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The Limiting Eigenvalue Density for the Ensemble of Period m–Circulant Matrices

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Classical
Random Matrix Theory

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & a_{N3} & \cdots & a_{NN} \end{pmatrix}$$

a_{ij} are functions of independent identically distributed random variables $b_1, \dots, b_{k_N}$.
Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le k_N} p(b_i).$$

Example: Real symmetric ensemble. Pick entries of the matrix, up to equivalence of $a_{ij} = a_{ji}$, independently from p. We have $k_N = \frac{N(N+1)}{2}$ degrees of freedom.

Eigenvalue Trace Formula

We want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \ = \ \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \ = \ \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_k i_1}.$$

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int_{\mathbb{R}} f(x)\delta(x - x_0)dx = f(x_0)$.

To *each* matrix A, attach a probability measure:

$$\mu_{A,N}(x) \quad := \quad \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{\sqrt{N}}\right)$$

$$\int_{\mathbb{R}} x^n \mu_{A_N}(x)dx \quad = \quad \sum_{i=1}^{N}\left(\frac{\lambda_i(A)}{\sqrt{N}}\right)^n$$

$$M_n(A, N) := \text{n}^{\text{th}} \text{ moment} \quad = \quad \frac{1}{N^{\frac{n}{2}+1}}\sum_{i=1}^{N}\lambda_i(A)^n \; = \; \frac{\text{Trace}(A^n)}{N^{\frac{n}{2}+1}}.$$

Averaging

Look at the *expected value* for the moments:

$$\begin{aligned} M_n(N) &:= \mathbb{E}(M_n(A,N)) \\ &= \frac{1}{N^{\frac{n}{2}+1}}\mathbb{E}(\mathrm{Trace}(A^n)) \\ &= \frac{1}{N^{\frac{n}{2}+1}}\sum_{1\le i_1,\dots,i_n\le N}\mathbb{E}(a_{i_1 i_2}a_{i_2 i_3}\cdots a_{i_n i_1}). \end{aligned}$$

- If moments converge as $N \to \infty$, they define a probability density called the limiting spectral density.
- For nice ensembles, typical large matrices approach this density.

Linked Ensembles

- Some of the entries of our matrices are always the same.
- Equivalence classes of entries are chosen i.i.d.r.v. from p with mean 0, variance 1, and finite higher moments.

$$
\begin{array}{ccc}
\{1,2,\dots,N\}^2 & \twoheadrightarrow & \{1,2,\dots,N\}^2/\simeq \\
\mathbb{R}^{\{1,2,\dots,N\}^2/\simeq} & \hookrightarrow & \mathbb{R}^{N^2}
\end{array}
$$

Matchings for a Linked Ensemble

We rewrite our formula for the moments as

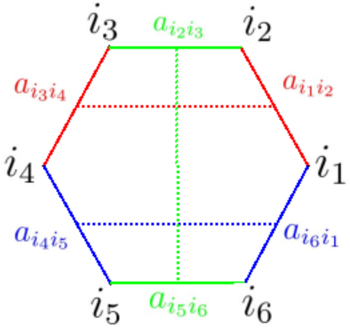
$$M_n(N) = \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)},$$

where the sum is over equivalence relations on $\{(1, 2), (2, 3), ..., (n, 1)\}$.

- $d_j(\sim)$: sizes of equivalence classes.
- m_d: moments of p.
- $\eta(\sim)$: number of $\{(i_1, i_2), (i_2, i_3), ..., (i_n, i_1)\}$ on which $\simeq$ induces $\sim$.

Matchings for a Linked Ensemble

Equivalence relations on $\{(1,2),(2,3),...,(n,1)\}$ are equivalence relations on the sides of an n-gon.

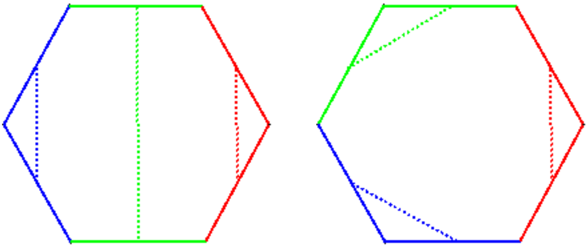


Matchings for a Linked Ensemble

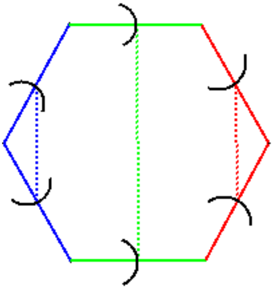
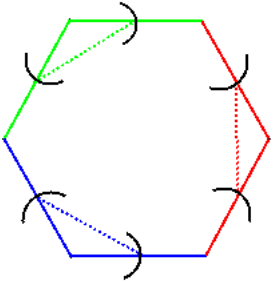
$$M_n(N) = \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)}.$$

- Relations with singletons vanish because the mean $m_1 = 0$.
- For ensembles that are nice enough, higher order pairings are lower order terms.
- If ensemble is real symmetric, non-crossing pairings (Catalan words) always contribute at least 1.

Non-Crossing Pairings



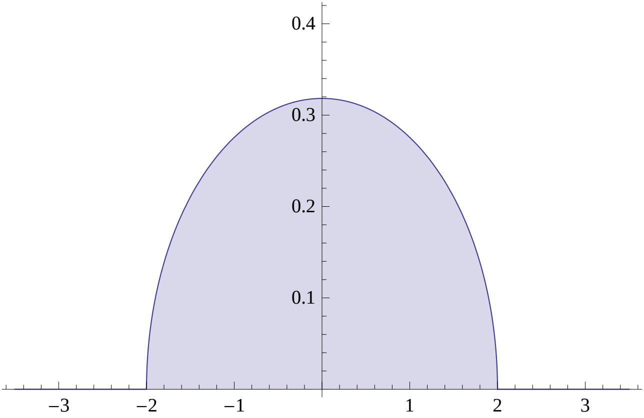
Non-Crossing Pairings

The number of non-crossing pairings is equal to the Catalan number $C_{n/2}$.

Non-Crossing Pairings

The Catalan numbers are the moments of the semi-circle density. This is the limiting spectral density for the full ensemble of real symmetric matrices.



Our ensemble:
Period m–Circulant Matrices

Toeplitz Matrices

- Toeplitz matrices are constant along diagonals.

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ c_{-1} & c_0 & c_1 & c_2 & c_3 & c_4 \\ c_{-2} & c_{-1} & c_0 & c_1 & c_2 & c_3 \\ c_{-3} & c_{-2} & c_{-1} & c_0 & c_1 & c_2 \\ c_{-4} & c_{-3} & c_{-2} & c_{-1} & c_0 & c_1 \\ c_{-5} & c_{-4} & c_{-3} & c_{-2} & c_{-1} & c_0 \end{pmatrix}.$$

- If we think of the indices of our basis elements modulo N, diagonals wrap around.
- Circulant matrices are constant along these "toroidal diagonals." More "fair."

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_4 & c_5 \\ c_5 & c_0 & c_1 & c_2 & c_3 & c_4 \\ c_4 & c_5 & c_0 & c_1 & c_2 & c_3 \\ c_3 & c_4 & c_5 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_4 & c_5 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_4 & c_5 & c_0 \end{pmatrix}.$$

The Ensemble of Real Symmetric Circulant Matrices

- Linked ensemble. Pick the first half of the first row i.i.d.r.v., and the rest of the matrix is determined.
- The limiting eigenvalue density is Gaussian.

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_2 & c_1 \\ c_1 & c_0 & c_1 & c_2 & c_3 & c_2 \\ c_2 & c_1 & c_0 & c_1 & c_2 & c_3 \\ c_3 & c_2 & c_1 & c_0 & c_1 & c_2 \\ c_2 & c_3 & c_2 & c_1 & c_0 & c_1 \\ c_1 & c_2 & c_3 & c_2 & c_1 & c_0 \end{pmatrix}.$$

The Ensemble of Real Symmetric Period m–Circulant Matrices

- Rather than constant, we impose the weaker condition that diagonals are periodic of period m.

- For our purposes, they provide an opportunity to see a transition between the ensemble of real symmetric circulant matrices and that of all real symmetric matrices.

$$\begin{pmatrix} c_0 & c_1 & c_2 & c_3 & c_2 & d_1 \\ c_1 & d_0 & d_1 & d_2 & c_3 & d_2 \\ c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\ c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\ c_2 & c_3 & c_2 & d_1 & c_0 & c_1 \\ d_1 & d_2 & c_3 & d_2 & c_1 & d_0 \end{pmatrix}.$$

Matchings

Recall our formula for the moments of a linked ensemble.

$$M_n(N) = \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)}.$$

where the sum is over equivalence relations on $\{(1,2),(2,3),...,(n,1)\}$.

For the ensemble of symmetric period m–circulant matrices, the coefficient $\eta(\sim)$ is the number of solutions to the system of Diophantine equations:
Whenever $(s, s+1) \sim (t, t+1)$,

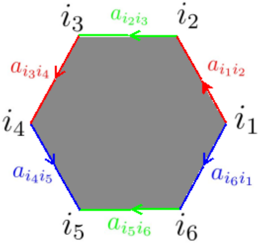
- $i_{s+1} - i_s \equiv i_{t+1} - i_t \pmod{N}$ and $i_s \equiv i_t \pmod{m}$, or
- $i_{s+1} - i_s \equiv -(i_{t+1} - i_t) \pmod{N}$ and $i_s \equiv i_{t+1} \pmod{m}$.

Whenever $(s, s+1) \sim (t, t+1)$,

- $i_{s+1} - i_s \equiv i_{t+1} - i_t \pmod{N}$ and $i_s \equiv i_t \pmod{m}$, or
- $i_{s+1} - i_s \equiv -(i_{t+1} - i_t) \pmod{N}$ and $i_s \equiv i_{t+1} \pmod{m}$.

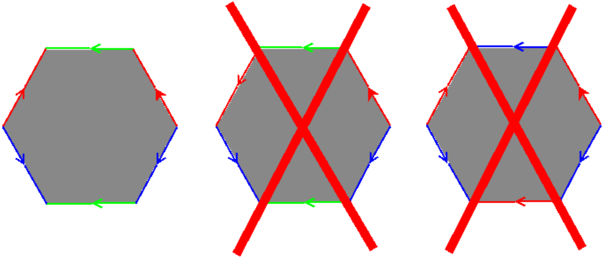
Split up sum further, based on if the first or second set of equations holds for each pair of equivalent edges.

When the first holds, edges are matched with the same orientation, and when the second holds, edges are matched with opposite orientation.



Contributing Terms

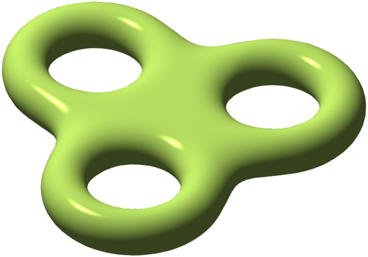
As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.



Therefore, the odd moments go to zero as $N^{-1/2}$.

Algebraic Topology

If we think of these pairings as topological identifications, the contributing ones are precisely those that give rise to orientable surfaces.



It turns out that the contribution from such a pairing is m^{-2g}, where g is the genus (number of holes) of the surface. The proof is a combinatorial argument involving Euler characteristic.

Computing the Even Moments

Our formula for the even moments becomes

$$M_{2k}(N) = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k\left(\frac{1}{N}\right),$$

with $\varepsilon_g(k)$ the number of pairings of the edges of a $(2k)$-gon giving rise to a genus g surface.

But J. Harer and D. Zagier (1986) gave generating functions for the $\varepsilon_g(k)$...

Harer and Zagier say:

$$\sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{k+1-2g} = (2k-1)!!\, c(k,m)$$

where

$$1 + 2\sum_{k=0}^{\infty} c(k,m) x^{k+1} = \left(\frac{1+x}{1-x}\right)^m.$$

Thus, we write

$$M_{2k} = m^{-(k+1)}(2k-1)!!\, c(k,m).$$

We can then use a multiplicative convolution and Cauchy's residue formula to find the *characteristic function* of the distribution (inverse Fourier transform of the density).

$$\begin{aligned}
\phi(t) &= \sum_{k=0}^{\infty} \frac{M_{2k}}{2k!}(it)^{2k} = \frac{1}{m}\sum_{k=0}^{\infty} c(k,m)\frac{1}{k!}\left(\frac{-t^2}{2m}\right)^k \\
&= \frac{1}{2\pi i m}\oint_{|z|=2} \frac{1}{2z^{-1}}\left(\left(\frac{1+z^{-1}}{1-z^{-1}}\right)^m - 1\right) e^{-t^2 z/2m}\frac{dz}{z} \\
&= \frac{1}{m} e^{\frac{-t^2}{2m}} \sum_{l=1}^{m} \binom{m}{l} \frac{1}{(l-1)!}\left(\frac{-t^2}{m}\right)^{l-1}
\end{aligned}$$

Results

Taking a Fourier transform and doing a bit of manipulation, we obtain our explicit formulas.

Theorem

The limiting spectral density function $f_m(x)$ of the real symmetric period m–circulant ensemble is given by the formula

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$

Results (continued)

Theorem

As $m \to \infty$, the limiting spectral densities approach the semicircle distribution.

Results (continued)

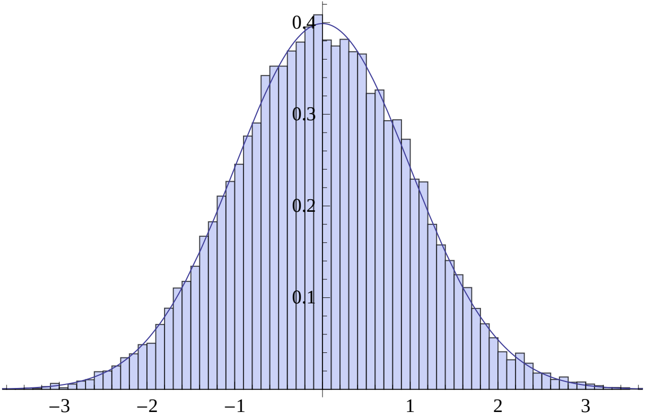


Figure: Plot for f_1 and histogram of eigenvalues of 100 circulant matrices of size 400×400.

Results (continued)

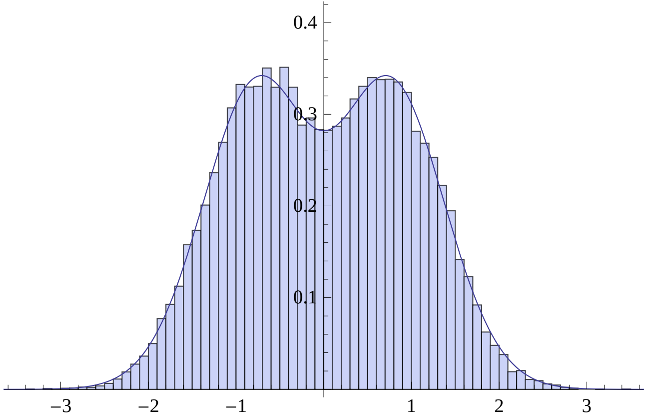


Figure: Plot for f_2 and histogram of eigenvalues of 100 period 2–circulant matrices of size 400×400.

Results (continued)

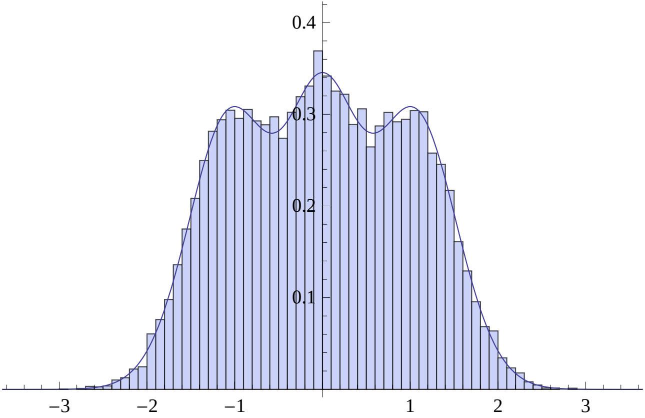


Figure: Plot for f_3 and histogram of eigenvalues of 100 period 3–circulant matrices of size 402×402.

Results (continued)

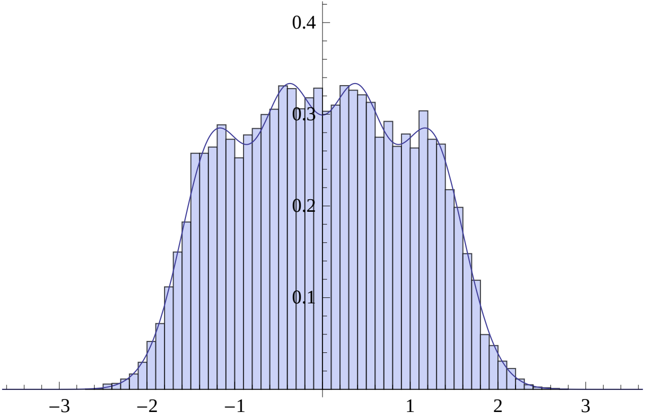


Figure: Plot for f_4 and histogram of eigenvalues of 100 period 4–circulant matrices of size 400×400.

Results (continued)

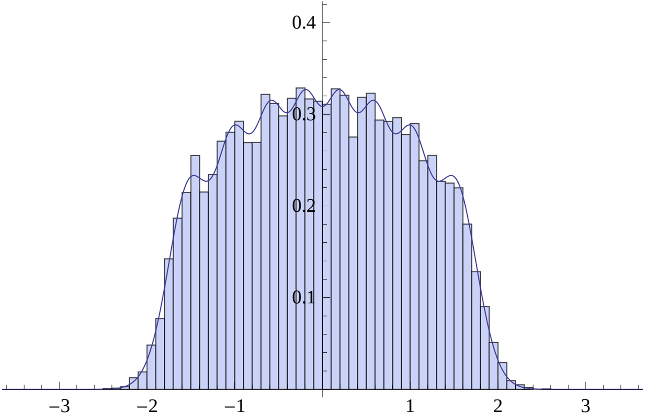


Figure: Plot for f_8 and histogram of eigenvalues of 100 period 8–circulant matrices of size 400×400.

Results (continued)

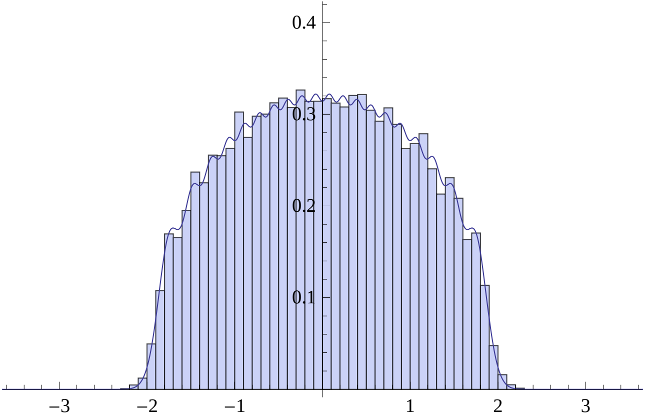


Figure: Plot for f_{20} and histogram of eigenvalues of 100 period 20–circulant matrices of size 400×400.

Results (continued)

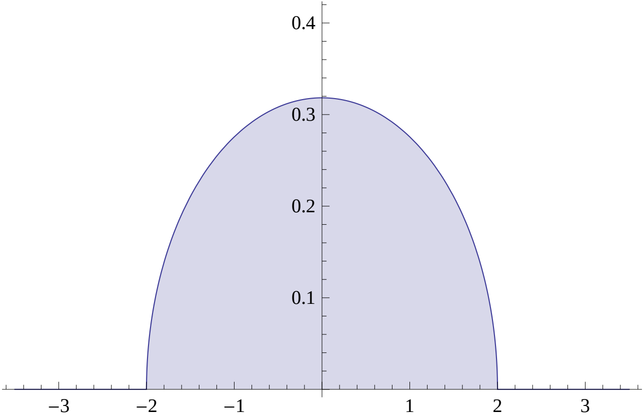


Figure: Plot for the density of the Wigner semi-circle distribution (“$m = \infty$”).

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