# Bounding ranks of cuspidal newforms through excised orthogonal ensembles

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Joint work with Santiago Miguel Velazquez Iannuzzelli, Andrew Keisling, Zoe McDonald, Annika Mauro, Jack Miller, and Steven Miller

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$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

This was formalized by Riemann $\zeta$-function, which is a well-known example of an $L$-function.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \Re(s) > 1,$$

# Riemann Zeta

The Riemann $\zeta$-function admits an analytic continuation by means of a functional equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

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Riemann Hypothesis: All non trivial zeros lie in the line $\Re(s) = 1/2$.

# General L-functions

An $L$-function is

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They have functional equation:

$$\Lambda(s,f)=\left(\frac{\sqrt{N}}{2\pi}\right)^s \Gamma\left(s+\frac{k-1}{2}\right) L(s,f)=\epsilon_f \Lambda(1-s,f)$$

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Generalized Riemann Hypothesis: All non trivial zeroes for all $L$-functions lie in the line $\Re(s)=1/2$.

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**Katz-Sarnak Philosophy**

In the limit, statistics of $L$-functions match statistics for large random matrices from particular classical compact groups.

- $U(N)$
- $O(N)$
- $USp(2N)$

# $L$-Functions Connection to Random Matrix Theory

**Key Takeaway!**

Random matrix models are extremely useful for comparisons of the behavior or zeros of $L$-Functions.

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- Characterize $L$-functions at the central point by their characteristic polynomials of matrix analogues evaluated at 1. The $L$-functions are discretized at the central point.
- Since the values of the $L$-function are discretized, we remove characteristic polynomials whose value at 1 is lower than a cut-off value $c$.
- Removal of matrices with small values is equivalent to the discretization of the central values of $L$-functions. This is our **excised ensemble**.

# Repulsion of critical zeros

- In 2006, S.J.Miller observed numerically that for families of elliptic L-functions, there always exists repulsion of critical zeros away from the critical strip.
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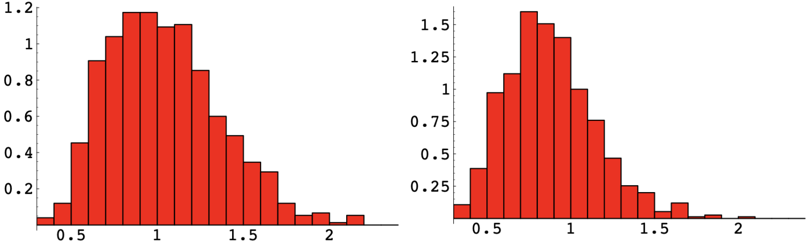


**Figure:** First normalized zero above the central point: Left: 750 rank 0 curves from $y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, $log(cond) \in [3.2, 12.6]$, median $= 1.00$, mean $= 1.04$, standard deviation about the mean $= .32$. Right: 750 rank 0 curves from $y_2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$, $log(cond) \in [12.6, 14.9]$, median $= .85$, mean $= .88$, standard deviation about the mean $= .27$

# Non-excised RMT Ensemble

Simply applying the Random Matrix model to such L-functions without excising the characteristic polynomial according to the discretization fails to detect the repulsion

zeckendorf_fibonacci/Finite Cutof

**Figure:** Probability density of normalized eigenvalue closest to 1 for SO(8) (solid), SO(6) (dashed) and SO(4) (dot-dashed).

# Comparison

Now we look at an explicit comparison between the distribution of the lowest zero for $L_{E_{11}}(s, \chi_d)$ with $0 < d \leq 400,000$, and 2 non-excised random matrix model with different size.

zeckendorf_fibonacci/Finite Cutof

**Figure:** Caption

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# Central Discretization

## Theorem (Kohnen-Zagier)

*Let $f \in S_k$ be a normalized Hecke eigenform, $g \in S^+_{(k+1)/2}$ be the Shimura correspondence of $f$, $d$ a fundamental discriminant with $(-1)^d > 0$, and $L(s, f_d)$ the L-series of $f$ twisted by the quadratic character with fundamental discriminant $d$, which is defined by analytic continuation of the Dirichlet L-series $\sum_{n\geq 1}(d/n)a(n)n^{-s}$. Then*

$$L_f(k, \psi_d) = \kappa_f \frac{c(|d|)^2}{|d|^{(k-1)/2}}, \text{ where } \kappa_f = \frac{(k-1)!}{\pi^{k/2}} \frac{\langle f, f\rangle}{\langle g, g\rangle}$$

*where $\langle \cdot, \cdot \rangle$ denotes the Petersson inner product, $c(|d|)$ is the Fourier coefficient of $g$.*

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Morally, Kohnen-Zagier explicitly relates the central value of twisted $L$-functions to the Fourier coefficient of the Shimura correspondence.

# Construction of a Cut-off

Let $P_{O^+}(N,x)$ be the probability density function for values of $\Lambda_A(1,N)$ with $A \in SO(2N)$, we first conjecture the probability density function of the central value at the dth twist by fundamental discriminant:$P_f(d,x)$

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$$P_f(d, x) \sim a_f(1/2) P_{O^+}(logd, x)$$

Now, let $\kappa_f \theta_f$ be the cutoff for each twist d. we further conjecture:

## Conjecture

$$|\{L_f(s, \psi_d) \in \mathcal{F}_f^+(X) : d \ prime, L_f(1/2, \psi_d) = 0\}| = \sum_{\substack{d \leq X \\ d \ prime}} Prob(0 \leq Y_d \leq \kappa_f \theta_f) = \sum_{\substack{d \leq X \\ d \ prime}} \int_0^{\kappa_f \theta_f} P_f(d, x)$$

# Construction of a Cut-off

On the other hand, by the discretization of central value, the value is forced to be 0 if it's less than some constant.

## Lemma

$$L_f(k, \psi_d) < \kappa_f \frac{c(|d|)^2}{|d|^{(k-1)/2}} \implies L_f(k, \psi_d) = 0$$

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**Lemma**

$$L_f(k,\psi_d) < \kappa_f \frac{c(|d|)^2}{|d|^{(k-1)/2}} \implies L_f(k,\psi_d) = 0$$

**Corollary**

$$|\{L_f(s,\psi_d) \in \mathcal{F}_f^+(X) : d \ prime, L_f(1/2,\psi_d) = 0\}| = \sum_{\substack{d \le X \\ d \ prime}} Prob(0 \le Y_d \le \tfrac{\kappa_f c(|d|)^2}{d^{(k-1)/2}})$$

Thus, confident with our control of $c(d)^2$, we propose our effective cutoff value:

**Theorem (SMALL 2022)**

*For weight 4 Hecke eigen form $f$, The cutoff value for the RMT model:* $C_{eff} = \delta_f \kappa_f a_f^{-2}(-1/2)(\frac{\sqrt{7}}{2\pi})^{-3/2}$, *where* $\delta_f = \mathbb{E}(\frac{c(|d|)^2}{d^{3/2}})$

# Explicit Analysis on Shimura Correspondence

In order to analyze new forms that doesn't arise from elliptic curve, we first study a specific form $f_{7k4A}$ of weight 4 and level 7(Label 7.4.a.a in LMFDB)

## Theorem (Rosson-Tonaria)

*let $g \in S^{+}_{5/2}$ be the Shimura-correspondence of $f_{7k4A}$, then:*

$$g = \frac{1}{4} \sum_{(x,y,z)\in\mathbb{Z}^3} x\omega_{11} q^{Q(x,y,z)/11} = \sum_{n=1}^{\infty} c(n) q^n$$

*Where:*

$$\omega_{11}(x,y,z) = \begin{cases} 0, & \text{if } 11 \nmid Q(x,y,z) \\ \left(\frac{-2x+z}{11}\right), & \text{if } 2 \not\equiv z \pmod{11} \\ \left(\frac{x}{11}\right), & \text{otherwise} \end{cases}$$

*and* $Q(x,y,z) = 4x^2 + 7y^2 + 8z^2 + 4xz$

# Data Analysis on $c(|d|)$

By explicitly coding for q series of $g$, we obtained the list of $c(d)$ for $d \in [0, 20000]$, we proceed to compute $\delta_f$ for different range of fundamental conductors.

**Theorem (SMALL 2022)**

$$\delta_f = \mathbb{E}\left(\frac{c(|d|)^2}{d^{3/2}}\right) = \begin{cases} 0.025056183493753\,93, & 0 \leq d \leq 5000 \\ 0.017724652470136\,76, & 0 \leq d \leq 10000 \\ 0.014527296268071\,65, & 0 \leq d \leq 15000 \\ 0.012595838778631\,81, & 0 \leq d \leq 20000 \end{cases}$$

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# Our RMT model for $f_{7k4A}$

We first compute the standard size of random matrix: $N_{std}$ for $f_{7k4A}$, suppose our fundamental discriminant satisfy $0 < d < X = 20,000$

**Theorem**

$$N_{std} = \log(\frac{\sqrt{7}X}{2\pi e}) \approx 8$$

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**Theorem**

$$N_{std} = \log(\frac{\sqrt{7}X}{2\pi e}) \approx 8$$

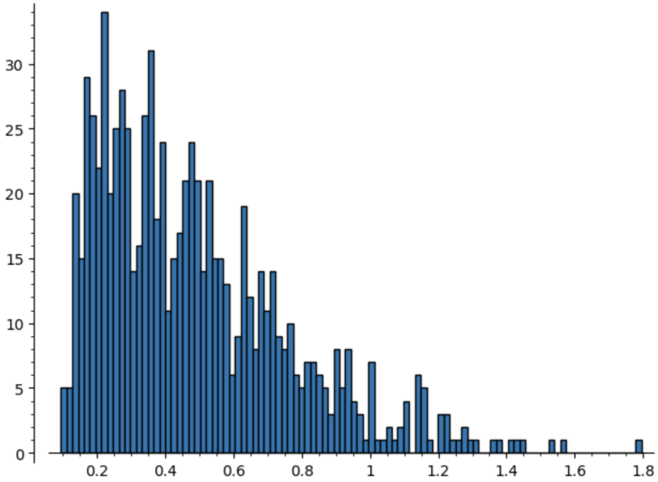
Then we compute the Cutoff for characteristic polynomial of RMT model: $C_{eff}$

**Theorem (SMALL 2022)**

$C_{eff} = \delta_f \kappa_f a_f^{-2}(-1/2)(\frac{\sqrt{7}}{2\pi})^{-3/2}$
*where:* $\delta_f \approx 0.0125958$, $\kappa_f \approx 0.599566158$

# Result

First eigenvalue of our excised random matrix model:



# Result

First zeros of twisted $f_{7k4K}$

zeckendorf_fibonacci/Finite Cutoff

**Figure:** Caption

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