Group Theory in Compound Families of L-functions

Steven J. Miller, Williams College

sjm1@williams.edu,
Steven.Miller.MC.96@aya.yale.edu
http://www.williams.edu/Mathematics/sjmiller/

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Talk online here: https://youtu.be/-dASMp-nK90

Introduction

Why study zeros of L-functions?

- Infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Distribution of zeros

- $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

Goals

- See similar behavior in different systems (random matrix theory).
- Discuss the tools and techniques needed to prove the results.
- Group Theory and Compound Families of L-Functions.
- Open Problems.

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the t_i?

Examples:

- Spacings b/w Energy Levels of Nuclei.
- Spacings b/w Eigenvalues of Matrices.
- Spacings b/w Primes.
- Spacings b/w $n^k\alpha \bmod 1$.
- Spacings b/w Zeros of L-functions.

Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.

2. Develop an explicit formula relating what we want to study to something we understand.

3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Classical
Random Matrix Theory

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

Heavy nuclei (Uranium: 200$+$ protons / neutrons) worse!

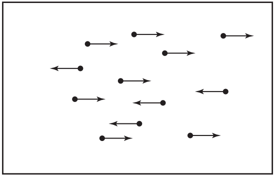
Get some info by shooting high-energy neutrons into nucleus, see what comes out.

Fundamental Equation:

$$H\psi_n \;=\; E_n\psi_n$$

H : matrix, entries depend on system
E_n : energy levels
ψ_n : energy eigenfunctions

Origins of Random Matrix Theory



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Classical Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Want to understand eigenvalues of A.

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int f(x)\delta(x - x_0)dx = f(x_0)$.

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$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

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$$\text{k}^{\text{th}} \text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

Wigner's Semi-Circle Law

Not most general case, gives flavor.

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

SKETCH OF PROOF: Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) = \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) = \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

SKETCH OF PROOF: Correct Scale

$$\text{Trace}(A^2) \;=\; \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\text{Trace}(A^2) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \;\sim\; N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \;\sim\; N^2$$

Gives $N\text{Ave}(\lambda_i(A)^2) \sim N^2$ or $\text{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k-th moment of $\mu_{A,N}(x)$ is $\text{Trace}(A^k)/2^k N^{k/2+1}$.

Average k-th moment is

$$\int \cdots \int \frac{\text{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \leq j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

$$\frac{1}{2^2 N^2} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i=1}^{N} \sum_{j=1}^{N} a_{ij}^2 \cdot p(a_{11}) da_{11} \cdots p(a_{NN}) da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty} a_{ij}^2 p(a_{ij}) da_{ij} \quad \cdot \prod_{\substack{(k,l)\neq(i,j) \\ k<l}} \int_{a_{kl}=-\infty}^{\infty} p(a_{kl}) da_{kl} \; = \; 1.$$

Higher moments involve more advanced combinatorics (Catalan numbers).

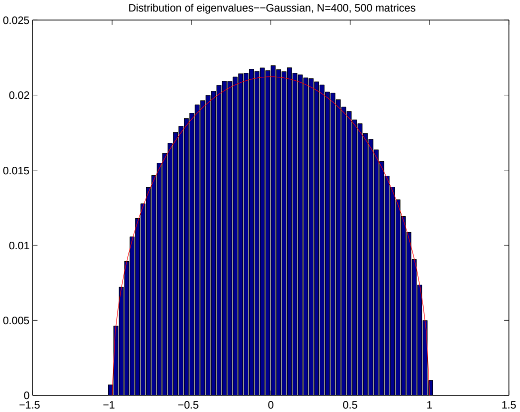
SKETCH OF PROOF: Averaging Formula for Higher Moments

Higher moments involve more advanced combinatorics (Catalan numbers).

$$\frac{1}{2^k N^{k/2+1}} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} \cdots a_{i_k i_1} \cdot \prod_{i \le j} p(a_{ij}) da_{ij}.$$

Main term $a_{i_\ell i_{\ell+1}}$'s matched in pairs, not all matchings contribute equally (if did have Gaussian, see in Real Symmetric Palindromic Toeplitz matrices; interesting results for circulant ensembles (joint with Gene Kopp, Murat Kologlu).

Numerical examples



500 Matrices: Gaussian 400×400

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

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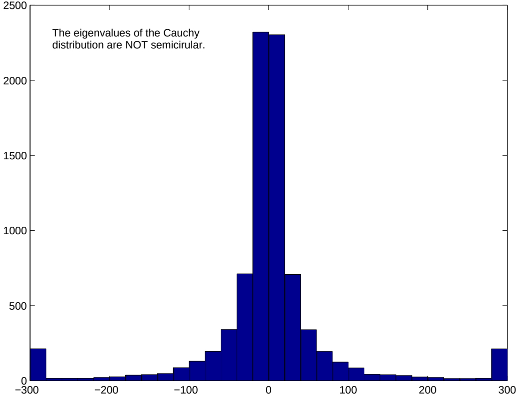

Cauchy Distribution: $p(x) = \frac{1}{\pi(1+x^2)}$

Introduction to L-Functions

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right) \pi^{-\frac{s}{2}} \zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p\left(s,f\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f)L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

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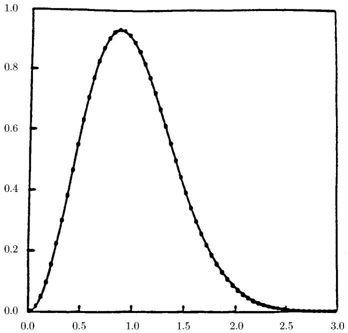
Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

$$= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right)$$

$$= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).$$

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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.$$

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Contour Integration (see Fourier Transform arising):

$$\int -\frac{\zeta'(s)}{\zeta(s)}\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma\log p} e^{-it\log p} ds.$$

Knowledge of Zeros $\Leftrightarrow$ Knowledge of Coefficients.

Explicit Formula: Examples

Dirichlet L-functions: Let h be an even Schwartz function and $L(s,\chi) = \sum_n \chi(n)/n^s$ a Dirichlet L-function from a non-trivial character χ with conductor m and zeros $\rho = \frac{1}{2} + i\gamma_\chi$; if the Generalized Riemann Hypothesis is true then $\gamma \in \mathbb{R}$. Then

$$\sum_\rho h\left(\gamma_\rho \frac{\log(m/\pi)}{2\pi}\right) = \int_{-\infty}^{\infty} h(y)dy$$
$$-2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(\frac{\log p}{\log(m/\pi)}\right) \frac{\chi(p)}{p^{1/2}}$$
$$-2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(2\frac{\log p}{\log(m/\pi)}\right) \frac{\chi^2(p)}{p} + O\Big(\frac{1}{\log m}\Big).$$

Explicit Formula: Examples

Cuspidal Newforms: Let $\mathcal{F}$ be a family of cuspidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0) + \frac{1}{2}\phi(0) - \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbb{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\dots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i \neq j_k\right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbb{R}^{n-1}$.

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
2. 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).
4. n-level correlations for the classical compact groups (Katz-Sarnak).
5. Insensitive to any finite set of zeros.

Measures of Spacings: n-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f\gamma_f^{(j_1)}\right)\cdots\phi_n\left(L_f\gamma_f^{(j_n)}\right)$$

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Katz-Sarnak Conjecture

For a ‘nice’ family of L-functions, the n-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1}\sum_{f\in\mathcal{F}_N}\log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathcal{F}|\to\infty$,

$$\begin{aligned}\frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N} D_{n,f}(\phi) &= \frac{1}{|\mathcal{F}_N|}\sum_{f\in\mathcal{F}_N}\sum_{\substack{j_1,\dots,j_n\\ j_i\neq\pm j_k}}\prod_i \phi_i\left(\frac{\log C_f}{2\pi}\gamma_E^{(j_i)}\right)\\ &\to \int\cdots\int \phi(x) W_{n,\mathcal{G}(\mathcal{F})}(x)dx.\end{aligned}$$

Katz-Sarnak Conjecture

As $C_f\to\infty$ the behavior of zeros near $1/2$ agrees with $N\to\infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Correspondences

Similarities between *L*-Functions and Nuclei:

Zeros	$\longleftrightarrow$	Energy Levels
Schwartz test function	$\longrightarrow$	Neutron
Support of test function	$\longleftrightarrow$	Neutron Energy.

Compound Families
Dueñez-Miller

Identifying the Symmetry Groups

- Often an analysis of the monodromy group in the function field case suggests the answer.
- All simple families studied to date are built from GL_1 or GL_2 *L*-functions.
- Tools: Explicit Formula, Orthogonality of Characters / Petersson Formula.
- How to identify symmetry group in general? One possibility is by the signs of the functional equation:
- **Folklore Conjecture:** If all signs are even and no corresponding family with odd signs, Symplectic symmetry; otherwise SO(even). (False!)

Explicit Formula

- π: cuspidal automorphic representation on GL_n.
- $Q_\pi > 0$: analytic conductor of $L(s,\pi) = \sum \lambda_\pi(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{\pi,j}$.
- Satake parameters $\{\alpha_{\pi,i}(p)\}_{i=1}^n$;
  $\lambda_\pi(p^\nu) = \sum_{i=1}^n \alpha_{\pi,i}(p)^\nu$.
- $L(s,\pi) = \sum_n \frac{\lambda_\pi(n)}{n^s} = \prod_p \prod_{i=1}^n \left(1 - \alpha_{\pi,i}(p)p^{-s}\right)^{-1}$.

$$\sum_j g\left(\gamma_{\pi,j}\frac{\log Q_\pi}{2\pi}\right) = \widehat{g}(0) - 2\sum_{p,\nu} \widehat{g}\left(\frac{\nu \log p}{\log Q_\pi}\right)\frac{\lambda_\pi(p^\nu)\log p}{p^{\nu/2}\log Q_\pi}$$

Some Results: Rankin-Selberg Convolution of Families

Symmetry constant: $c_{\mathcal{L}} = 0$ (resp, 1 or -1) if family $\mathcal{L}$ has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for $\pi_{1,p} \times \pi_{2,p}$ are

$$\{\alpha_{\pi_1 \times \pi_2}(k)\}_{k=1}^{nm} \ = \ \{\alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j)\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}.$$

Theorem (Dueñez-Miller)

If $\mathcal{F}$ and $\mathcal{G}$ are *nice* families of L-functions, then $c_{\mathcal{F} \times \mathcal{G}} = c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

1-Level Density

Assuming conductors constant in family $\mathcal{F}$, have to study

$$\lambda_f(p^\nu) = \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu$$

$$S_1(\mathcal{F}) = -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p)\right]$$

$$S_2(\mathcal{F}) = -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2)\right]$$

The corresponding classical compact group is determined by

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2) = c_{\mathcal{F}} = \begin{cases} 0 & \text{Unitary} \\ 1 & \text{Symplectic} \\ -1 & \text{Orthogonal.} \end{cases}$$

1-Level Density for Rankin-Selberg Convolution of Families

Families $\mathcal{F}$ and $\mathcal{G}$.
Satake parameters $\{\alpha_{f,i}(p)\}_{i=1}^{n}$ and $\{\beta_{g,j}(p)\}_{j=1}^{m}$.
Family $\mathcal{F} \times \mathcal{G}$, $L(s, f \times g)$ has parameters $\{\alpha_{f,i}(p)\beta_{g,j}(p)\}_{i=1 \dots n, j=1 \dots m}$.

$$\begin{aligned} a_{f \times g}(p^\nu) &= \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{f,i}(p)^\nu \beta_{g,j}(p)^\nu \\ &= \sum_{i=1}^{n} \alpha_{f,i}(p)^\nu \sum_{j=1}^{m} \beta_{g,j}(p)^\nu \\ &= \lambda_f(p^\nu) \cdot \lambda_g(p^\nu). \end{aligned}$$

Technical restriction: need f and g unrelated (i.e., g is not the contragredient of f) for our applications.

1-Level Density for Rankin-Selberg Convolution of Families (cont)

To analyze $S_\nu(\mathcal{F} \times \mathcal{G})$ we must study

$$\frac{1}{|\mathcal{F} \times \mathcal{G}|} \sum_{f \times g \in \mathcal{F} \times \mathcal{G}} \lambda_f(p^\nu) \cdot \lambda_g(p^\nu) = \left[\frac{1}{|\mathcal{F}|} \sum_{f \in \mathcal{F}} \lambda_f(p^\nu)\right] \cdot \left[\frac{1}{|\mathcal{G}|} \sum_{g \in \mathcal{G}} \lambda_g(p^\nu)\right]$$

- $\nu = 1$: If one of the families is rank zero, so is $\mathcal{F} \times \mathcal{G}$; $S_1(\mathcal{F} \times \mathcal{G})$ will not contribute.
- $\nu = 2$: $c_{\mathcal{F} \times \mathcal{G}} \ = \ c_{\mathcal{F}} \cdot c_{\mathcal{G}}$.

Proves if each family is of rank 0, the symmetry type of the convolution is the product of the symmetry types. □

Symplectic leaves alone, Orthogonal flips symmetry.

Future Work

RMT Ensembles and Convolution

Is there a procedure to combine two RMT ensembles similar to convolution?

Ralph Morrison's Williams Senior Thesis: Tried Hadamard, Kronecker products, no luck.

- Hadamard: $A, B \mapsto A \bigodot B$, $(A \bigodot B)_{ij} = A_{ij} B_{ij}$.
- Kronecker: $A, B \mapsto A \otimes B$,

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{n1}B \\ \vdots & \ddots & \vdots \\ a_{1n}B & \cdots & a_{nn}B \end{pmatrix}.$$

Disco

Keller Blackwell, Neelima Borade, Arup Bose, Charles Devlin Vi, Noah Luntzlara, Renyuan Ma, Steven J. Miller, Soumendu Sundar Mukherjee, Mengxi Wang, Wanqiao Xu.

Consider the “disco” concatenation:

$$\mathcal{D}_1(A, B) = \begin{bmatrix} A & B \\ B & A \end{bmatrix}.$$

Disco

d-Disco of A and $\mathbf{B} = \{B_k\}$, denoted $\mathcal{D}_d(A, \mathbf{B})$, is the $2^d N \times 2^d N$ matrix

$$\left[\begin{array}{ccc} \begin{array}{cc} \begin{array}{cc} A & B_1 \\ B_1 & A \end{array} & B_2 \\ B_2 & \begin{array}{cc} A & B_1 \\ B_1 & A \end{array} \end{array} & \cdots & B_d \\ \vdots & \ddots & \vdots \\ B_d & \cdots & \begin{array}{cc} \begin{array}{cc} A & B_1 \\ B_1 & A \end{array} & B_2 \\ B_2 & \begin{array}{cc} A & B_1 \\ B_1 & A \end{array} \end{array} \end{array}\right].$$

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Thank you!

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