

Leslie Population Models in Predator-Prey and Competitive Populations

Predator-Prey Modelling Group (SMALL 2024)

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Video Recording: <https://www.youtube.com/watch?v=-hGJV4t-Lrk&t=9s>

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Lotka-Volterra Model

Continuous prey/predator densities $x(t), y(t) \in [0, 1]$:

$$\frac{dx}{dt} = rx(1 - x) - axy, \quad \frac{dy}{dt} = ay(x - y). \quad (1)$$

Theorem (Merdan)

If $r - \alpha\beta > 0$, the system converges to the positive stable state

$$(x_*, y_*) = \left(\frac{r - \alpha\beta}{a + r}, \frac{r - \alpha\beta}{a + r} \right). \quad (2)$$

Assuming an homogenous predator and prey population, stability is guaranteed.

Leslie Matrices

We substitute the classical continuous model with finite-difference methods.

Definition (Simple Leslie matrix, fertility f)

$$L_f := \begin{bmatrix} f & f & \cdots & f & f \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}, \quad \vec{p}_n = L_f^n \vec{p}_0. \quad (3)$$

Example Model (Eqs. 1.2–1.7)

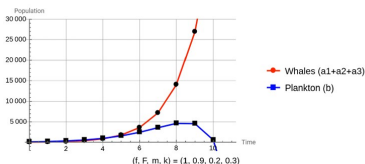
Whales (3 age groups $a_n^{(i)}$) consuming plankton b_n :

$$\vec{p}_{n+1} = \tilde{L} \vec{p}_n = \begin{bmatrix} f & f & f & m \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -k & -k & -k & 1 + F \end{bmatrix} \vec{p}_n, \quad \vec{p}_n = (a_n^{(1)}, a_n^{(2)}, a_n^{(3)}, b_n)^T. \quad (4)$$

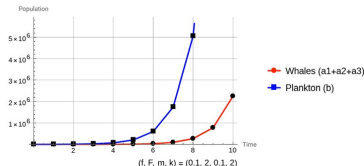
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(A) High predation and prey exhaustion



(B) Low predation and mutual population growth

Predator-Prey Model (Def. 3.1)

Definition

Predator $\vec{\alpha}_n$, prey $\vec{\beta}_n \in \mathbb{R}_{\geq 0}^N$:

$$\vec{\alpha}_{n+1} = L_a \vec{\alpha}_n + km \vec{\beta}_n, \quad (5)$$

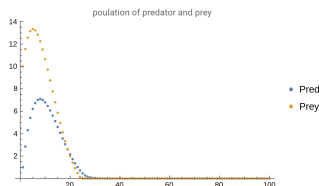
$$\vec{\beta}_{n+1} = L_b \vec{\beta}_n - k \vec{\alpha}_n. \quad (6)$$

k : predation ratio, m : nurturing ratio ($k, m > 0$).

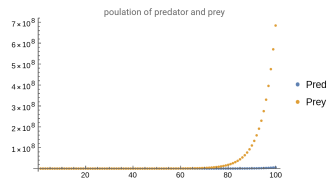
L_a : dominant eigenvalue < 1 ; L_b : dominant eigenvalue > 1 .

Behavior

If consumption is too high, both populations die out.



Overconsumption: extinction



Small consumption: growth

Goal

Goal

A necessary and sufficient condition on k for population growth.

Lemma (Prop. 3.5)

$$\vec{\alpha}_n = (L_a + L_b)\vec{\alpha}_{n-1} - L_b L_a \vec{\alpha}_{n-2} - m k^2 \vec{\alpha}_{n-2}. \quad (7)$$

Scalar case $N = 1$: $l_a < 1$, $l_b > 1$.

Theorem 3.9 (1×1 case)

Theorem

Neither population vanishes as $n \rightarrow \infty$ iff

$$k \leq \sqrt{\frac{(1 - l_b)(l_a - 1)}{m}}. \quad (8)$$

Proof.

Companion matrix eigenvalue (discriminant ≥ 0):

$$\lambda = \frac{l_a + l_b}{2} + \frac{\sqrt{D}}{2}, \quad D = \frac{(l_a + l_b)^2}{4} - k^2 m - l_a l_b. \quad (9)$$

$$\text{Non-extinction} \iff \lambda \geq 1 \iff k \leq \sqrt{(1 - l_b)(l_a - 1)/m}. \quad \square$$

Theorem 3.11 ($L_a = \rho L_b$)

Theorem

For $L_a = \rho L_b$,

$$\vec{\alpha}_n = \Lambda_1^n \vec{v}_1 + \Lambda_2^n \vec{v}_2, \quad (10)$$

where Λ_1, Λ_2 are matrices that are defined in terms of L_a, L_b and with $\lambda_{\max}$ the dominant eigenvalue of L_b . The dominant eigenvalue of Λ_1, Λ_2 are bounded over by

$$\|\Lambda_{\max}\| = \frac{(\rho + 1)\lambda_{\max} + \sqrt{(\rho + 1)^2 \lambda_{\max}^2 + 4mk^2}}{2}. \quad (11)$$

Open Question

The case $[L_a, L_b] \neq 0$ remains open.

Competitive Model (Def. 4.1)

Definition

Same growth matrix $L = L_f$ ($f \geq 1/N$):

$$\vec{\alpha}_{n+1} = \max(L\vec{\alpha}_n - km\vec{\beta}_n, \vec{0}), \quad (12)$$

$$\vec{\beta}_{n+1} = \max(L\vec{\beta}_n - k\vec{\alpha}_n, \vec{0}). \quad (13)$$

$k, m \in (0, 1)$: interaction and competitive-advantage ratios.

Theorem 4.2 (Last Species Standing)

Theorem

With $\vec{\alpha}_0 = (\alpha_0, \dots, \alpha_0)$, $\vec{\beta}_0 = (\beta_0, \dots, \beta_0)$, let $D := \alpha_0 - \sqrt{m} \beta_0$.

- $D > 0$: β vanishes, α grows.
- $D < 0$: α vanishes, β grows.
- $D = 0$: both vanish or both grow.

Approach: reduce to eigenvalues, use modulus as $n \rightarrow \infty$:

$$\Lambda_{1,2} = L \pm k\sqrt{m} I, \quad \vec{\alpha}_n \approx \Lambda_1^n \vec{v}_1, \quad \vec{v}_1 = -\frac{D}{2\sqrt{m}}(1, \dots, 1). \quad (14)$$

P. aurelia and *P. caudatum*



Figure: *Paramecium aurelia* (left), *Paramecium caudatum* (right). Source: wikipedia.

Goal

Fit population data of *P. aurelia* and *P. caudatum* to the competitive model and find underlying parameters.

Error Function

Definition (Error of the fit)

Let T be the number of discrete timesteps in the experimental data, and $\bar{\alpha}_t, \bar{\beta}_t$ be the total population of respective species predicted by the competitive model. Then,

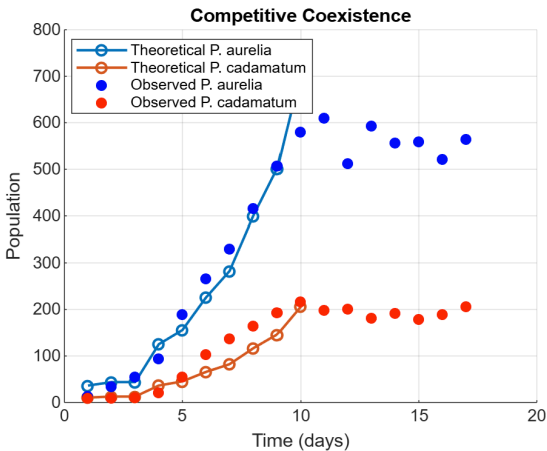
$$\chi(f, k, \beta_0, m) := \left(\sum_{t \leq T} \left(\frac{\bar{\alpha}_t - p(t)}{p(t)} \right)^2 + \sum_{t \leq T} \left(\frac{\bar{\beta}_t - q(t)}{q(t)} \right)^2 \right). \quad (15)$$

Machine Learning Scheme

MACHINE LEARNING SCHEME

- 1 Evaluate χ for random choice of the four parameters. Among the random tuples, choose the tuple that minimizes the value of χ .
- 2 Perform a general gradient descent involving all four parameters, starting from the point chosen in step 1. Find the admissible value of k, m .
- 3 Define a subroutine, Opt_g , that performs a gradient descent solely on parameter g with all other parameters fixed.
- 4 Finally perform an optimized gradient descent on parameter m . In the beginning of each loop, invoke the subroutine Opt_g to adjust the parameter g .

Admissible Fit



$$\chi = 3.11 < 10.351 \quad (16)$$

Open Problems

- 1 Can we provide a sufficient condition for population growth of a Predator-Prey model where $[L_a, L_b] \neq 0$?
- 2 Is it possible to create a Leslie matrix model that displays saturation?
- 3 Formulate a Leslie Population Model of open systems with migration.

Thank You!

Questions & Answers

Paper

Status: *Accepted for publication to the Fibonacci Quarterly.*

arXiv: <https://arxiv.org/abs/2412.19831>

Acknowledgements

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