Winning Strategy of Two-Player, Multiplayer and Multialliance Generalized Zeckendorf Game

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Fibonacci Sequence

According to "The Zeckendorf Game" paper [1], the Fibonacci Sequence in this presentation is defined as: $F_1 = 1, F_2 = 2, F_i = F_{i-1} + F_{i-2}$ for any $i \geq 3$.

Zeckendorf's Theorem

Every positive integer can be uniquely written as a sum of one or more non-consecutive Fibonacci numbers.

Example: $2021 = 1597 + 377 + 34 + 13 = F_{16} + F_{13} + F_8 + F_6$.

The Fibonacci Zeckendorf Game

This game is introduced in "The Zeckendorf Game" paper.[1]

Game Rules: At the beginning of the game, there is an unordered list of n 1's. The initial list is $\{F_1^n\}$.

On each turn, a player can do one of the following moves:

1. $F_{i-1} \wedge F_i \to F_{i+1}$ (Combining Move)
2. If the list has two of the same Fibonacci number, $F_i \wedge F_i$ then
   a. if $i = 1$, $F_1 \wedge F_1 \to F_2$ (Combining Move)
   b. if $i = 2$, $F_2 \wedge F_2 \to F_1 \wedge F_3$ (Splitting Move)
   c. if $i \geq 3$, $F_i \wedge F_i \to F_{i-2} \wedge F_{i+1}$ (Splitting Move)

The game always terminates at the Zeckendorf decomposition (no more moves are possible). The last player to move wins.

Game Ends in Finitely Many moves

Theorem (Baird-Smith, P., Epstein, A., Flint, K., & Miller, S. J. (2018, May). *"The Zeckendorf Game"*.[1])

All games end in finitely many moves.

Proof: The sum of the square roots of the indices is a strict monovariant.

- Adding consecutive terms: $\left(\sqrt{k}+\sqrt{k}\right)-\sqrt{k+2}<0.$
- Splitting: $2\sqrt{k}-\left(\sqrt{k+1}+\sqrt{k+1}\right)<0.$
- Adding 1's: $2\sqrt{1}-\sqrt{2}<0.$
- Splitting 2's: $2\sqrt{2}-\left(\sqrt{3}+\sqrt{1}\right)<0.$

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Tribonacci Sequence

According to "The Generalized Zeckendorf Game" paper [2], the Tribonacci Sequence in this presentation is defined as:
$T_1 = 1, T_2 = 2, T_3 = 4, T_i = T_{i-1} + T_{i-2} + T_{i-3}$ for any $i \geq 4$

The Tribonacci Zeckendorf Game

This game is introduced in "The Generalized Zeckendorf Game" paper.[2]
Game Rules: At the beginning of the game, there is an unordered list of n 1's. The initial list is $\{T_1^n\}$.
On each turn, a player can do one of the following moves:

1. $R_{i-1} \wedge R_{i-2} \wedge R_{i-3} \to R_i$ for any $i \geq 4$
2. $R_1 \wedge R_1 \to R_2$
3. $R_1 \wedge R_1 \wedge R_2 \to R_3$
4. $R_2 \wedge R_2 \to R_3$
5. $R_3 \wedge R_3 \to R_1 + R_4$
6. $R_i \wedge R_i \to R_{i+1} + R_{i-3}$ for any $i \geq 4$

The game always terminates at a unique legal decomposition (no more moves are possible). The last player to move wins.

Sample Game

Start with 9 pieces at T_1, everything else is empty.

9	0	0	0	0
$[T_1 = 1]$	$[T_2 = 2]$	$[T_3 = 4]$	$[T_4 = 7]$	$[T_5 = 13]$

Sample Game

Turn 1:
Player 1: $T_1 + T_1 = T_2$

7	1	0	0	0
$[T_1 = 1]$	$[T_2 = 2]$	$[T_3 = 4]$	$[T_4 = 7]$	$[T_5 = 13]$

Sample Game

Turn 2:
Player 2: $T_1 + T_1 + T_2 = T_3$

5	0	1	0	0
$[T_1 = 1]$	$[T_2 = 2]$	$[T_3 = 4]$	$[T_4 = 7]$	$[T_5 = 13]$

Sample Game

Turn 3:
Player 1: $T_1 + T_1 = T_2$

3	1	1	0	0
$[T_1 = 1]$	$[T_2 = 2]$	$[T_3 = 4]$	$[T_4 = 7]$	$[T_5 = 13]$

Sample Game

Turn 4:

Player 2: $T_1 + T_2 + T_3 = T_4$

2	0	0	1	0
$[T_1 = 1]$	$[T_2 = 2]$	$[T_3 = 4]$	$[T_4 = 7]$	$[T_5 = 13]$

Sample Game

Turn 5:
Player 1: $T_1 + T_1 = T_2$

0	1	0	1	0
$[T_1 = 1]$	$[T_2 = 2]$	$[T_3 = 4]$	$[T_4 = 7]$	$[T_5 = 13]$

No moves left, Player 1 wins in 5 moves.

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Generalized (c,k)-nacci Sequences

According to "The Generalized Zeckendorf Game" paper [2], (c, k)-nacci sequences in this presentation are defined as $S_1 = 1$; for any $i \leq k, S_{i+1} = cS_i + cS_{i-1} + \cdots cS_1 + 1$; for any $i \geq k+1$, $S_{i+1} = cS_i + cS_{i-1} + \cdots cS_{i-k}$.

The (c,k)-nacci Zeckendorf Game

This game is introduced in "The Generalized Zeckendorf Game" paper.[2]
Game Rules: At the beginning of the game, there is an unordered list of n 1's. The initial list is $\{S_1^n\}$.
On each turn, a player can do one of the following moves:

1. if $i \leq k, cS_i \wedge cS_{i-1} \wedge \cdots \wedge cS_2 \wedge (c+1)S_1 \to S_{i+1}$;
2. if $i \geq k+1,\ cS_i \wedge cS_{i-1} \wedge \cdots \wedge cS_{i-k} \to S_{i+1}$;
3. if $1 \geq i \geq k, (c+1)S_i \to S_{i+1}$;
4. if $i = k+1, (c+1)S_i \to S_{i+1} \wedge S_1$;
5. if $i \geq k+2, (c+1)S_i \to S_i \wedge cS_{i-k-1}$.

The game always terminates at a unique legal decomposition (no more moves are possible). The last player to move wins.

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Theorem (Baird-Smith, P., Epstein, A., Flint, K., & Miller, S. J. (2018, May). *"The Zeckendorf Game"*.[1])

For all $n > 2$, Player 2 has the winning strategy for the 2-player Fibonacci Zeckendorf Game.

- **Idea:** If we assume player 1 has the winning strategy, player 2 could steal player 1's winning strategy, which is a contradiction. Interestingly, the proof in non-constructive so we do not know what the winning strategy for player 2 is

Winning strategies

Previous Results 2

Theorem (E. Boldyriew, A. Cusenza, L. Dai, P. Ding, A. Dunkelberg, J. Haviland, K. Huffman, D. Ke, D. Kleber, J. Kuretski, J. Lentfer, T. Luo, C. Mizgerd, V. Tiwari, J. Ye, Y. Zhang, X. Zheng and Weiduo Zhu (2020). *"Generalizing Zeckendorf's Theorem to a Non-constant Recurrence"*.[3])

For all $n \geq 5$, in a Multi-player Fibonacci Game with $p \geq 3$, no player has a winning strategy.

- **Idea:** Use the stealing strategy to show that if any player had a winning strategy, another player could steal the winning strategy which is a contradiction.

Result 1:

For all $n \geq 7$, Player 2 has the winning strategy for 2-player Tribonacci Zeckendorf Game.

- **Idea:** Non-constructive proof by contradiction.

Result 2:

For all $n \geq 7$, $p \geq 3$ Multi-player Tribonacci Game, no player has a winning strategy.

- **Idea:** Suppose player m has the winning strategy $(1 \leq m \leq p)$. Then player m-1 can steal player m's winning strategy.
    1. For all $n \geq 7$, $p \geq 3$ Tribonacci games, any player m's winning path does not contain the following 3 consecutive steps. If it did, the player in step 2 can do $1+1+2=4$ instead and player $m-1$ can steal the winning strategy:
       Step $1: 1+1=2$ (Combine two 1s into one 2)
       Step $2: 1+1=2$ (Combine two 1s into one 2)
       Step $3: 2+2=4$ (Combine two 2s into one 4)
    2. Then we construct the other $m-1$ players' moves containing these 3 consecutive steps, which leads to a contradiction, so no player m can have a winning strategy.

Result 3:

In a generalized (c,k)-nacci Zeckendorf Game, for any $c \geq 1$, $k \geq 1$ and $n \geq (c+1)^3 + c + 1$, when c is odd, player 2 always has a winning strategy; when c is even, player 1 always has a winning strategy.

Idea:

1. Prove by contradiction.
2. It is more convenient to prove the equivalent statement for the (c-1,k)-nacci game.
3. Treat every c steps as a round.
4. Find a general pattern for the first c rounds.
5. After the first c rounds, find the contradiction from there.

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Future Directions

Future Direction

Zeckendorf Game:

1. Find constructive winning strategies in two-player Fibonacci, Tribonacci, and (c,k)-nacci games.
2. Find constructive winning strategies for alliances in multiplayer games.
3. Further tighten the lower bound for generalized (c,k)-nacci games

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References I

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3. E. Boldyriew, A. Cusenza, L. Dai, P. Ding, A. Dunkelberg, J. Haviland, K. Huffman, D. Ke, D. Kleber, J. Kuretski, J. Lentfer, T. Luo, C. Mizgerd, V. Tiwari, J. Ye, Y. Zhang, X. Zheng and W. Zhu (2020). Generalizing Zeckendorf's Theorem to a Non-constant Recurrence. The Fibonacci Quarterly, Volume 58, no.5, pp.55-76.
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References II

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