}$$

A multiplicative convolution and Cauchy's residue formula yield the characteristic function of the distribution.

$$\begin{aligned}\phi(t) &= \sum_{k=0}^{\infty}\frac{(it)^{2k}M_{2k}}{(2k)!} = \frac{1}{m}\sum_{k=0}^{\infty}\frac{(-t^2/2m)^k}{k!}c(k,m)\\ &= \frac{1}{2\pi i m}\oint_{|z|=2}\frac{1}{2z^{-1}}\left(\left(\frac{1+z^{-1}}{1-z^{-1}}\right)^m-1\right)e^{-t^2z/2m}\frac{dz}{z}\\ &= \frac{1}{m}e^{\frac{-t^2}{2m}}\sum_{\ell=1}^{m}\binom{m}{\ell}\frac{1}{(\ell-1)!}\left(\frac{-t^2}{m}\right)^{\ell-1}\end{aligned}$$

Results

Fourier transform and algebra yields

Theorem: Koloğlu, Kopp and Miller

The limiting spectral density function $f_m(x)$ of the real symmetric m-block circulant ensemble is given by the formula

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$

As $m \to \infty$, the limiting spectral densities approach the semicircle distribution.

Results (continued)

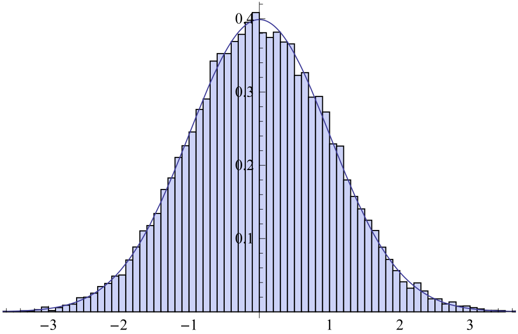


Figure: Plot for f_1 and histogram of eigenvalues of 100 circulant matrices of size 400×400.

Results (continued)

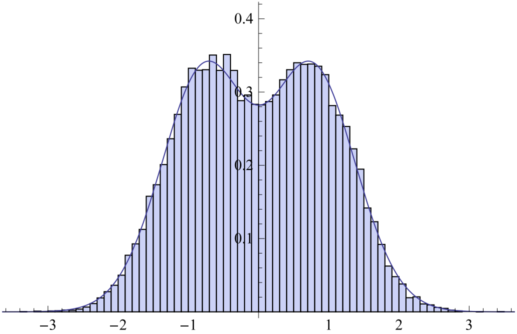


Figure: Plot for f_2 and histogram of eigenvalues of 100 2-block circulant matrices of size 400×400.

Results (continued)

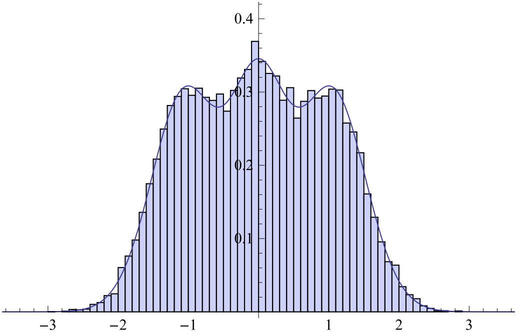


Figure: Plot for f_3 and histogram of eigenvalues of 100 3-block circulant matrices of size 402×402.

Results (continued)

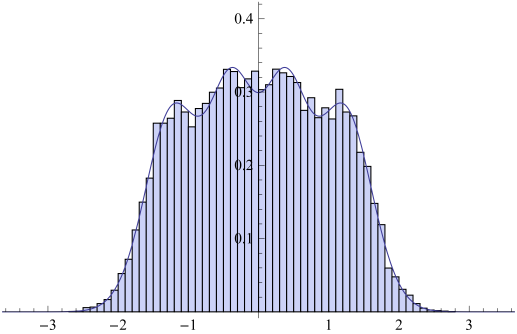


Figure: Plot for f_4 and histogram of eigenvalues of 100 4-block circulant matrices of size 400×400.

Results (continued)

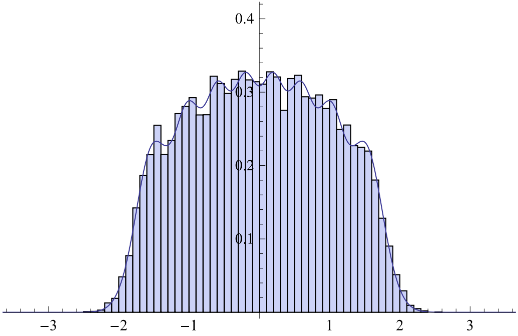


Figure: Plot for f_8 and histogram of eigenvalues of 100 8-block circulant matrices of size 400×400.

Results (continued)

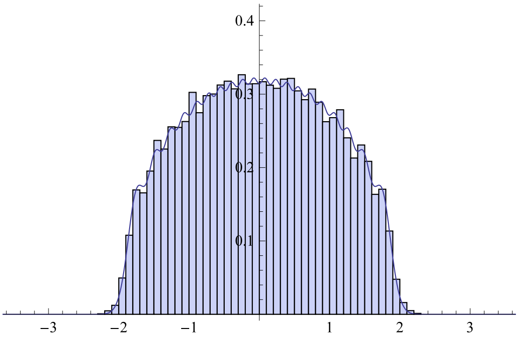


Figure: Plot for f_{20} and histogram of eigenvalues of 100 20-block circulant matrices of size 400×400.

Results (continued)

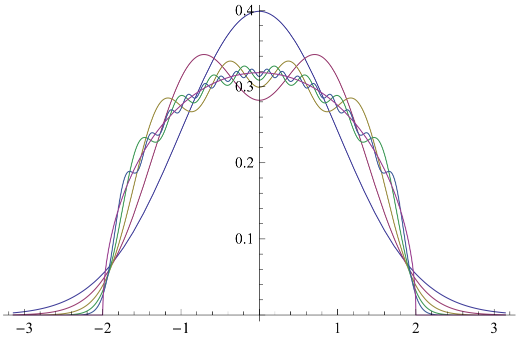


Figure: Plot of convergence to the semi-circle.

Weighted Real Symmetric Toeplitz Matrices

Olivia Beckwith, Steven J. Miller and Karen Shen

New Ensemble: Signed Toeplitz and Palindromic Toeplitz Matrices

For each entry, multiply by a randomly chosen $\epsilon_{ij} = \{1, -1\}$ with $p = \mathbb{P}(\epsilon_{ij} = 1)$ such that $\epsilon_{ij} = \epsilon_{ji}$.

New Ensemble: Signed Toeplitz and Palindromic Toeplitz Matrices

For each entry, multiply by a randomly chosen $\epsilon_{ij} = \{1, -1\}$ with $p = \mathbb{P}(\epsilon_{ij} = 1)$ such that $\epsilon_{ij} = \epsilon_{ji}$.

Varying p allows us to *continuously* interpolate between:

- Real Symmetric at $p = \frac{1}{2}$ (less structured)
- Unsigned Toeplitz/Palindromic Toeplitz at $p = 1$ (more structured)

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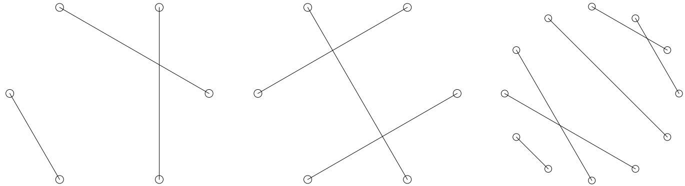
What is the eigenvalue distribution of these signed ensembles?

Weighted Contributions

Theorem:

Each configuration weighted by $(2p-1)^{2m}$, where $2m$ is the number of points on the circle whose edge crosses another edge.

Example:



2m = 4 2m=6 2m=8

Proof of Weighted Contributions Theorem

We compute the average k^{th} moment to be:

$$\frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \le i_1, \dots, i_k \le N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \cdots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

where the b's are matched in pairs.

Proof of Weighted Contributions Theorem

We compute the average k^{th} moment to be:

$$\frac{1}{N^{\frac{k}{2}+1}} \sum_{1 \leq i_1, \dots, i_k \leq N} \mathbb{E}\left(\epsilon_{i_1 i_2} b_{|i_1 - i_2|} \epsilon_{i_2 i_3} b_{|i_2 - i_3|} \dots \epsilon_{i_k i_1} b_{|i_k - i_1|}\right)$$

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Can show two ϵ's are matched if and only if their b's are not in a crossing.

Counting Crossing Configurations

Problem: Out of the $(2k-1)!!$ ways to pair $2k$ vertices, how many will have $2m$ vertices crossing ($Cross_{2k,2m}$)?

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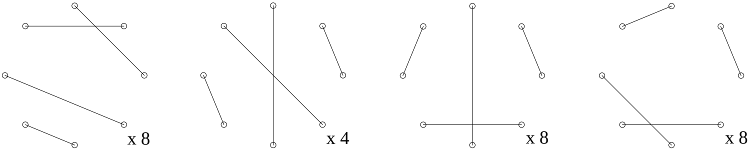

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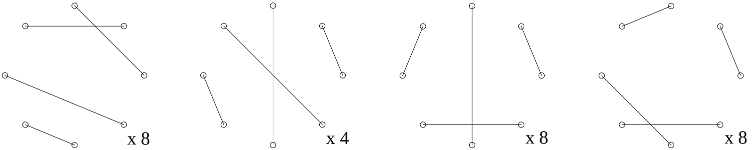

Fact:

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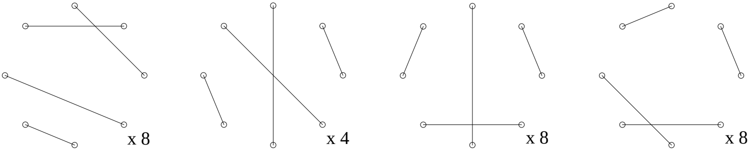

Fact:

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What about for higher m?

Counting Crossing Configurations

To calculate $Cross_{2k,2m}$, we write it as the following sum:

$$Cross_{2k,2m} = \sum_{p=1}^{\lfloor \frac{m}{4} \rfloor} P_{2k,2m,p}.$$

where $P_{2k,2m,p}$ is the number of configurations of $2k$ vertices with $2m$ vertices crossing in p partitions.

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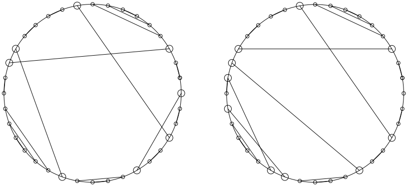
For example:



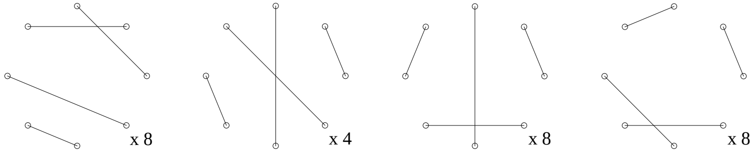
p = 1 p = 2

Non-Crossing Regions

Theorem:

If $2m$ vertices are already paired, the number of ways to pair and place the remaining $2k - 2m$ vertices as non-crossing non-partitioning edges is $\binom{2k}{k-m}$.

Example: $\binom{8}{2} = 28$ pairings with 4 crossing vertices.



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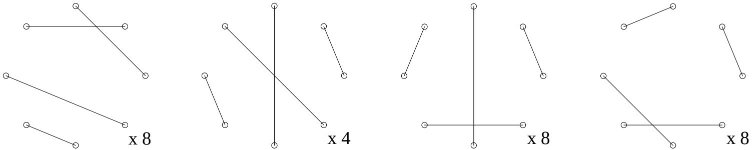


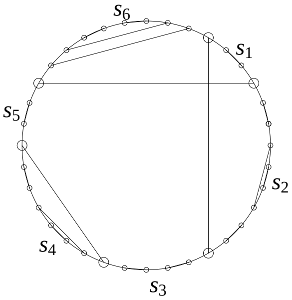
Lemma:

$P_{2k,2m,1} = Cross_{2m,2m}\binom{2k}{k-m}$.

Proof of Non-Crossing Regions Theorem

We showed the following equivalence:

$$\sum_{s_1+s_2+\cdots+s_{2m}=2k-2m} C_{s_1}C_{s_2}\cdots C_{s_{2m}} = \binom{2k}{k-m}.$$



Summary of Results

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  - A way to count the number of configurations with $2m$ vertices crossing for small m
- Tight bounds on the moments in the limit
  - The expected number of vertices involved in a crossing is

$$\frac{2k}{2k-1}\left(2k-2-\frac{{}_2F_1(1,\frac{3}{2},\frac{5}{2}-k;-1)}{2k-3}-(2k-1)\,{}_2F_1(1,\frac{1}{2}+k,\frac{3}{2};-1)\right),$$

    which is $2k-2-\frac{2}{k}+O\left(\frac{1}{k^2}\right)$ as $k \to \infty$.
  - The variance tends to 4 as $k \to \infty$.

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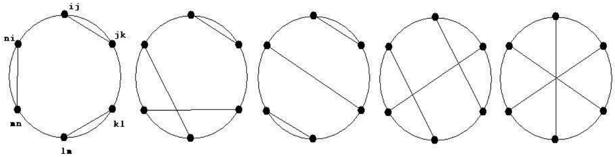
- M. Koloğlu, G. S. Kopp, S. J. Miller, F. W. Strauch and W. Xiong, *The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices*, to appear in the Journal of Theoretical Probability.

Appendices

Higher Toeplitz Moments: Brute Force

For sixth moment, five configurations occurring (respectively) $2, 6, 3, 3$ and 1 times.



$M_6(N) = 11$ (Gaussian's is 15).
$M_8(N) = 64\frac{4}{15}$ (Gaussian's is 105).

Lemma: For $2k \geq 4$, $\lim_{N\to\infty} M_{2k}(N) < (2k-1)!!$.

Higher Toeplitz Moments: Unbounded support

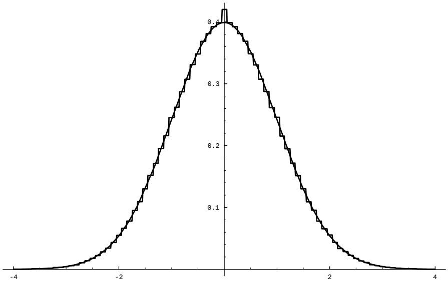
Lemma: Moments' growth implies unbounded support.

Proof: Main idea:

$$\begin{aligned} i_2 &= i_1 - \widetilde{x}_1 \\ i_3 &= i_1 - \widetilde{x}_1 - \widetilde{x}_2 \\ &\vdots \\ i_{2k} &= i_1 - \widetilde{x}_1 - \cdots - \widetilde{x}_{2k}. \end{aligned}$$

Once specify i_1 and $\widetilde{x}_1$ through $\widetilde{x}_{2k}$, all indices fixed.
If matched in pairs and each $i_j \in \{1, \dots, N\}$, have a valid configuration, contributes $+1$.
Problem: a running sum $i_1 - \widetilde{x}_1 - \cdots - \widetilde{x}_m \notin \{1, \dots, N\}$.
Lots of freedom in locating positive and negative signs, use CLT to show "most" configurations are valid.

Real Symmetric Palindromic Toeplitz



500 Real Symmetric Palindromic Toeplitz, 1000×1000.

Note the bump at the zeroth bin is due to the forced eigenvalues at 0.

Palindromic Toeplitz: Effects of Palindromicity on Matchings

$a_{i_m i_{m+1}}$ paired with $a_{i_n i_{n+1}}$ implies one of the following hold:

$$\begin{aligned} i_{m+1} - i_m &= \pm(i_{n+1} - i_n) \\ i_{m+1} - i_m &= \pm(i_{n+1} - i_n) + (N-1) \\ i_{m+1} - i_m &= \pm(i_{n+1} - i_n) - (N-1). \end{aligned}$$

Concisely: There is a $C \in \{0, \pm(N-1)\}$ such that

$$i_{m+1} - i_m = \pm(i_{n+1} - i_n) + C.$$

Palindromic Toeplitz: Fourth Moment

Highlights the effect of palindromicity.

Still matched in pairs, but more diagonals now lead to valid matchings.

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Non-adjacent case was $x_1 = x_3$ and $x_2 = x_4$:

$$i_1 - i_2 \ = \ -(i_3 - i_4) \quad \text{and} \quad i_2 - i_3 \ = \ -(i_4 - i_1).$$

This yields

$$i_1 \ = \ i_2 + i_4 - i_3, \quad i_1, i_2, i_3, i_4 \in \{1, \dots, N\}.$$

Palindromic Toeplitz: Fourth Moment

Highlights the effect of palindromicity.

Still matched in pairs, but more diagonals now lead to valid matchings.

Non-adjacent case now $x_1 = x_3$ and $x_2 = x_4$:

$$j - i \;=\; -(l - k) + C_1 \qquad\qquad k - j \;=\; -(i - l) + C_2,$$

or equivalently

$$j \;=\; i + k - l + C_1 \;=\; i + k - l - C_2.$$

We see that $C_1 = -C_2$, or $C_1 + C_2 = 0$.

Highly Palindromic: Key Lemmas

Much of analysis similar to previous ensembles (though combinatorics more involved).

For the fourth moment: both the adjacent and non-adjacent matchings contribute the same.

Lemma: As $N \to \infty$ the fourth moment tends to

$$M_{4,n} \ = \ 2^{n+1} + 2^{-n}.$$

Note: Number of palindromes is 2^n; thus smallest is $2^0 = 1$ (and do recover 3 for palindromic Toeplitz).

Highly Palindromic: Conjectures

Conjecture

In the limit, all matchings contribute equally.

Very hard to test; numerics hard to analyze.

To avoid simulating ever-larger matrices, noticed Diophantine analysis suggests average $2m^{\text{th}}$ moment of $N \times N$ matrices should satisfy

$$M_{2m,n;N} \;=\; M_{2m,n} + \frac{C_{1,n}}{N} + \frac{C_{2,n}}{N^2} + \cdots + \frac{C_{m,n}}{N^m}.$$

Instead of simulating prohibitively large matrices, simulate large numbers of several sizes of smaller matrices, do a least squares analysis to estimate $M_{2m,n}$.

Highly Palindromic: Conjectures

Table: Conjectured and observed moments for 1000 real symmetric doubly palindromic 2048×2048 Toeplitz matrices. The conjectured values come from assuming Conjecture.

Moment	Conjectured	Observed	Observed/Predicted
2	1.000	1.001	1.001
4	4.500	4.521	1.005
6	37.500	37.887	1.010
8	433.125	468.53	1.082
10	6260.63	107717.3	17.206

Highly Palindromic: Conjectures

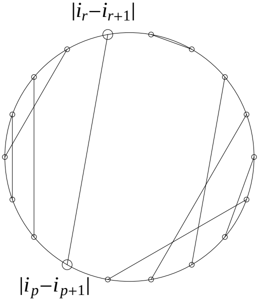
Table: Observed moments for doubly palindromic Toeplitz matrices. Conjectured values from assuming Conjecture.

N	#sims	2nd	4th	6th	8th	10th
8	1,000,000	1.000	8.583	150.246	3984.36	141270.00
12	1,000,000	1.000	7.178	110.847	2709.61	90816.60
16	1,000,000	1.001	6.529	93.311	2195.78	73780.00
20	1,000,000	1.001	6.090	80.892	1790.39	57062.50
24	1,000,000	1.000	5.818	73.741	1577.42	49221.50
28	1,000,000	1.000	5.621	68.040	1396.50	42619.90
64	250,000	1.001	4.992	50.719	858.58	22012.90
68	250,000	1.000	4.955	49.813	831.66	20949.60
72	250,000	1.000	4.933	49.168	811.50	20221.20
76	250,000	1.000	4.903	48.474	794.10	19924.10
80	250,000	1.000	4.888	47.951	773.31	18817.00
84	250,000	1.001	4.876	47.615	764.84	18548.00
128	125,000	1.000	4.745	44.155	659.00	14570.60
132	125,000	1.000	4.739	43.901	651.18	14325.30
136	125,000	0.999	4.718	43.456	637.70	13788.10
140	125,000	1.000	4.718	43.320	638.74	14440.40
144	125,000	1.001	4.727	43.674	647.05	14221.80
148	125,000	1.000	4.716	43.172	628.02	13648.10
Conjectured		1.000	4.500	37.500	433.125	6260.63
Best Fit $M_{2m,2}$		1.000	4.496	38.186	490.334	6120.94

Weighted Toeplitz: Proof of Weighted Contributions Theorem

A non-crossing pair of b's must have matched ϵs:

Assume $b_{|i_r - i_{r+1}|}$ and $b_{|i_p - i_{p+1}|}$ are a non-crossing pair.



$$\sum_{k=r}^{p} (i_k - i_{k+1}) = 0$$
$$= i_r - i_{r+1} + i_{r+1} \cdots + i_p - i_{p+1} = i_r - i_{p+1}$$

This implies that $i_r = i_{p+1}$.
Similarly, $i_{r+1} = i_p$

Thus, $\epsilon_{i_r i_{r+1}} = \epsilon_{i_p i_{p+1}}$.

Weighted Toeplitz: Proof of Weighted Contributions Theorem

A matched pair of ϵ's must not be in a crossing:

Suppose $\epsilon_{i_a i_{a+1}} = \epsilon_{i_b i_{b+1}}$, with $a < b$.

$$\sum_{k=a}^{b} (i_k - i_{k+1}) = i_a - i_{b+1} = 0$$

$$= \sum_{k=b}^{d} \delta_k \left| i_k - i_{k+1} \right|$$

where $\delta_k = 0$ if and only if the vertex k is paired with is between a and b.

Need N^{k+1} degrees of freedom, so $\delta_k = 0$ for all k. Thus, $\epsilon_{i_a i_{a+1}}$ and $\epsilon_{i_b i_{b+1}}$ are not in a crossing.