Extending Agreement in the Katz-Sarnak Density Conjecture

Peter Cohen and Carsten Sprunger

Young Mathematician's Conference
The Ohio State University, August 20^{th}, 2016

Introduction

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as between eigenvalues of Complex Hermitian matrices.

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f) L(s,f) = \epsilon_f \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (GRH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings between zeros appear same as b/w eigenvalues of Complex Hermitian matrices.

Gaussian Unitary Ensemble

- The GUE: complex Hermitian matrices

$$A = \begin{cases} X_{ij} \sim \mathcal{N}(0, 1/\sqrt{2}) + i\mathcal{N}(0, 1/\sqrt{2}) & \text{if } i \neq j \\ X_{ij} \sim \mathcal{N}(0, 1) & \text{if } i = j \end{cases}$$

Zeros of $\zeta(s)$ and Pair Correlation

- Given zeros of $\zeta(s)$ of the form $\frac{1}{2} + i\gamma_n$ for $n \in \mathbb{N}$.

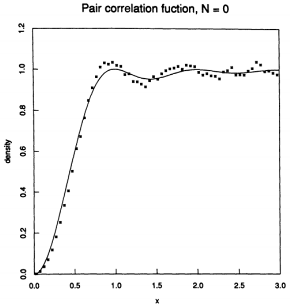
$$\delta_n = (\gamma_{n+1} - \gamma_n)\frac{\log \gamma_n}{4\pi^2}$$

- Montgomery's Pair Correlation Conjecture:

$$N^{-1}\left|\left\{(n,k) : 1 \leq n \leq N,\ k \geq 0,\ \sum_{i=n}^{n+k} \delta_i \in [\alpha, \beta]\right\}\right|$$
$$\sim \int_{\alpha}^{\beta} \left(1 - \left(\frac{sin \pi u}{\pi u}\right)^2\right)\, du$$

- Dyson noticed something extraordinary [2]

Zeros of $\zeta(s)$ vs GUE



Pair correlation of zeros of the zeta function vs. GUE prediction (solid line). Scatter plot is empirical data based on γ_n for $1 \leq n \leq 10^5$.[1]

Other statistics

- Pair correlation fails to discriminate between different families of L-functions and different classical compact groups.

Other statistics

- Pair correlation fails to discriminate between different families of L-functions and different classical compact groups.
- It is also insensitive to finitely many zeros.

Other statistics

- Pair correlation fails to discriminate between different families of L-functions and different classical compact groups.
- It is also insensitive to finitely many zeros.
- In order to discriminate and also preserve information about low-lying zeros, need to study different statistics.

Katz-Sarnak density conjecture

- The Katz-Sarnak density conjecture states that the scaling limits of the distributions of zeros of families of automorphic L-functions near the central point agree with the scaling limits of eigenvalue distributions near 1 of classical subgroups of the unitary groups $U(N)$.

Katz-Sarnak density conjecture

- The Katz-Sarnak density conjecture states that the scaling limits of the distributions of zeros of families of automorphic L-functions near the central point agree with the scaling limits of eigenvalue distributions near 1 of classical subgroups of the unitary groups $U(N)$.
- This conjecture is often tested by way of computing particular statistics, like the n-level density.

1-Level Density

- We want to study the behavior of zeros for L-functions near the point $s = \frac{1}{2}$.
- We define the **1-level density** for an L-function $L(s,f)$ and ϕ an even Schwartz function, where $\widehat{\phi}$ is compactly supported, by

$$D_f(\phi) = \sum_{\gamma_f} \phi\left(\gamma_f \frac{\log R}{2\pi}\right)$$

Random Matrix Theory Analogue

- For an even Schwartz function ϕ on $\mathbb{R}$ define

$$F_M(\theta) := \sum_{j=-\infty}^{\infty} \phi\left(\frac{M}{2\pi}(\theta + 2\pi j)\right).$$

- For U and $M \times M$ unitary matrix with eigenvalues $e^{i\theta_n}$ let

$$Z_\phi(U) := \sum_{n=1}^{M} F_M(\theta_n)$$

The Question

- What are the moments of $Z_\phi(U)$ for matrices from the classical compact groups?
- Katz and Sarnak: Compute for any test function, but there is a catch.

The Question

- Rather than moments, we study cumulants.

$$\mu_n' = \sum \left(\frac{C_2}{2!}\right)^{k_2} \cdots \left(\frac{C_n}{n!}\right)^{k_n} \frac{n!}{k_2! \cdots k_n!},$$

summing over k_j such that $\sum_{j=2}^{n} jk_j = n$.

Cumulants and the Classical Compact Groups

- For $\phi \in \mathcal{S}(\mathbb{R})$ with $\operatorname{supp}(\widehat{\phi}) \subseteq \left[-\frac{2}{n}, \frac{2}{n}\right]$ and $n \geq 3$, we have

$$C_n^U(\phi) = 0$$
$$C_n^{SO(even)}(\phi) = 2^n Q_n(\phi)$$
$$C_n^{SO(odd)}(\phi) = -2^n Q_n(\phi)$$

What is $Q_n(\phi)$

$$Q_n(\phi) = -\frac{1}{2}\sum_{m=1}^{n} \sum_{\substack{\lambda_1+\ldots+\lambda_m=n \\ \lambda_j\geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}$$

$$\int_{\mathbb{R}^m} \left(\prod_{j=1}^{m} \phi^{\lambda_j}(x_j)\right) \times S(x_1 - x_2)\cdots S(x_{m-1} - x_m)$$

$$\times S(x_m + x_1) dx_1 \cdots dx_m$$

where $S(x) = \frac{\sin(\pi x)}{\pi x}$.

What is $Q_n(\phi)$

$$Q_n(\phi) = \frac{1}{4}\int_0^\infty \cdots \int_0^\infty \widehat{\phi}(y_1)\cdots\widehat{\phi}(y_n) K(y_1,\dots,y_n)dy_1\cdots dy_n,$$

What is $Q_n(\phi)$

$$K(y_1,\ldots,y_n) = \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\ldots+\lambda_m=n \\ \lambda_j\geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1!\cdots\lambda_m!}$$

$$\sum_{\epsilon_1,\ldots,\epsilon_n=\pm 1} \prod_{\ell=1}^{m} \chi_{\left\{\left|\sum_{j=1}^{n} \eta(\ell,j)\epsilon_j y_j\right|\leq 1\right\}}$$

and

$$\eta(\ell,j) = \begin{cases} +1 & \text{if } j \leq \sum_{k=1}^{\ell} \lambda_k \\ -1 & \text{if } j > \sum_{k=1}^{\ell} \lambda_k. \end{cases}$$

Proceeding Combinatorially

- Simplifying integrals of products of indicator functions becomes combinatorial.

Proceeding Combinatorially

- Simplifying integrals of products of indicator functions becomes combinatorial.
- Can attack specific cases *ad hoc* using some form of inclusion-exclusion.

Proceeding Combinatorially

- Simplifying integrals of products of indicator functions becomes combinatorial.
- Can attack specific cases *ad hoc* using some form of inclusion-exclusion.
- For fixed $(y_1, \ldots, y_n)$ take sum of all terms of $K(y_1, \ldots, y_n)$, subtract those which have certain vanishing χ's in them, add those which have certain pairs of vanishing χ's in them, etc.

Technical Obstructions to Inclusion-Exclusion

- The number of terms in the inclusion-exclusion calculation is dependent on $\text{supp}(\widehat{\phi})$, and it grows too quickly to be manageable.

Technical Obstructions to Inclusion-Exclusion

- The number of terms in the inclusion-exclusion calculation is dependent on $\text{supp}(\widehat{\phi})$, and it grows too quickly to be manageable.
- Moreover, the indicator functions that come out become more and more complicated (hard to integrate).

Technical Obstructions to Inclusion-Exclusion

- The number of terms in the inclusion-exclusion calculation is dependent on $\mathrm{supp}(\widehat{\phi})$, and it grows too quickly to be manageable.
- Moreover, the indicator functions that come out become more and more complicated (hard to integrate).
- Need to be more organized and ditch inclusion-exclusion.

Developing combinatorial framework

- We developed a combinatorial framework for the problem which allows us to write $Q_n(\phi)$ as a linear combination of integrals over distinct classes of indicator functions.

Developing combinatorial framework

- We developed a combinatorial framework for the problem which allows us to write $Q_n(\phi)$ as a linear combination of integrals over distinct classes of indicator functions.
- In this framework, can show elegant cancellation of most terms.

Developing combinatorial framework

- We developed a combinatorial framework for the problem which allows us to write $Q_n(\phi)$ as a linear combination of integrals over distinct classes of indicator functions.
- In this framework, can show elegant cancellation of most terms.
- Remaining terms have simple indicator functions and can be simplified nicely.

Final expression

Theorem

For $\phi \in \mathcal{S}(\mathbb{R})$ even such that supp$(\widehat{\phi}) \subseteq \left[-\frac{1}{n-w}, \frac{1}{n-w}\right]$ with $w \leq n/2$, we have

$$Q_n(\phi) = \sum_{\ell=0}^{w-1} \frac{(-1)^{n+\ell+1}\binom{n}{\ell}}{2} \left(\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \widehat{\phi}(x_{\ell+1}) \cdots \widehat{\phi}(x_2) \right.$$

$$\left. \int_{-\infty}^{\infty} \phi^{n-\ell}(x_1) \frac{\sin(2\pi x_1(1 + |x_2| + \cdots + |x_{\ell+1}|))}{2\pi x_1} dx_1 \cdots dx_{\ell+1} - \frac{1}{2}\phi^n(0) \right)$$

Comparing with number theory

- Again, the point of all this simplification is to compare with L-functions.

Comparing with number theory

- Again, the point of all this simplification is to compare with L-functions.
- On the number theory side, we look at L-functions associated to cuspidal newforms, splitting by the sign of their functional equaitons.

Comparing with number theory

- Again, the point of all this simplification is to compare with L-functions.
- On the number theory side, we look at L-functions associated to cuspidal newforms, splitting by the sign of their functional equaitons.
- Can extend what is known there to test functions ϕ with $\operatorname{supp}(\widehat{\phi}) \subseteq \left[-\frac{1}{n-3}, \frac{1}{n-3}\right]$ and show agreement with random matrix theory using the previous theorem.

Big picture

- It is not understood why there is such a strong connection between random matrix theory and families of L-functions.

Big picture

- It is not understood why there is such a strong connection between random matrix theory and families of L-functions.
- Many connections are proven, some only strongly believed (we should prove them).

Big picture

- It is not understood why there is such a strong connection between random matrix theory and families of L-functions.
- Many connections are proven, some only strongly believed (we should prove them).
- Random matrix theory provides models for a wide range of statistical behavior of these families.

Big picture

- It is not understood why there is such a strong connection between random matrix theory and families of L-functions.
- Many connections are proven, some only strongly believed (we should prove them).
- Random matrix theory provides models for a wide range of statistical behavior of these families.
- Consequently, we can gain information about questions about L-functions that we couldn't before, and we can confidently predict the answer to new questions.

Acknowledgements

We thank Williams College and NSF Grants DMS1265673, DMS1561945, and DMS1347804 for support.
This work was joint work with Geoffrey Iyer, Nicholas Triantafillou, Nhi Truong, Roger Van Peski under the visage of Professor Steven J. Miller at the SMALL REU.

Contact Information

Peter Cohen
Bowdoin College
pcohen@bowdoin.edu

Carsten Sprunger
University of Michigan
csprun@umich.edu

Bibliography

- Odlyzko, A. M. "On the Distribution of Spacings between Zeros of the Zeta Function." *Mathematics of Computation Math. Comp.* 48.177 (1987): 273. Web.
- Montgomery, H. L. "The Pair Correlation of Zeros of the Zeta Function", *Proc. Sympos. Pure Mat.*, vol. 24, Amer. Math. Soc., Providence, R. I. 1973, pp 181-193
- C. Hughes, S. J. Miller, *Low-Lying Zeroes of L-Functions with Orthogonal Symmetry*, Duke Mathematical Journal **136** (2007), no. 1, 115 - 172.

How Does Soshnikov's Trick Work

We use the identities that

$$z = \log(1 + (e^z - 1)) = \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\ldots+\lambda_m=n \\ \lambda_j \geq 1}} \frac{(-1)^{m+1}}{m} \frac{1}{\lambda_1! \cdots \lambda_m!}$$

and

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^n = e^{-z} = \frac{1}{1 + (e^z - 1)}$$

$$= \sum_{n=1}^{\infty} z^n \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\ldots+\lambda_m=n \\ \lambda_j \geq 1}} (-1)^m \frac{1}{\lambda_1! \cdots \lambda_m!}$$

The $\left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$ Case

- Suppose that we want $Q_n(\phi)$ for $\text{supp}(\widehat{\phi}) \subseteq \left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$.

The $\left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$ Case

- Suppose that we want $Q_n(\phi)$ for $\text{supp}(\widehat{\phi}) \subseteq \left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$.
- Suffices to analyze $K(y_1, \dots, y_n)$ when $0 \le y_j \le \frac{1}{n-1}$ for all j.

The $\left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$ Case

- Suppose that we want $Q_n(\phi)$ for $\text{supp}(\widehat{\phi}) \subseteq \left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$.
- Suffices to analyze $K(y_1, \dots, y_n)$ when $0 \leq y_j \leq \frac{1}{n-1}$ for all j.
- If $\sum_i y_i > 1$ then $\chi_{\left\{\left|\sum_{j=1}^n \eta(\ell,j)\epsilon_j y_j\right| \leq 1\right\}} = 0$ if and only if all $\eta(\ell, j)\epsilon_j$ have same sign.

The $\left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$ Case

- We have exactly $2m$ choices for $(\epsilon_1, \dots, \epsilon_n)$ which cause the product to vanish.

The $\left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$ Case

- We have exactly $2m$ choices for $(\epsilon_1, \dots, \epsilon_n)$ which cause the product to vanish.
- So

$$K(y_1, \dots, y_n) = \sum_{m=1}^{n} \sum_{\substack{\lambda_1+\ldots+\lambda_m=n \\ \lambda_j \geq 1}} \frac{(-1)^{m+1}}{m} \frac{n!}{\lambda_1! \cdots \lambda_m!} \times \left(2^n - 2m\chi_{\left\{\left|\sum_{j=1}^{n} \eta(\ell,j)\epsilon_j y_j\right| \geq 1\right\}}\right).$$

The $\left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$ Case

- Using a combinatorial trick from Soshnikov, we use generating functions to evaluate the sum above. This gives

$$K(y_1, \ldots, y_n) = 2(-1)^n \chi_{\left\{\left|\sum_{j=1}^n \eta(\ell,j)\epsilon_j y_j\right| \geq 1\right\}}.$$

The $\left[-\frac{1}{n-1}, \frac{1}{n-1}\right]$ Case

- Integration using standard techniques from Fourier analysis gives us,

$$Q_n(\phi) = \frac{(-1)^{n-1}}{2}\left(\int_{\mathbb{R}} \phi(x)^n \frac{\sin 2\pi x}{2\pi x} - \frac{1}{2}\phi(0)^n\right)$$