On the density of low-lying zeros of a large family of automorphic L-functions

Timothy Cheek (timcheek@umich.edu)
Marie-Hélène Tomé (mariehelene.tome@duke.edu)

(joint with Pico Gilman and Kareem Jaber)

Advisor: Steven J. Miller
SMALL REU at Williams College

Conférence de théorie des nombres Québec-Maine

October 5, 2024

Duality Between Primes and the Zeros of the Riemann Zeta Function

Theorem (Riemann-von Mangoldt Explicit Formula)

For $X > 1$,

$$\sum_{p^j < X} \log p = X - \sum_{\rho} \frac{X^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - X^{-2}).$$

RH $\implies [X, X + X^{1/2+\epsilon}]$ contains primes, so RH “knows” about patterns in primes.

Montgomery Pair Correlation Conjecture

Conjecture (Montgomery-Dyson 1973)

The zeros of the Riemann zeta function on the critical strip are distributed like the eigenvalues of random Hermitian matrices from the Gaussian Unitary Ensemble (GUE).

Montgomery Pair Correlation Conjecture

Conjecture (Montgomery-Dyson 1973)

The zeros of the Riemann zeta function on the critical strip are distributed like the eigenvalues of random Hermitian matrices from the Gaussian Unitary Ensemble (GUE).

Essentially, predicts that for f a Schwartz test function whose Fourier tranform has arbitrary compact support

$$\frac{1}{N(T)} \sum_{\substack{0 \leq \gamma, \gamma' \leq T \\ \gamma \neq \gamma'}} f\left((\gamma - \gamma')\frac{\log T}{2\pi}\right) \longrightarrow \int_{-\infty}^{\infty} f(x)W(x)\, dx, \quad T \to \infty.$$

Montgomery Pair Correlation Conjecture

Conjecture (Montgomery-Dyson 1973)

The zeros of the Riemann zeta function on the critical strip are distributed like the eigenvalues of random Hermitian matrices from the Gaussian Unitary Ensemble (GUE).

Essentially, predicts that for f a Schwartz test function whose Fourier tranform has arbitrary compact support

$$\frac{1}{N(T)} \sum_{\substack{0 \leq \gamma, \gamma' \leq T \\ \gamma \neq \gamma'}} f\left((\gamma - \gamma') \frac{\log T}{2\pi} \right) \longrightarrow \int_{-\infty}^{\infty} f(x) W(x)\, dx, \quad T \to \infty.$$

- Rudnick-Sarnak ('94, '96): introduced and extended n-level correlations to L-functions, showing universality for all automorphic cuspidal L-functions (agree with GUE).

Montgomery Pair Correlation Conjecture

Conjecture (Montgomery-Dyson 1973)

The zeros of the Riemann zeta function on the critical strip are distributed like the eigenvalues of random Hermitian matrices from the Gaussian Unitary Ensemble (GUE).

Essentially, predicts that for f a Schwartz test function whose Fourier tranform has arbitrary compact support

$$\frac{1}{N(T)} \sum_{\substack{0 \leq \gamma, \gamma' \leq T \\ \gamma \neq \gamma'}} f\left((\gamma - \gamma')\frac{\log T}{2\pi}\right) \longrightarrow \int_{-\infty}^{\infty} f(x)W(x)\, dx, \quad T \to \infty.$$

- Rudnick-Sarnak ('94, '96): introduced and extended n-level correlations to L-functions, showing universality for all automorphic cuspidal L-functions (agree with GUE).
- Also agree with classical compact groups $O(N)$, $SO(\text{even})$, $SO(\text{odd})$, $U(N)$, $Sp(2N)$.

Spectral interpretation of the zeros of L-functions

Question

What is the correct operator for linking the zero statistics of general L-functions to random matrix theory?

Spectral interpretation of the zeros of L-functions

Question

What is the correct operator for linking the zero statistics of general L-functions to random matrix theory?

- Katz-Sarnak (1999): introduced n-level density, distinguishes the classical compact groups, depends on behavior of eigenvalues near 1.

Spectral interpretation of the zeros of L-functions

Question

What is the correct operator for linking the zero statistics of general L-functions to random matrix theory?

- Katz-Sarnak (1999): introduced n-level density, distinguishes the classical compact groups, depends on behavior of eigenvalues near 1.
- Katz-Sarnak density conjecture: behavior of low-lying zeros of a family of L-functions governed by behavior of eigenvalues of a classical compact group.

Spectral interpretation of the zeros of L-functions

Question

What is the correct operator for linking the zero statistics of general L-functions to random matrix theory?

- Katz-Sarnak (1999): introduced n-level density, distinguishes the classical compact groups, depends on behavior of eigenvalues near 1.
- Katz-Sarnak density conjecture: behavior of low-lying zeros of a family of L-functions governed by behavior of eigenvalues of a classical compact group.
- Low-lying zeros related to infinitude of primes, Chebyshev's bias, Birch and Swinnerton-Dyer conjecture, class number bounds.

Distribution of the Low-Lying Zeros of L-functions

Riemann Zeta function $\rightsquigarrow$ Families L-functions

Distribution of the Low-Lying Zeros of L-functions

Riemann Zeta function	$\rightsquigarrow$	Families L-functions
Vertical distribution of zeros for which one L-function is enough	$\rightsquigarrow$ $\rightsquigarrow$	Studying low-lying zeros which requires studying L-functions in families

Distribution of the Low-Lying Zeros of L-functions

Riemann Zeta function	$\rightsquigarrow$	Families L-functions
Vertical distribution of zeros for which one L-function is enough	$\rightsquigarrow$ $\rightsquigarrow$	Studying low-lying zeros which requires studying L-functions in families
Riemann Hypothesis (RH)	$\rightsquigarrow$	Generalized Riemann Hypothesis (GRH)

Distribution of the Low-Lying Zeros of L-functions

Riemann Zeta function	$\rightsquigarrow$	Families L-functions
Vertical distribution of zeros for which one L-function is enough	$\rightsquigarrow$ $\rightsquigarrow$	Studying low-lying zeros which requires studying L-functions in families
Riemann Hypothesis (RH)	$\rightsquigarrow$	Generalized Riemann Hypothesis (GRH)
Studying zeros in an n-dimensional box (n-level correlations)	$\rightsquigarrow$	Studying sums of compactly supported Schwartz test functions evaluated at zeros (n-level densities)

Distribution of the Low-Lying Zeros of L-functions

Riemann Zeta function	$\rightsquigarrow$	Families L-functions
Vertical distribution of zeros for which one L-function is enough	$\rightsquigarrow$ $\rightsquigarrow$	Studying low-lying zeros which requires studying L-functions in families
Riemann Hypothesis (RH)	$\rightsquigarrow$	Generalized Riemann Hypothesis (GRH)
Studying zeros in an n-dimensional box (n-level correlations)	$\rightsquigarrow$	Studying sums of compactly supported Schwartz test functions evaluated at zeros (n-level densities)
Montgomery pair correlation conjecture	$\rightsquigarrow$	Katz-Sarnak density conjecture

Density of low-lying zeros

Definition (1-level density)

Let Φ be a Schwartz function with $\operatorname{supp}(\widehat{\Phi}) \subset (-\sigma, \sigma)$. Assume GRH and write $\rho_f = 1/2 + i\gamma_f$ for the non-trivial zeros of $L(s, f)$ counted with multiplicity. Then

$$\mathscr{OD}(f;\Phi) := \sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi} \log c_f\right),$$

is the *1-level density*, where c_f is the analytic conductor of f.

- 1-level density captures density of the zeros within height $O(1/\log c_f)$ of $s = 1/2$.
- Cannot asymptotically evaluate $\mathscr{OD}(f;\Phi)$ for a single f, must perform averaging over the family ordered by analytic conductor.

Katz-Sarnak Density Conjecture

Katz-Sarnak Density Conjecture

Let $\mathscr{F}(Q) := \{f \in \mathscr{F} : c_f = Q\}$ *or* $\mathscr{F}(Q) := \{f \in \mathscr{F} : c_f \leq Q\}$*. Then for a Schwartz test function* Φ *whose Fourier transform has arbitrary compact support, we have that*

$$\frac{1}{|\mathscr{F}(Q)|} \sum_{f \in \mathscr{F}(Q)} \mathscr{OD}(f;\Phi) \longrightarrow \int_{-\infty}^{\infty} \Phi(x) W(G_{\mathscr{F}})(x)\, dx \quad \text{as} \quad Q \to \infty,$$

where $W(G_{\mathscr{F}})(x)$ *is a distribution depending on the underlying symmetry group* $G_{\mathscr{F}}$ *associated to the family.*

n-level density

Definition

In the setting as before, define the n*-level density* as

$$\mathscr{D}_n(f;\Phi) := \sum_{\substack{j_1,\ldots,j_n \\ j_i \neq \pm j_k}} \prod_{i=1}^{n} \Phi_i\left(\frac{\gamma_f(j_i)}{2\pi}\log c_f\right).$$

- Computing n-level density for $n > 2$ requires knowledge of distribution of signs of the functional equation of each $L(s,f)$, which is beyond current theory.
- Hughes-Rudnick (2003): introduced n-th centered moments.

L-functions Attached to Cuspidal Newforms

Fix $f \in \mathcal{S}_k^{\text{new}}(q)$. Then for $\Re(s) > 1$, we define

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right)^{-1}$$
$$= \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1},$$

where χ_0 is the principal character mod q. Note, $L(s,f)$ can be analytically continued to an entire function on $\mathbb{C}$. Moreover, $L(s,f) = L(s,\overline{f})$.

Katz-Sarnak Density Conjecture for Orthogonal Symmetry

The symmetry type of the family of automorphic L-functions attached to holomorphic cuspidal newforms is orthogonal. Thus, the Katz-Sarnak density conjecture predicts that for test functions Φ whose Fourier transform has arbitrary compact support,

$$\frac{1}{|\mathscr{H}_k(Q)|} \sum_{f \in \mathscr{H}_k(Q)} \mathscr{O}\mathscr{D}(f;\Phi) \longrightarrow \int_{-\infty}^{\infty} \Phi(x) W(O)(x)\, dx \quad \text{as } Q \to \infty,$$

where O is the scaling limit of the group of square orthogonal matrices with density

$$W(O)(x) = 1 + \frac{1}{2}\delta_0(x),$$

where $\delta_0(x)$ denotes the Dirac delta function at $x = 0$.

Extending the Support

Theorem (Iwaniec-Luo-Sarnak '00)

Assume GRH. Then for Φ any even Schwartz function with $\mathrm{supp}(\widehat{\Phi}) \subset (-2, 2)$, we have that

$$\lim_{\substack{q\to\infty \\ \text{square-free}}} \frac{1}{|\mathscr{H}_k(q)|} \sum_{f\in\mathscr{H}_k(q)} \mathscr{O}\mathscr{D}(f;\Phi) = \int_{-\infty}^{\infty} \Phi(x)W(O)(x)\,dx,$$

where O denotes the orthogonal type, showing agreement with the Katz-Sarnak philosophy predictions.

Recent Breakthrough

Theorem (Baluyot-Chandee-Li '23)

Assume GRH. Let Φ be an even Schwartz function such that $\mathrm{supp}(\widehat{\Phi}) \subset (-4, 4)$, *and let Ψ be any smooth function compactly supported on $\mathbb{R}^+$ with $\widehat{\Psi}(0) \neq 0$. Then we have that*

$$\langle \mathscr{O}\mathscr{D}(f;\Phi)\rangle_* := \lim_{Q\to\infty} \frac{1}{N(Q)} \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{f\in\mathscr{H}_k(q)}^h \mathscr{O}\mathscr{D}(f;\Phi) = \int_{-\infty}^{\infty} \Phi(x) W(O)(x)dx,$$

where $N(Q)$ is a normalizing factor, showing agreement with the Katz-Sarnak philosophy predictions.

The n-th Centered Moments of the 1-level Density

We study the n-th centered moments of the 1-level density averaged over levels $q \asymp Q$.

Definition (level-averaged n-th centered moments of the 1-level density)

In the setting as above, define the *n-th centered moment of the 1-level density* to be

$$\left\langle \prod_{i=1}^{n} \left[\mathscr{O}\mathscr{D}(f;\Phi_i) - \langle \mathscr{O}\mathscr{D}(f;\Phi_i)\rangle_*\right] \right\rangle_* .$$

Remark

Note that because of the additional averaging over the level our n-th are a variant of the classical n-th centered moments of the 1-level density studied in previous work.

Main Results

Theorem (Cheek-Gilman-Jaber-Miller-Tomé '24)

Assume GRH. For Ψ non-negative and Φ_i even Schwartz functions with $\operatorname{supp}(\widehat{\Phi}) \subset (-\sigma, \sigma)$ and $\sigma \leq \min\left\{\frac{3}{2(n-1)}, \frac{4}{2n-\mathbf{1}_{2\nmid n}}\right\}$ we have that

$$\left\langle \prod_{i=1}^{n} \left(\mathscr{OD}(f;\Phi_i) - \langle \mathscr{OD}(f;\Phi_i)\rangle_*\right) \right\rangle_* = \frac{\mathbf{1}_{2|n}}{(n/2)!} \sum_{\tau \in S_n} \prod_{i=1}^{n/2} \int_{-\infty}^{\infty} |u| \widehat{\Phi}_{\tau(2i-1)}(u) \widehat{\Phi}_{\tau(2i)}(u)\, du.$$

As such, our work is a generalization of the BCL '23 $n = 1, \sigma = 4$ result.

Main Results

Theorem (Cheek-Gilman-Jaber-Miller-Tomé '24)

Assume GRH. For Ψ *non-negative and* Φ_i *even Schwartz functions with* $\operatorname{supp}(\widehat{\Phi}) \subset (-\sigma, \sigma)$ *and* $\sigma \leq \min\left\{\frac{3}{2(n-1)}, \frac{4}{2n-\mathbf{1}_{2\nmid n}}\right\}$ *we have that*

$$\left\langle \prod_{i=1}^{n} \left(\mathscr{O}\mathscr{D}(f;\Phi_i) - \langle \mathscr{O}\mathscr{D}(f;\Phi_i)\rangle_*\right) \right\rangle_* = \frac{\mathbf{1}_{2|n}}{(n/2)!} \sum_{\tau \in S_n} \prod_{i=1}^{n/2} \int_{-\infty}^{\infty} |u| \widehat{\Phi}_{\tau(2i-1)}(u) \widehat{\Phi}_{\tau(2i)}(u)\, du.$$

As such, our work is a generalization of the BCL '23 $n = 1, \sigma = 4$ result.

Remark

Notably, for $n = 3$*, we achieve* $\sigma = \sigma_i = 3/4$*, greater than currently best known* $\sigma = \sigma_i = 2/3$*. We also have the additional flexibility of taking our test functions to be different.*

Main results

Corollary (Cheek-Gilman-Jaber-Miller-Tomé '24)

Let $\sigma_1 = 3/2$ *and* $\sigma_2 = 5/6$. *Then the two-level density*

$$\left\langle \sum\sum_{j_1 \neq \pm j_2} \Phi_1\left(\gamma_f(j_1)\right)\Phi_2\left(\gamma_f(j_2)\right) \right\rangle_* = 2\int_{-\infty}^{\infty} |u|\widehat{\Phi}_1(u)\widehat{\Phi}_2(u)\,du + \prod_{i=1}^{2}\left(\frac{1}{2}\Phi_i(0) + \widehat{\Phi}_i(0)\right)$$
$$- \Phi_1\Phi_2(0) - 2\widehat{\Phi_1\Phi_2}(0) + P_{\mathrm{odd}}\ \Phi_1\Phi_2(0),$$

where $P_{\mathrm{odd}} := \langle (1-\epsilon_f)/2\rangle_*$ *denotes the proportion of forms with odd functional equation.*

Main results

Corollary (Cheek-Gilman-Jaber-Miller-Tomé '24)

Let $\sigma_1 = 3/2$ *and* $\sigma_2 = 5/6$. *Then the two-level density*

$$\left\langle \sum_{j_1 \neq \pm j_2} \sum \Phi_1\left(\gamma_f(j_1)\right)\Phi_2\left(\gamma_f(j_2)\right)\right\rangle_* = 2\int_{-\infty}^{\infty} |u|\widehat{\Phi}_1(u)\widehat{\Phi}_2(u)\,du + \prod_{i=1}^{2}\left(\frac{1}{2}\Phi_i(0) + \widehat{\Phi}_i(0)\right)$$
$$- \Phi_1\Phi_2(0) - 2\widehat{\Phi_1\Phi_2}(0) + P_{\text{odd}}\ \Phi_1\Phi_2(0),$$

where $P_{\text{odd}} := \langle (1-\epsilon_f)/2\rangle_*$ *denotes the proportion of forms with odd functional equation.*

Remark

This is the first evidence of an interesting new phenomenon: only by taking different test functions are we able to extend the range in which the Katz-Sarnak density predictions hold. In particular, $\sigma_1 + \sigma_2 = 7/3 \ > \ 2$, *where* $\sigma_1 + \sigma_2 = 2$ *was the previously best known.*

Duality Between Primes and Zeros of L-functions

Using an explicit formula relating sums over zeros to sums of prime power coefficients of $L(s,f)$, we deduce that

$$\sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi}\log q\right) = \widehat{\Phi}(0) + \frac{1}{2}\Phi(0) - \frac{2}{\log q}\sum_{p\nmid q} \frac{\lambda_f(p)\log p}{\sqrt{p}}\widehat{\Phi}\left(\frac{\log p}{\log q}\right) + O\left(\frac{\log\log q}{\log q}\right).$$

Duality Between Primes and Zeros of L-functions

Using an explicit formula relating sums over zeros to sums of prime power coefficients of $L(s, f)$, we deduce that

$$\sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi}\log q\right) = \widehat{\Phi}(0) + \frac{1}{2}\Phi(0) - \frac{2}{\log q}\sum_{p\nmid q}\frac{\lambda_f(p)\log p}{\sqrt{p}}\widehat{\Phi}\left(\frac{\log p}{\log q}\right) + O\left(\frac{\log\log q}{\log q}\right).$$

We introduce averaging over the level and split the sums into sums over powers of distinct primes

$$\frac{1}{N(Q)}\sum_q \Psi\left(\frac{q}{Q}\right)\sum_{f\in\mathscr{H}_k(q)}^h \sum_{\substack{p_1,\ldots,p_\ell\nmid q\\ p_i\neq p_j}}\cdots\sum\prod_{i=1}^{\ell}\left(\frac{\lambda_f(p_i)\log p_i}{\sqrt{p_i}\log q}\Phi\left(\frac{\log p_i}{\log q}\right)\right)^{a_i}$$

where $a_1 \geq a_2 \geq \cdots \geq a_\ell$ is a partition of n.

Reducing to the case of distinct primes

Using GRH for $L(s, \mathrm{sym}^2 f)$ we show that

$$\frac{1}{(\log q)^2} \sum_{(p,q)=1} \frac{(\log p)^2 \lambda_f(p^2)}{p} \widehat{\Phi}_i^2 \left(\frac{\log p}{\log q} \right) \ll \frac{\log \log q}{\log q}.$$

Reducing to the case of distinct primes

Using GRH for $L(s, \mathrm{sym}^2 f)$ we show that

$$\frac{1}{(\log q)^2} \sum_{(p,q)=1} \frac{(\log p)^2 \lambda_f(p^2)}{p} \widehat{\Phi}_i^2\left(\frac{\log p}{\log q}\right) \ll \frac{\log\log q}{\log q}.$$

Together with splitting into cases based on the partition of n, this allows us to reduce to studying sums over *distinct* primes:

$$\sum_{\substack{p_1,\ldots,p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^{n} \frac{\lambda_f(p_i)\log p_i}{\sqrt{p_i}} \widehat{\Phi}_i\left(\frac{\log p_i}{\log q}\right).$$

Averaging Over the Extended Orthogonal Family

We average over $f \in \mathscr{H}_k(q)$ with $q \asymp Q$ and study

$$\frac{1}{N(Q)} \sum_{q} \Psi\left(\frac{q}{Q}\right) \frac{1}{(\log q)^n} \sum_{f \in \mathscr{H}_k(q)}^{h} \sum_{\substack{p_1,\dots,p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^{n} \frac{\lambda_f(p_i) \log p_i}{\sqrt{p_i}} \widehat{\Phi}_i\left(\frac{\log p_i}{\log q}\right)$$

$$= \frac{1}{N(Q)} \sum_{q} \Psi\left(\frac{q}{Q}\right) \frac{1}{(\log q)^n} \sum_{\substack{p_1,\dots,p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^{n} \frac{\log p_i}{\sqrt{p_i}} \widehat{\Phi}_i\left(\frac{\log p_i}{\log q}\right) \sum_{f \in \mathscr{H}_k(q)}^{h} \lambda_f(1) \lambda_f\left(\prod_{i=1}^{n} p_i\right).$$

Converting sums over primes to spectral terms

- Ng's work allows us to convert sums over $\mathscr{H}_k(q)$ to a linear combination of sums over an orthogonal basis $\mathscr{B}_k(d)$ for the space $\mathcal{S}_k(d)$, $d \mid q$: Morally, if $(m, n, q) = 1$ and for A a specific arithmetic function, then

$$\sum_{f \in \mathscr{H}_k(q)}^h \lambda_f(m)\lambda_f(n) = \sum_{\substack{q = L_1 L_2 d \\ L_1 | q_1 \\ L_2 | q_2 \\ q_2 \text{ square-free}}} A(L_1, L_2, d) \sum_{e | L_2^\infty} \frac{1}{e} \sum_{f \in \mathscr{B}_k(d)}^h \lambda_f(e^2 m)\lambda_f(n).$$

Converting sums over primes to spectral terms

- Ng's work allows us to convert sums over $\mathscr{H}_k(q)$ to a linear combination of sums over an orthogonal basis $\mathscr{B}_k(d)$ for the space $\mathcal{S}_k(d)$, $d \mid q$: Morally, if $(m,n,q)=1$ and for A a specific arithmetic function, then

$$\sum_{f \in \mathscr{H}_k(q)}^{h} \lambda_f(m)\lambda_f(n) = \sum_{\substack{q = L_1 L_2 d \\ L_1 | q_1 \\ L_2 | q_2 \\ q_2 \text{ square-free}}} A(L_1, L_2, d) \sum_{e | L_2^\infty} \frac{1}{e} \sum_{f \in \mathscr{B}_k(d)}^{h} \lambda_f(e^2 m)\lambda_f(n).$$

- Then we apply the Petersson trace formula

$$\sum_{f \in \mathscr{B}_k(d)}^{h} \lambda_f(m)\lambda_f(n) = \delta(m,n) + \sum_{c \geq 1} \frac{S(m,n;cq)}{cq} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{cq}\right).$$

The Kuznetsov Trace Formula

Let $x := \prod p_i$. We are essentially left to analyze

$$\sum_{c \geq 1} \sum_{\substack{p_1,\ldots,p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^{n} \frac{\log p_i}{\sqrt{p_i}} V\left(\frac{p_i}{P_i}\right) \mathrm{e}\left(v_i \frac{p_i}{P_i}\right) \sum_{s} \frac{S(e^2, x; cL_1rds)}{cL_1rds} h\left(\frac{4\pi\sqrt{e^2x}}{cL_1rds}\right)$$

where V is smooth and compactly supported and h is essentially a smooth truncation of J_{k-1}. We use the Kuznetsov trace formula to convert an average over $f \in \mathscr{B}_k(d)$ into spectral terms:

Holomorphic cuspforms $+$ Maass cuspforms $+$ Eisenstein series.

Acknowledgements

We are grateful to Professor Steven J Miller for suggesting this problem and for his mentorship. We would like to thank Arijit Paul and Zijie Zhou for traveling to Williams College to provide mentorship and guidance. We would like to thank Siegfried Baluyot, Vorrapan Chandee, Xiannian Li, Wenzhi Luo, and Jesse Thorner for helpful conversations.

This research was supported by the National Science Foundation, under grant DMS2241623, Duke University, Princeton University, the University of Michigan, and Williams College.

References

[1] A.O.L Atkin and J. Leher, *Hecke Operators on* $\Gamma_0(m)$, in Mathematische Annalen 185, pp. 134-160.

[2] S. Baluyot, V. Chandee, and X. Li, *Low-lying zeros of a large family of automorphic* L*-functions with orthogonal symmetry*, https://arxiv.org/pdf/2310.07606.

[3] O. Barrett, F. Firk, S. J. Miller, and C. Turnage-Butterbaugh, *From Quantum Systems to* L*-Functions: Pair Correlation Statistics and Beyond*, in Open Problems in Mathematics (editors John Nash Jr. and Michael Th. Rassias), Springer-Verlag, 2016. https://arxiv.org/abs/1505.07481.

[4] T. Cheek, P. Gilman, K. Jaber, S. J. Miller, and M. Tomé, *On the distribution of low-lying zeros of a family of automorphic* L*-functions*, in preparation.

[5] C. Hughes and S. J. Miller, *Low lying zeros of L–functions with orthogonal symmetry*, Duke Mathematical Journal 136 (2007), no. 1, 115–172. https://arxiv.org/abs/math/0507450v1.

[6] H. Iwaniec, W. Luo, and P. Sarnak, *Low lying zeros of families of L-functions*, Inst. Hautes Études Sci. Publ. Math. 91 (2000), 55–131. https://arxiv.org/abs/math/9901141.

[7] N. Katz and P. Sarnak, *Zeros of zeta functions and symmetries*, Bull. AMS **36** (1999), 1–26. http://www.ams.org/journals/bull/1999-36-01/S0273-0979-99-00766-1/home.html.

[8] M. Rubinstein, *Low-lying zeros of L-functions and random matrix theory*, Duke Math J. **109** (2001), 147–181. 10.1215/S0012-7094-01-10916-2.

[9] Z. Rudnick and P. Sarnak, *Zeros of principal* L*-functions and random matrix theory*, Duke Math. J. **81** (1996), 269–322.