On the density of low-lying zeros of a large family of automorphic L-functions

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Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at t_1, t_2, $t_3, \dots$.

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Examples: Spacings between

- ◇ Energy Levels of Nuclei.
- ◇ Eigenvalues of Matrices.
- ◇ Zeros of L-functions.

Sketch of proofs

In studying many statistics, often three key steps:

- ◇ Determine the correct scale for events.
- ◇ Develop an explicit formula relating what want to study to what can study.
- ◇ Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\text{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \text{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f) L(s,f) = \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

$$\text{All non-trivial zeros have } \text{Re}(s) = \frac{1}{2}; \text{ can write zeros as } \frac{1}{2} + i\gamma.$$

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Distribution of zeros

- ◇ $\zeta(s) \neq 0$ for $\Re(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- ◇ GRH: error terms.
- ◇ GSH: Chebyshev's bias.
- ◇ Analytic rank, adjacent spacings: $h(D)$.

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

Explicit Formula (Contour Integration)

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&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p\int\left(\frac{x}{p}\right)^s\frac{ds}{s}.$$

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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.$$

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\end{aligned}
$$

Contour Integration (see Fourier Transform arising):

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma\log p} e^{-it\log p} ds.
$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Examples

Cuspidal Newforms: Let $\mathscr{F}$ be a family of cuspidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f)=\sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathscr{F}|}\sum_{f\in\mathscr{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0)+\frac{1}{2}\phi(0)-\frac{1}{|\mathscr{F}|}\sum_{f\in\mathscr{F}}P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{ \left(\alpha_{j_1} - \alpha_{j_2}, \ldots, \alpha_{j_{n-1}} - \alpha_{j_n}\right) \in B, j_i \neq j_k \right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

- ◇ Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
- ◇ 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
- ◇ n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).
- ◇ n-level correlations for the classical compact groups (Katz-Sarnak).
- ◇ Insensitive to any finite set of zeros.

Measures of Spacings: n-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

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- ◇ Individual zeros contribute in limit.
- ◇ Most of contribution is from low zeros.
- ◇ Average over similar curves (family).

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathscr{F}_N|^{-1} \sum_{f \in \mathscr{F}_N} \log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathscr{F}| \to \infty$,

$$\frac{1}{|\mathscr{F}_N|} \sum_{f \in \mathscr{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathscr{F}_N|} \sum_{f \in \mathscr{F}_N} \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i\left(\frac{\log C_f}{2\pi} \gamma_E^{(j_i)}\right)$$

$$\to \int \cdots \int \phi(x) W_{n,\mathscr{G}(\mathscr{F})}(x) dx.$$

Katz-Sarnak Conjecture

As $C_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1. \end{cases}$$

Density of low-lying zeros (Slight Notational Change)

Definition (1-level density)

Let Φ be a Schwartz function with $\text{supp}(\widehat{\Phi}) \subset (-\sigma, \sigma)$. Assume GRH and write $\rho_f = 1/2 + i\gamma_f$ for the non-trivial zeros of $L(s, f)$ counted with multiplicity. Then

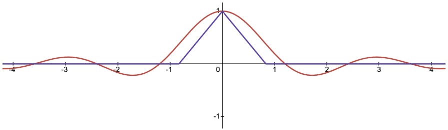
$$\mathcal{OD}(f; \Phi) \ := \ \sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi} \log c_f\right),$$

is the *1-level density*, where c_f is the analytic conductor of f.

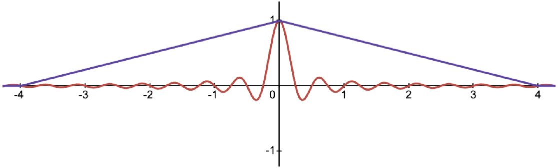
- 1-level density captures density of the zeros within height $O(1/\log c_f)$ of $s = 1/2$; since gaps between zeros are approximately c_f, this is counting (morally) a small number of zeros.
- Cannot asymptotically evaluate $\mathcal{OD}(f; \Phi)$ for a single f, must perform averaging over the family ordered by analytic conductor.

Extending the Support

Taking the support of $\widehat{\Phi}$ (purple) to be bounded yet arbitrarily large corresponds to taking Φ (red) close to a Dirac delta function at $s = 1/2$.



Smaller support = less precise information



Larger support = more precise information

As the support of $\hat{\Phi}$ gets larger, this approaches a delta spike, and thus (morally) allows us to measure the zeros near $s = 1/2$. Hence, larger support allows finer measurement of zeros.

n-level density

Definition

In the setting as before, define the n-*level density* as

$$\mathscr{D}_n(f;\Phi) \;:=\; \sum_{\substack{j_1,\dots,j_n \\ j_i\neq\pm j_k}} \prod_{i=1}^{n} \Phi_i\left(\frac{\gamma_f(j_i)}{2\pi}\log c_f\right).$$

- Computing n-level density for $n > 2$ requires knowledge of distribution of signs of the functional equation of each $L(s,f)$, which is beyond current theory.
- Hughes-Rudnick (2003): introduced n-th centered moments.
  - Similar combiniatorially, but often easier to analyze

Modular Forms

Definition (Modular form of trivial nebentypus)

We write $f \in M_k(q)$ and say f is a *modular form* of level q, even weight k, and trivial nebentypus if $f : \mathbb{H} \to \mathbb{C}$ is holomorphic and

1. for each $\tau \in \Gamma_0(q) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{q} \right\}$ we have

$$f(\tau z) \;:=\; f\left(\frac{az+b}{cz+d}\right) \;=\; (cz+d)^k f(z).$$

2. for $\tau \in \mathrm{SL}_2(\mathbb{Z})$, as $\Im(z) \to +\infty$ we have $(cz+d)^{-k} f(\tau z) \ll 1$.

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With $\tau = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $f(z) = f(z+1)$ so f is 1-periodic and thus has a Fourier expansion at ∞:
$$f(z) \;=\; \sum_{n=0}^{\infty} a_f(n) q^n, \quad q \;=\; e^{2\pi i z}.$$

Holomorphic Cuspforms

Definition (Cuspform)

If $f \in M_k(q)$ vanishes at all cusps of $\Gamma_0(q)$ we say f is a *cuspform* and denote by $\mathcal{S}_k(q) \subset M_k(q)$ the space of holomorphic cuspforms.

- By Atkin-Lehner Theory, we have the orthogonal decomposition

$$\mathcal{S}_k(q) = \mathcal{S}_k^{\text{old}}(q) \oplus \mathcal{S}_k^{\text{new}}(q).$$

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- By Atkin-Lehner Theory, we have the orthogonal decomposition

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- A cuspform $f \in \mathcal{S}_k(q)$ is an eigenfunction of the Hecke operators T_n for $(n, q) = 1$ and $T_n f = \lambda_f(n) f$.

The Space of Cuspidal Newforms

Definition (Newform)

If f is an eigenform of *all* the Hecke operators and the Atkin-Lehner involutions $|_k W(q)$ and $|_k W(Q_p)$ for all the primes $p \mid q$, then we say that f is a *newform* and if, in addition, f is normalized so that $\psi_f(1) = 1$ we say that f is *primitive*.

- The space $\mathcal{S}_k^{\text{new}}(q)$ of newforms has an orthogonal basis $\mathscr{H}_k(q)$ of primitive newforms.
- Trivial nebentypus $\implies$ T_n's are self-adjoint $\implies$ $\lambda_f(n) \in \mathbb{R}$ for all n.

L-functions Attached to Cuspidal Newforms

Fix $f \in \mathcal{S}_k^{\text{new}}(q)$. Then for $\Re(s) > 1$, we define

$$L(s,f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right)^{-1}$$
$$= \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1},$$

where χ_0 is the principal character mod q. Note, $L(s,f)$ can be analytically continued to an entire function on $\mathbb{C}$. Moreover, $L(s,f) = L(s,\overline{f})$.

Katz-Sarnak Density Conjecture for Orthogonal Symmetry

The symmetry type of the family of automorphic L-functions attached to holomorphic cuspidal newforms is orthogonal. Thus, the Katz-Sarnak density conjecture predicts that for test functions Φ whose Fourier transform has arbitrary compact support,

$$\frac{1}{|\mathscr{H}_k(Q)|}\sum_{f\in\mathscr{H}_k(Q)}\mathscr{OD}(f;\Phi) \longrightarrow \int_{-\infty}^{\infty}\Phi(x)W(O)(x)\,dx \quad \text{as } Q\to\infty,$$

where O is the scaling limit of the group of square orthogonal matrices. It has density

$$W(O)(x) \;=\; 1+\frac{1}{2}\delta_0(x),$$

where $\delta_0(x)$ denotes the Dirac delta function at $x=0$.

Extending the Support

Theorem (Iwaniec-Luo-Sarnak '00)

Assume GRH. Then for Φ any even Schwartz function with $\operatorname{supp}(\widehat{\Phi}) \subset (-2,2)$, we have that

$$\lim_{\substack{q\to\infty \\ \square-\text{free}}} \frac{1}{|\mathcal{H}_k(q)|} \sum_{f\in\mathcal{H}_k(q)} \mathcal{OD}(f;\Phi) \;=\; \int_{-\infty}^{\infty} \Phi(x) W(O)(x)\, dx,$$

where O denotes the orthogonal type, showing agreement with the Katz-Sarnak philosophy predictions.

Recent Breakthrough

Theorem (Baluyot-Chandee-Li '23)

Assume GRH. Let Φ *be an even Schwartz function such that* $\mathrm{supp}(\widehat{\Phi}) \subset (-4,4)$, *and let* Ψ *be any smooth function compactly supported on* $\mathbb{R}^+$ *with* $\widehat{\Psi}(0) \neq 0$. *Then we have that*

$$\langle \mathscr{O}\mathscr{D}(f;\Phi)\rangle_* \; := \; \lim_{Q\to\infty} \frac{1}{N(Q)} \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{f\in\mathscr{H}_k(q)}^h \mathscr{O}\mathscr{D}(f;\Phi) \; = \; \int_{-\infty}^{\infty} \Phi(x) W(O)(x) dx,$$

where $N(Q)$ *is a normalizing factor, showing agreement with the Katz-Sarnak philosophy predictions.*

This doubling of support uses averaging over the level (q) to double the support, but many of the necessary manipulations rely on this being the 1-level density.

The n-th Centered Moments of the 1-level Density

We study the n-th centered moments of the 1-level density averaged over levels $q \asymp Q$.

Definition (n-th centered moments of the 1-level density)

In the setting as above, define the n*-th centered moment of the 1-level density* to be

$$\left\langle \prod_{i=1}^{n} \left[\mathscr{OD}(f;\Phi_i) - \langle \mathscr{OD}(f;\Phi_i)\rangle_*\right] \right\rangle_*,$$

where $\langle f \rangle_*$ means averaging f over q as described previously.

Remark

Previous work occassionally split forms based on their sign, $\epsilon(f) \in \{1,-1\}$*; we do not.*

Main Results

Theorem (Cheek-Gilman-Jaber-Miller-Tomé '24)

Assume GRH. For Ψ non-negative and Φ_i even Schwartz functions with $\operatorname{supp}(\widehat{\Phi}) \subset (-\sigma, \sigma)$ and $\sigma \leq \min\left\{\frac{3}{2(n-1)}, \frac{4}{2n-\mathbf{1}_{2\nmid n}}\right\}$ we have that

$$\left\langle \prod_{i=1}^{n} \left(\mathscr{O}\mathscr{D}(f;\Phi_i) - \langle \mathscr{O}\mathscr{D}(f;\Phi_i)\rangle_*\right) \right\rangle_* = \frac{\mathbf{1}_{2|n}}{(n/2)!} \sum_{\tau \in S_n} \prod_{i=1}^{n/2} \int_{-\infty}^{\infty} |u| \widehat{\Phi}_{\tau(2i-1)}(u) \widehat{\Phi}_{\tau(2i)}(u)\, du.$$

As such, our work is a generalization of the BCL '23 $n = 1, \sigma = 4$ result.

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As such, our work is a generalization of the BCL '23 $n = 1, \sigma = 4$ result.

Remark

Notably, for $n = 3$*, we achieve* $\sigma = \sigma_i = 3/4$*, greater than previous best* $\sigma = \sigma_i = 2/3$*.*

Main results ($n = 2$)

Corollary (Cheek-Gilman-Jaber-Miller-Tomé '24)

Let $\sigma_1 = 3/2$ *and* $\sigma_2 = 5/6$. *Then the two-level density*

$$\left\langle \sum_{j_1 \neq \pm j_2} \sum \Phi_1\Big(\gamma_f(j_1)\Big)\Phi_2\Big(\gamma_f(j_2)\Big) \right\rangle_* = 2\int_{-\infty}^{\infty} |u|\widehat{\Phi}_1(u)\widehat{\Phi}_2(u)\,du + \prod_{i=1}^{2}\left(\frac{1}{2}\Phi_i(0) + \widehat{\Phi}_i(0)\right)$$
$$- \Phi_1\Phi_2(0) - 2\widehat{\Phi_1\Phi_2}(0) + \mathscr{O}\mathscr{D}\mathscr{D}\ \Phi_1\Phi_2(0),$$

where $\mathscr{O}\mathscr{D}\mathscr{D} := \langle (1-\epsilon_f)/2\rangle_*$ *denotes the proportion of forms with odd functional equation. This agrees with the predictions from random matrix theory.*

Main results ($n = 2$)

This is the first evidence of an interesting new phenomenon: only by taking different test functions are we able to extend the range in which the Katz-Sarnak density predictions hold. In particular, $\sigma_1 + \sigma_2 = 7/3 \; > \; 2$, where $\sigma_1 + \sigma_2 = 2$ was the previously best known.

Remark

More generally, one can use $\sigma_1 \geq \sigma_2$ such that $\sigma_1 \leq 3/2$ and $\sigma_1 + 3\sigma_2 \leq 4$. The above choice maximizes $\sigma_1 + \sigma_2$.

Duality Between Primes and Zeros of L-functions

Using an explicit formula relating sums over zeros to sums of prime power coefficients of $L(s, f)$, we deduce that

$$\sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi}\log q\right) = \widehat{\Phi}(0) + \frac{1}{2}\Phi(0) - \frac{2}{\log q}\sum_{p\nmid q}\frac{\lambda_f(p)\log p}{\sqrt{p}}\widehat{\Phi}\left(\frac{\log p}{\log q}\right) + O\left(\frac{\log\log q}{\log q}\right).$$

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We use a combinatorial argument together with GRH for $L(s, \mathrm{sym}^2 f)$ to reduce our task to bounding sums over *distinct* primes:

$$\sum_{\substack{p_1,\ldots,p_n\nmid q \\ p_i\neq p_j}} \prod_{i=1}^{n} \frac{\lambda_f(p_i)\log p_i}{\sqrt{p_i}}\widehat{\Phi}_i\left(\frac{\log p_i}{\log q}\right).$$

Averaging Over the Extended Orthogonal Family

We average over $f \in \mathscr{H}_k(q)$ with $q \asymp Q$ and study

$$\frac{1}{N(Q)} \sum_q \Psi\left(\frac{q}{Q}\right) \frac{1}{(\log q)^n} \sum_{f \in \mathscr{H}_k(q)}^h \sum_{\substack{p_1, \ldots, p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^n \frac{\lambda_f(p_i) \log p_i}{\sqrt{p_i}} \widehat{\Phi}_i \left(\frac{\log p_i}{\log q}\right)$$

$$= \frac{1}{N(Q)} \sum_q \Psi\left(\frac{q}{Q}\right) \frac{1}{(\log q)^n} \sum_{\substack{p_1, \ldots, p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^n \frac{\log p_i}{\sqrt{p_i}} \widehat{\Phi}_i \left(\frac{\log p_i}{\log q}\right) \sum_{f \in \mathscr{H}_k(q)}^h \lambda_f(1) \lambda_f\left(\prod_{i=1}^n p_i\right).$$

Trace formulae

- Ng's work allows us to convert sums over $\mathscr{H}_k(q)$ to a linear combination of sums over an orthogonal basis $\mathscr{B}_k(d)$ for the space $\mathcal{S}_k(d)$, $d \mid q$: Morally, if $(m,n,q)=1$ and for A a specific arithmetic function, then

$$\sum_{f\in\mathscr{H}_k(q)}^h \lambda_f(m)\lambda_f(n) = \sum_{\substack{q=L_1L_2d \\ L_1|q_1 \\ L_2|q_2 \\ q_2 \ \square-\text{free}}} A(L_1,L_2,d) \sum_{e|L_2^\infty} \frac{1}{e} \sum_{f\in\mathscr{B}_k(d)}^h \lambda_f(e^2m)\lambda_f(n).$$

Trace formulae

- Ng's work allows us to convert sums over $\mathscr{H}_k(q)$ to a linear combination of sums over an orthogonal basis $\mathscr{B}_k(d)$ for the space $\mathcal{S}_k(d)$, $d \mid q$: Morally, if $(m, n, q) = 1$ and for A a specific arithmetic function, then

$$\sum_{f \in \mathscr{H}_k(q)}^{h} \lambda_f(m)\lambda_f(n) = \sum_{\substack{q = L_1 L_2 d \\ L_1 | q_1 \\ L_2 | q_2 \\ q_2 \ \square-\text{free}}} A(L_1, L_2, d) \sum_{e | L_2^\infty} \frac{1}{e} \sum_{f \in \mathscr{B}_k(d)}^{h} \lambda_f(e^2 m)\lambda_f(n).$$

- Petersson trace formula, a quasi-orthogonality relation for GL_2

$$\sum_{f \in \mathscr{B}_k(d)}^{h} \lambda_f(m)\lambda_f(n) = \delta(m, n) + \sum_{c \geq 1} \frac{S(m, n; cq)}{cq} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{cq}\right).$$

The Kuznetsov Trace Formula

Let $x := \prod p_i$. We are essentially left to analyze

$$\sum_{c \geq 1} \sum_{\substack{p_1, \ldots, p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^{n} \frac{\log p_i}{\sqrt{p_i}} V\left(\frac{p_i}{P_i}\right) \mathrm{e}\left(v_i \frac{p_i}{P_i}\right) \sum_{s} \frac{S(e^2, x; cL_1 rds)}{cL_1 rds} h\left(\frac{4\pi\sqrt{e^2 x}}{cL_1 rds}\right)$$

where V is smooth and compactly supported and h is essentially a smooth truncation of J_{k-1}. We use the Kuznetsov trace formula to convert an average over $f \in \mathscr{B}_k(d)$ into spectral terms:

Holomorphic cuspforms $+$ Maass cuspforms $+$ Eisenstein series.

Origin of restirction on σ

To preform the above manipulations, we technically need to sum over primes $p_1, \ldots, p_n$ without restriction (i.e. not dividing q). For $n = 1$, this is only adding back when $p_1 \mid q$, which is $O(\log Q)$, but when $n > 1$, we need to add back $p_1 \mid q, p_2, \ldots, p_n \nmid q$, so this is adding back more than $Q^{n-1-\epsilon}$ many terms. This results in the $\sigma \leq \frac{3}{2(n-1)}$ restriction.

To analyze the terms from Holomorphic and Maass cuspforms, similar techniques require $\sigma \leq \frac{4}{n}$ (the expected bound; the sum of supports is 4). On the other hand, a contour shift for the Eisenstein series term no longer in general achieves any cancellation with n even and only minimal cancellation with n odd. Thus, we need $\sigma \leq \frac{4}{2n - \mathbf{1}_{2\nmid n}}$.

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