How Low Can We Go? Understanding Zeros of L-Functions Near The Central Point

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Introduction

Why study zeros of L-functions?

- Infinitude of primes, primes in arithmetic progression.
- Chebyshev's bias: $\pi_{3,4}(x) \geq \pi_{1,4}(x)$ 'most' of the time.
- Birch and Swinnerton-Dyer conjecture.
- Goldfeld, Gross-Zagier: bound for $h(D)$ from L-functions with many central point zeros.
- Even better estimates for $h(D)$ if a positive percentage of zeros of $\zeta(s)$ are at most $1/2 - \epsilon$ of the average spacing to the next zero.

Distribution of zeros

- $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- GRH: error terms.
- GSH: Chebyshev's bias.
- Analytic rank, adjacent spacings: $h(D)$.

Goals

- Determine correct scale and statistics to study zeros of L-functions.

- See similar behavior in different systems (random matrix theory).

- Discuss the tools and techniques needed.

- Explain the repulsion of zeros / lower order terms near the central point.

- Bound order of vanishing at the central point.

Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.

2. Develop an explicit formula relating what we want to study to something we understand.

3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Introduction to L-Functions

Riemann Zeta Function

$$\zeta(s) \;=\; \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \operatorname{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) \;=\; \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) \;=\; \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\operatorname{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) \;=\; \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} \;=\; \prod_{p \text{ prime}} L_p\left(s,f\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f)L(s,f) \;=\; \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

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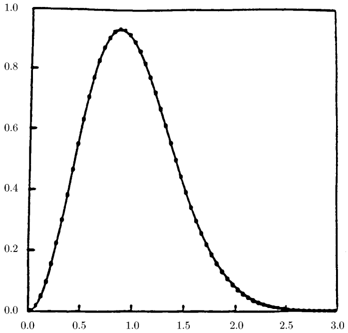
Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Zeros of $\zeta(s)$ vs GUE



70 million spacings b/w adjacent zeros of $\zeta(s)$, starting at the $10^{20\text{th}}$ zero (from Odlyzko).

Explicit Formula (Contour Integration)

$$
\begin{aligned}
-\frac{\zeta'(s)}{\zeta(s)} &= -\frac{\mathrm{d}}{\mathrm{d}s}\log\zeta(s) = -\frac{\mathrm{d}}{\mathrm{d}s}\log\prod_p\left(1-p^{-s}\right)^{-1} \\
&= \frac{\mathrm{d}}{\mathrm{d}s}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p \cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
\end{aligned}
$$

Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)p^{-s}ds.
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\end{aligned}
$$

Contour Integration (see Fourier Transform arising):

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\,\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s) e^{-\sigma\log p} e^{-it\log p} ds.
$$

Knowledge of zeros gives info on coefficients.

Explicit Formula: Examples

Riemann Zeta Function: Let $\sum_\rho$ denote the sum over the zeros of $\zeta(s)$ in the critical strip, g an even Schwartz function of compact support and $\phi(r) = \int_{-\infty}^{\infty} g(u) e^{iru} du$. Then

$$\begin{aligned} \sum_\rho \phi(\gamma_\rho) &= 2\phi\left(\frac{i}{2}\right) - \sum_p \sum_{k=1}^{\infty} \frac{2\log p}{p^{k/2}} g\left(k \log p\right) \\ &\quad + \frac{1}{\pi}\int_{-\infty}^{\infty} \left(\frac{1}{iy - \frac{1}{2}} + \frac{\Gamma'(\frac{iy}{2}+\frac{5}{4})}{\Gamma(\frac{iy}{2}+\frac{5}{4})} - \frac{1}{2}\log\pi\right)\phi(y)\, dy. \end{aligned}$$

Explicit Formula: Examples

Dirichlet L-functions: Let h be an even Schwartz function and $L(s,\chi) = \sum_n \chi(n)/n^s$ a Dirichlet L-function from a non-trivial character χ with conductor m and zeros $\rho = \frac{1}{2} + i\gamma_\chi$; if the Generalized Riemann Hypothesis is true then $\gamma \in \mathbb{R}$. Then

$$\begin{aligned}\sum_\rho h\left(\gamma_\rho \frac{\log(m/\pi)}{2\pi}\right) \ &= \ \int_{-\infty}^{\infty} h(y)dy \\ &-2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(\frac{\log p}{\log(m/\pi)}\right) \frac{\chi(p)}{p^{1/2}} \\ &-2\sum_p \frac{\log p}{\log(m/\pi)} \widehat{h}\left(2\frac{\log p}{\log(m/\pi)}\right) \frac{\chi^2(p)}{p} + O\Big(\frac{1}{\log m}\Big).\end{aligned}$$

Explicit Formula: Examples

Cuspidal Newforms: Let $\mathscr{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\begin{aligned}
\frac{1}{|\mathscr{F}|}\sum_{f\in\mathscr{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) &= \widehat{\phi}(0)+\frac{1}{2}\phi(0)-\frac{1}{|\mathscr{F}|}\sum_{f\in\mathscr{F}}P(f;\phi) \\
&\quad + O\left(\frac{\log\log R}{\log R}\right) \\
P(f;\phi) &= \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.
\end{aligned}$$

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\dots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i \neq j_k\right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

1. Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
2. 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
3. n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).
4. n-level correlations for the classical compact groups (Katz-Sarnak).
5. Insensitive to any finite set of zeros.

Measures of Spacings: n-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) = \sum_{\substack{j_1,\dots,j_n \\ \text{distinct}}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

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1. Individual zeros contribute in limit.
2. Most of contribution is from low zeros.
3. Average over similar curves (family).

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Katz-Sarnak Conjecture

For a 'nice' family of L-functions, the n-level density depends only on a symmetry group attached to the family.

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathscr{F}_N|^{-1}\sum_{f\in\mathscr{F}_N}\log C_f$). Hope: ϕ a good even test function with compact support, as $|\mathscr{F}| \to \infty$,

$$\begin{aligned}\frac{1}{|\mathscr{F}_N|}\sum_{f\in\mathscr{F}_N} D_{n,f}(\phi) &= \frac{1}{|\mathscr{F}_N|}\sum_{f\in\mathscr{F}_N}\sum_{\substack{j_1,\dots,j_n\\ j_i\neq\pm j_k}}\prod_i \phi_i\left(\frac{\log C_f}{2\pi}\gamma_E^{(j_i)}\right)\\ &\to \int\cdots\int \phi(x)W_{n,\mathscr{G}(\mathscr{F})}(x)dx.\end{aligned}$$

Katz-Sarnak Conjecture

As $C_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1 \end{cases}$$

Some Number Theory Results

- Orthogonal: Iwaniec-Luo-Sarnak, Ricotta-Royer: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign).
- Symplectic: Rubinstein, Gao, Levinson-Miller, and Entin, Roddity-Gershon and Rudnick: n-level densities for twists $L(s, \chi_d)$ of the zeta-function.
- Unitary: Fiorilli-Miller, Hughes-Rudnick: Families of Primitive Dirichlet Characters.
- Orthogonal: Miller, Young: One and two-parameter families of elliptic curves.

Some Results II: Rankin-Selberg Convolution of Families

Symmetry constant: $c_{\mathscr{L}} = 0$ (resp, 1 or -1) if family $\mathscr{L}$ has unitary (resp, symplectic or orthogonal) symmetry.

Rankin-Selberg convolution: Satake parameters for $\pi_{1,p} \times \pi_{2,p}$ are

$$\{\alpha_{\pi_1 \times \pi_2}(k)\}_{k=1}^{nm} \ = \ \{\alpha_{\pi_1}(i) \cdot \alpha_{\pi_2}(j)\}_{\substack{1 \le i \le n \\ 1 \le j \le m}}.$$

Theorem (Dueñez-Miller)

If $\mathscr{F}$ and $\mathscr{G}$ are *nice* families of L-functions, then $c_{\mathscr{F} \times \mathscr{G}} = c_{\mathscr{F}} \cdot c_{\mathscr{G}}$.

1-Level Density

Assuming conductors constant in family $\mathcal{F}$, have to study

$$\begin{aligned}
\lambda_f(p^\nu) &= \alpha_{f,1}(p)^\nu + \cdots + \alpha_{f,n}(p)^\nu \\
S_1(\mathcal{F}) &= -2\sum_p \hat{g}\left(\frac{\log p}{\log R}\right)\frac{\log p}{\sqrt{p}\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p)\right] \\
S_2(\mathcal{F}) &= -2\sum_p \hat{g}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p\log R}\left[\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2)\right]
\end{aligned}$$

The corresponding classical compact group is determined by

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\lambda_f(p^2) = c_{\mathcal{F}} = \begin{cases} 0 & \text{Unitary} \\ 1 & \text{Symplectic} \\ -1 & \text{Orthogonal.} \end{cases}$$

1-Level Density for Rankin-Selberg Convolution of Families

Families $\mathscr{F}$ and $\mathscr{G}$.
Satake parameters $\{\alpha_{f,i}(p)\}_{i=1}^{n}$ and $\{\beta_{g,j}(p)\}_{j=1}^{m}$.
Family $\mathscr{F} \times \mathscr{G}$, $L(s, f \times g)$ has parameters $\{\alpha_{f,i}(p)\beta_{g,j}(p)\}_{i=1\dots n, j=1\dots m}$.

$$\begin{aligned} a_{f\times g}(p^\nu) &= \sum_{i=1}^{n}\sum_{j=1}^{m} \alpha_{f,i}(p)^\nu \beta_{g,j}(p)^\nu \\ &= \sum_{i=1}^{n} \alpha_{f,i}(p)^\nu \sum_{j=1}^{m} \beta_{g,j}(p)^\nu \\ &= \lambda_f(p^\nu)\cdot\lambda_g(p^\nu). \end{aligned}$$

Technical restriction: need f and g unrelated (i.e., g is not the contragredient of f) for our applications.

1-Level Density for Rankin-Selberg Convolution of Families (cont)

To analyze $S_\nu(\mathscr{F} \times \mathscr{G})$ we must study

$$\frac{1}{|\mathscr{F} \times \mathscr{G}|} \sum_{f \times g \in \mathscr{F} \times \mathscr{G}} \lambda_f(p^\nu) \cdot \lambda_g(p^\nu) = \left[\frac{1}{|\mathscr{F}|} \sum_{f \in \mathscr{F}} \lambda_f(p^\nu)\right] \cdot \left[\frac{1}{|\mathscr{G}|} \sum_{g \in \mathscr{G}} \lambda_g(p^\nu)\right]$$

- $\nu = 1$: If one of the families is rank zero, so is $\mathscr{F} \times \mathscr{G}$; $S_1(\mathscr{F} \times \mathscr{G})$ will not contribute.
- $\nu = 2$: $c_{\mathscr{F} \times \mathscr{G}} \; = \; c_{\mathscr{F}} \cdot c_{\mathscr{G}}$.

Proves if each family is of rank 0, the symmetry type of the convolution is the product of the symmetry types. $\square$

Applications of n-level density

One application: bounding the order of vanishing at the central point.

$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathscr{F})}(x) dx$ if ϕ non-negative.

Applications of n-level density

One application: bounding the order of vanishing at the central point.
$\text{Average rank} \cdot \phi(0) \ \leq \ \int \phi(x) W_{G(\mathscr{F})}(x) dx$ if ϕ non-negative.
Can also use to bound the percentage that vanish to order r for any r.

Theorem (Miller, Hughes-Miller)

Using n-level arguments, for the family of cuspidal newforms of prime level $N \to \infty$ (split or not split by sign), for any r there is a c_r such that probability of at least r zeros at the central point is at most $c_n r^{-n}$.

Better results using 2-level than Iwaniec-Luo-Sarnak using the 1-level for $r \geq 5$.

Correspondences

Similarities between L-Functions and Nuclei:

$$\text{Zeros} \quad \longleftrightarrow \quad \text{Energy Levels}$$

$$\text{Schwartz test function} \quad \longrightarrow \quad \text{Neutron}$$

$$\text{Support of test function} \quad \longleftrightarrow \quad \text{Neutron Energy.}$$

Similar to Central Limit Theorem: Main term from first and second moment, higher moments rate of convergence / lower order terms.

Example:
Dirichlet L-functions

Dirichlet Characters (m prime)

$(\mathbb{Z}/m\mathbb{Z})^*$ is cyclic of order $m-1$ with generator g. Let $\zeta_{m-1} = e^{2\pi i/(m-1)}$. The principal character χ_0 is given by

$$\chi_0(k) = \begin{cases} 1 & (k,m) = 1 \\ 0 & (k,m) > 1. \end{cases}$$

The $m-2$ primitive characters are determined (by multiplicativity) by action on g.

As each $\chi : (\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$, for each χ there exists an l such that $\chi(g) = \zeta_{m-1}^{l}$. Hence for each l, $1 \leq l \leq m-2$ we have

$$\chi_l(k) = \begin{cases} \zeta_{m-1}^{la} & k \equiv g^a(m) \\ 0 & (k,m) > 0 \end{cases}$$

Dirichlet L-Functions

Let χ be a primitive character mod m. Let

$$c(m,\chi) = \sum_{k=0}^{m-1} \chi(k) e^{2\pi i k/m}.$$

$c(m,\chi)$ is a Gauss sum of modulus $\sqrt{m}$.

$$\begin{aligned} L(s,\chi) &= \prod_p (1-\chi(p)p^{-s})^{-1} \\ \Lambda(s,\chi) &= \pi^{-\frac{1}{2}(s+\epsilon)} \Gamma\left(\frac{s+\epsilon}{2}\right) m^{\frac{1}{2}(s+\epsilon)} L(s,\chi), \end{aligned}$$

where

$$\epsilon = \begin{cases} 0 & \text{if } \chi(-1) = 1 \\ 1 & \text{if } \chi(-1) = -1 \end{cases}$$

Explicit Formula

Let ϕ be an even Schwartz function with compact support $(-\sigma, \sigma)$, let χ be a non-trivial primitive Dirichlet character of conductor m.

$$\begin{aligned}
&\sum \phi\left(\gamma \frac{\log(\frac{m}{\pi})}{2\pi}\right) \\
= &\int_{-\infty}^{\infty} \phi(y)dy \\
&- \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right)[\chi(p) + \bar{\chi}(p)]p^{-\frac{1}{2}} \\
&- \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)[\chi^2(p) + \bar{\chi}^2(p)]p^{-1} \\
&+ O\left(\frac{1}{\log m}\right).
\end{aligned}$$

Expansion

$\{\chi_0\} \cup \{\chi_l\}_{1 \le l \le m-2}$ are all the characters mod m.
Consider the family of primitive characters mod a prime m ($m-2$ characters):

$$\begin{aligned} & \int_{-\infty}^{\infty} \phi(y) dy \\ - & \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}} \\ - & \frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_p \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right) [\chi^2(p) + \bar{\chi}^2(p)] p^{-1} \\ + & O\left(\frac{1}{\log m}\right). \end{aligned}$$

Note can pass Character Sum through Test Function.

Character Sums

$$\sum_{\chi} \chi(k) = \begin{cases} m-1 & k \equiv 1(m) \\ 0 & \text{otherwise.} \end{cases}$$

For any prime $p \neq m$

$$\sum_{\chi \neq \chi_0} \chi(p) \;=\; \begin{cases} -1 + m - 1 & p \equiv 1(m) \\ -1 & \text{otherwise.} \end{cases}$$

Substitute into

$$\frac{1}{m-2} \sum_{\chi \neq \chi_0} \sum_{p} \frac{\log p}{\log(m/\pi)} \widehat{\phi}\left(\frac{\log p}{\log(m/\pi)}\right) [\chi(p) + \bar{\chi}(p)] p^{-\frac{1}{2}}$$

First Sum: no contribution if $\sigma < 2$

$$\frac{-2}{m-2}\sum_{p}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac12}$$

$$+\quad 2\frac{m-1}{m-2}\sum_{p\equiv 1(m)}^{m^\sigma}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\Big(\frac{\log p}{\log(m/\pi)}\Big)p^{-\frac12}$$

$$\ll\quad \frac{1}{m}\sum_{p}^{m^\sigma}p^{-\frac12}+\sum_{p\equiv 1(m)}^{m^\sigma}p^{-\frac12}\ll\frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac12}+\sum_{\substack{k\equiv 1(m)\\ k\geq m+1}}^{m^\sigma}k^{-\frac12}$$

$$\ll\quad \frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac12}+\frac{1}{m}\sum_{k}^{m^\sigma}k^{-\frac12}\ll\frac{1}{m}m^{\sigma/2}.$$

Second Sum

$$\frac{1}{m-2}\sum_{\chi\neq\chi_0}\sum_{p}\frac{\log p}{\log(m/\pi)}\widehat{\phi}\left(2\frac{\log p}{\log(m/\pi)}\right)\frac{\chi^2(p)+\bar{\chi}^2(p)}{p}.$$

$$\sum_{\chi\neq\chi_0}[\chi^2(p)+\bar{\chi}^2(p)] \;=\; \begin{cases} 2(m-2) & p\equiv\pm1(m) \\ -2 & p\not\equiv\pm1(m)\end{cases}$$

Up to $O\left(\frac{1}{\log m}\right)$ we find that

$$\ll \quad \frac{1}{m-2}\sum_{p}^{m^{\sigma/2}} p^{-1} \;+\; \frac{2m-2}{m-2}\sum_{p\equiv\pm1(m)}^{m^{\sigma/2}} p^{-1}$$

$$\ll \quad \frac{1}{m-2}\sum_{k}^{m^{\sigma/2}} k^{-1} \;+\; \sum_{k\equiv1(m)}^{m^{\sigma/2}} k^{-1} \;+\; \sum_{k\equiv-1(m)}^{m^{\sigma/2}} k^{-1}$$

Summary

Agrees with Unitary for $\sigma < 2$ for square-free $m \in [N, 2N]$ (larger support related to distribution of primes congruent to 1).

Theorem

- *m square-free odd integer with $r = r(m)$ factors;*
- $m = \prod_{i=1}^{r} m_i$;
- $M_2 = \prod_{i=1}^{r}(m_i - 2)$.

Then family $\mathscr{F}_m$ of primitive characters mod m has

$$\begin{aligned} \textit{First Sum} &\ll \frac{1}{M_2} 2^r m^{\frac{1}{2}\sigma} \\ \textit{Second Sum} &\ll \frac{1}{M_2} 3^r m^{\frac{1}{2}}. \end{aligned}$$

Cuspidal Newforms

Joint with Chris Hughes, several Williams College SMALL REUs (ongoing); removing square-free with Owen Barrett, Paula Burkhardt, Jon DeWitt and Robert Dorward)

Results from Iwaniec-Luo-Sarnak

- Orthogonal: Iwaniec-Luo-Sarnak: 1-level density for holomorphic even weight k cuspidal newforms of square-free level N (SO(even) and SO(odd) if split by sign) in $(-2, 2)$.

- Symplectic: Iwaniec-Luo-Sarnak: 1-level density for $\text{sym}^2(f)$, f holomorphic cuspidal newform.

Will review Orthogonal case and talk about extensions (joint with Chris Hughes, SMALL REUs).

Modular Form Preliminaries

$$\Gamma_0(N) \;=\; \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} ad - bc \;=\; 1 \\ c \equiv 0(N) \end{array} \right\}$$

f is a weight k holomorphic cuspform of level N if

$$\forall \gamma \in \Gamma_0(N), \;\; f(\gamma z) \;=\; (cz + d)^k f(z).$$

- Fourier Expansion: $f(z) \;=\; \sum_{n=1}^{\infty} a_f(n) e^{2\pi i z}$, $L(s, f) = \sum_{n=1}^{\infty} a_n n^{-s}$.
- Petersson Norm: $\langle f, g \rangle \;=\; \int_{\Gamma_0(N)\backslash \mathbf{H}} f(z)\overline{g(z)} y^{k-2} dx dy$.
- Normalized coefficients:

$$\psi_f(n) \;=\; \sqrt{\frac{\Gamma(k-1)}{(4\pi n)^{k-1}} \frac{1}{||f||}} a_f(n).$$

Modular Form Preliminaries: Petersson Formula

$B_k(N)$ an orthonormal basis for weight k level N. Define

$$\Delta_{k,N}(m,n) \;=\; \sum_{f \in B_k(N)} \psi_f(m)\overline{\psi}_f(n).$$

Petersson Formula

$$\Delta_{k,N}(m,n) \quad = \quad 2\pi i^k \sum_{c \equiv 0(N)} \frac{S(m,n,c)}{c} J_{k-1}\left(4\pi\frac{\sqrt{mn}}{c}\right) + \;\delta(m,n).$$

Modular Form Preliminaries: Explicit Formula

Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\begin{aligned}
\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) &= \widehat{\phi}(0)+\frac{1}{2}\phi(0)-\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}P(f;\phi) \\
&\quad + O\left(\frac{\log\log R}{\log R}\right) \\
P(f;\phi) &= \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.
\end{aligned}$$

Modular Form Preliminaries:Fourier Coefficient Review

$$\begin{aligned}\lambda_f(n) &= a_f(n)n^{\frac{k-1}{2}} \\ \lambda_f(m)\lambda_f(n) &= \sum_{\substack{d|(m,n)\\(d,M)=1}} \lambda_f\left(\frac{mn}{d}\right).\end{aligned}$$

For a newform of level N, $\lambda_f(N)$ is trivially related to the sign of the form:

$$\epsilon_f = i^k \mu(N)\lambda_f(N)\sqrt{N}.$$

The above will allow us to split into even and odd families: $1 \pm \epsilon_f$.

Key Kloosterman-Bessel integral from ILS

Ramanujan sum:

$$R(n,q) \ = \ \sum_{a \bmod q}^{*} e(an/q) \ = \ \sum_{d|(n,q)} \mu(q/d)d,$$

where $*$ restricts the summation to be over all a relatively prime to q.

Theorem (ILS)

Let Ψ be an even Schwartz function with $\mathrm{supp}(\widehat{\Psi}) \subset (-2,2)$. Then

$$\sum_{m \le N^{\epsilon}} \frac{1}{m^2} \sum_{(b,N)=1} \frac{R(m^2,b)R(1,b)}{\varphi(b)} \int_{y=0}^{\infty} J_{k-1}(y)\widehat{\Psi}\left(\frac{2\log(by\sqrt{N}/4\pi m)}{\log R}\right)\frac{dy}{\log R}$$
$$= -\frac{1}{2}\left[\int_{-\infty}^{\infty} \Psi(x)\frac{\sin 2\pi x}{2\pi x}dx - \frac{1}{2}\Psi(0)\right] + O\left(\frac{k \log\log kN}{\log kN}\right),$$

where $R = k^2N$ and φ is Euler's totient function.

Limited Support ($\sigma < 1$): Sketch of proof

- Estimate Kloosterman-Bessel terms trivially.
  $\diamond$ Kloosterman sum: $d\bar{d} \equiv 1 \bmod q$, $\tau(q)$ is the number of divisors of q,

$$S(m,n;q) \;=\; \sum_{d \bmod q}^{*} e\left(\frac{md}{q} + \frac{n\bar{d}}{q}\right)$$

$$|S(m,n;q)| \;\leq\; (m,n,q)\sqrt{\min\left\{\frac{q}{(m,q)}, \frac{q}{(n,q)}\right\}}\;\tau(q).$$

  $\diamond$ Bessel function: integer $k \geq 2$,
  $J_{k-1}(x) \ll \min\left(x, x^{k-1}, x^{-1/2}\right)$.

- Use Fourier Coefficients to split by sign: N fixed:
  $\pm \sum_f \lambda_f(N) * (\cdots)$.

2-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)J_{k-1}\left(4\pi\frac{\sqrt{m^2x_1x_2N}}{c}\right)\frac{dx_1dx_2}{\sqrt{x_1x_2}}$$

Change of variables and Jacobian:

$$\begin{aligned} u_2 &= x_1x_2 & x_2 &= \frac{u_2}{u_1} \\ u_1 &= x_1 & x_1 &= u_1 \end{aligned}$$

$$\left|\frac{\partial x}{\partial u}\right| = \left|\begin{matrix} 1 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1}\end{matrix}\right| = \frac{1}{u_1}.$$

Left with

$$\int\int\widehat{\phi}\left(\frac{\log u_1}{\log R}\right)\widehat{\phi}\left(\frac{\log\left(\frac{u_2}{u_1}\right)}{\log R}\right)\frac{1}{\sqrt{u_2}}J_{k-1}\left(4\pi\frac{\sqrt{m^2u_2N}}{c}\right)\frac{du_1du_2}{u_1}$$

2-Level Density

Changing variables, u_1-integral is

$$\int_{w_1=\frac{\log u_2}{\log R}-\sigma}^{\sigma} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right) dw_1.$$

Support conditions imply

$$\Psi_2\left(\frac{\log u_2}{\log R}\right) = \int_{w_1=-\infty}^{\infty} \widehat{\phi}(w_1)\,\widehat{\phi}\left(\frac{\log u_2}{\log R}-w_1\right) dw_1.$$

Substituting gives

$$\int_{u_2=0}^{\infty} J_{k-1}\left(4\pi\frac{\sqrt{m^2 u_2 N}}{c}\right)\Psi_2\left(\frac{\log u_2}{\log R}\right)\frac{du_2}{\sqrt{u_2}}$$

3-Level Density

$$\int_{x_1=2}^{R^\sigma}\int_{x_2=2}^{R^\sigma}\int_{x_3=2}^{R^\sigma}\widehat{\phi}\left(\frac{\log x_1}{\log R}\right)\widehat{\phi}\left(\frac{\log x_2}{\log R}\right)\widehat{\phi}\left(\frac{\log x_3}{\log R}\right)$$

$$* \quad J_{k-1}\left(4\pi\frac{\sqrt{m^2 x_1 x_2 x_3 N}}{c}\right)\frac{dx_1\,dx_2\,dx_3}{\sqrt{x_1 x_2 x_3}}$$

Change variables as below and get Jacobian:

$$\begin{array}{rclcrcl} u_3 & = & x_1 x_2 x_3 & \qquad & x_3 & = & \frac{u_3}{u_2} \\ u_2 & = & x_1 x_2 & & x_2 & = & \frac{u_2}{u_1} \\ u_1 & = & x_1 & & x_1 & = & u_1 \end{array}$$

$$\left|\frac{\partial x}{\partial u}\right| = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{u_2}{u_1^2} & \frac{1}{u_1} & 0 \\ 0 & -\frac{u_3}{u_2^2} & \frac{1}{u_2} \end{vmatrix} = \frac{1}{u_1 u_2}.$$

n-Level Density: Determinant Expansions from RMT

- U(N), $\mathrm{U}_k(N)$: $\det\left(K_0(x_j,x_k)\right)_{1\le j,k\le n}$
- USp(N): $\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\le n}$
- SO(even): $\det\left(K_1(x_j,x_k)\right)_{1\le j,k\le n}$
- SO(odd): $\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\le n} + \sum_{\nu=1}^{n}\delta(x_\nu)\det\left(K_{-1}(x_j,x_k)\right)_{1\le j,k\ne\nu\le n}$

  where

$$K_\epsilon(x,y) = \frac{\sin\left(\pi(x-y)\right)}{\pi(x-y)} + \epsilon\frac{\sin\left(\pi(x+y)\right)}{\pi(x+y)}.$$

n-Level Density: Sketch of proof

Expand Bessel-Kloosterman piece, use GRH to drop non-principal characters, change variables, main term is

$$\frac{b\sqrt{N}}{2\pi m}\int_0^\infty J_{k-1}(x)\widehat{\Phi_n}\left(\frac{2\log(bx\sqrt{N}/4\pi m)}{\log R}\right)\frac{\mathrm{d}x}{\log R}$$

with $\Phi_n(x) = \phi(x)^n$.

Main Idea

Difficulty in comparison with classical RMT is that instead of having an n-dimensional integral of $\phi_1(x_1)\cdots\phi_n(x_n)$ we have a 1-dimensional integral of a new test function. This leads to harder combinatorics but allows us to appeal to the result from ILS.

Support for n-Level Density

Careful book-keeping gives $\sigma_n < \frac{1}{n-1}$.

n-Level Density is trivial for $\sigma_n < \frac{1}{n}$, non-trivial up to $\frac{1}{n-1}$.

Expected $\frac{2}{n}$. Obstruction from partial summation on primes.

New terms emerge at $\frac{1}{n-k}$.

Elliptic Curves: First Zero Above Central Point

Eduardo Dueñez, Duc Khim Huynh, Jon P. Keating and Nina Snaith, extending with Owen Barrett.

Theoretical results: $y^2 = x^3 + A(T)x + B(T)$

Theorem: M– '04

For small support, one-param family of rank r over $\mathbb{Q}(T)$:

$$\lim_{N\to\infty} \frac{1}{|\mathscr{F}_N|} \sum_{E_t \in \mathscr{F}_N} \sum_j \varphi\left(\frac{\log C_{E_t}}{2\pi} \gamma_{E_t,j}\right) = \int \varphi(x)\rho_{\mathscr{G}}(x)dx + r\varphi(0)$$

$$\text{where} \quad \mathscr{G} = \begin{cases} \mathrm{SO(odd)} & \text{if half odd} \\ \mathrm{SO(even)} & \text{if all even} \\ \text{weighted average} & \text{otherwise.} \end{cases}$$

Supports Katz-Sarnak, B-SD, and Independent model in limit.

Independent Model:

$$\mathscr{A}_{2N,2r} = \left\{ \begin{pmatrix} I_{2r\times 2r} & \\ & g \end{pmatrix} : g \in SO(2N-2r) \right\}.$$

Interesting Families

Let $\mathscr{E} : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank r over $\mathbb{Q}(T)$.
Natural sub-families:

- Curves of rank r.
- Curves of rank $r + 2$.

Interesting Families

Let $\mathscr{E} : y^2 = x^3 + A(T)x + B(T)$ be a one-parameter family of elliptic curves of rank r over $\mathbb{Q}(T)$.
Natural sub-families:

- Curves of rank r.
- Curves of rank $r + 2$.

Question: Does the sub-family of rank $r + 2$ curves in a rank r family behave like the sub-family of rank $r + 2$ curves in a rank $r + 2$ family?

Equivalently, does it matter how one conditions on a curve being rank $r + 2$?

Testing Random Matrix Theory Predictions

Know the right model for large conductors, searching for the correct model for finite conductors.

In the limit must recover the independent model, and want to explain data on:

1. **Excess Rank:** Rank r one-parameter family over $\mathbb{Q}(T)$: observed percentages with rank $\geq r + 2$.
2. **First (Normalized) Zero above Central Point:** Influence of zeros at the central point on the distribution of zeros near the central point.

Excess Rank

One-parameter family, rank r over $\mathbb{Q}(T)$.
Density Conjecture (Generic Family) $\Longrightarrow$ 50% rank r, r+1.

For many families, observe

Percent with rank r $\approx$ 32%
Percent with rank r+1 $\approx$ 48%
Percent with rank r+2 $\approx$ 18%
Percent with rank r+3 $\approx$ 2%

Problem: small data sets, sub-families, convergence rate $\log$(conductor).

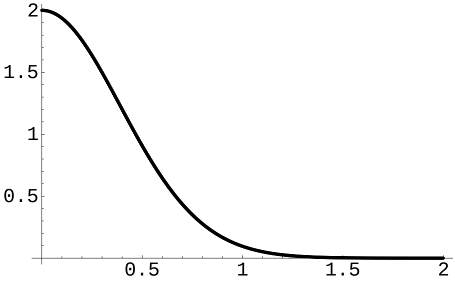
Excess Rank

$$y^2 + y = x^3 + Tx$$

Each set is 2000 curves, last has conductors of size 10^{17}, (small on logarithmic scale).

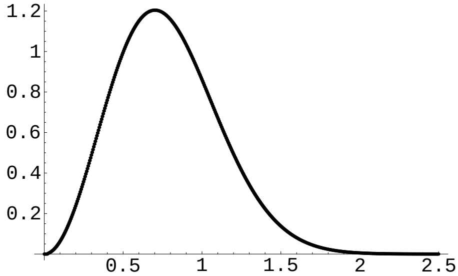
t-Start	Rk 0	Rk 1	Rk 2	Rk 3	Time (hrs)
-1000	39.4	47.8	12.3	0.6	<1
1000	38.4	47.3	13.6	0.6	<1
4000	37.4	47.8	13.7	1.1	1
8000	37.3	48.8	12.9	1.0	2.5
24000	35.1	50.1	13.9	0.8	6.8
50000	36.7	48.3	13.8	1.2	51.8

RMT: Theoretical Results ($N \to \infty$)



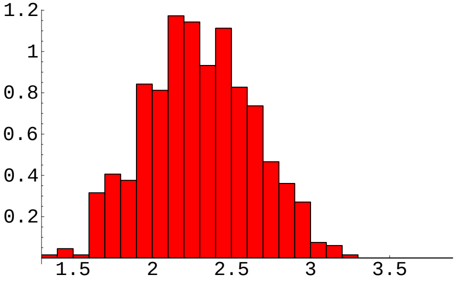
1st normalized evalue above 1: SO(even)

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(odd)

Rank 2 Curves: 1st Norm. Zero above the Central Point

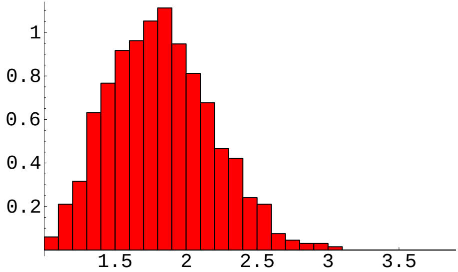


665 rank 2 curves from

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6.$$

$\log(\text{cond}) \in [10, 10.3125]$, median $= 2.29$, mean $= 2.30$

Rank 2 Curves: 1st Norm. Zero above the Central Point

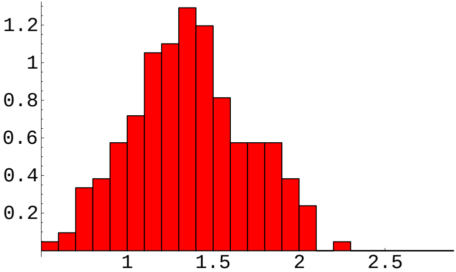


665 rank 2 curves from

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$

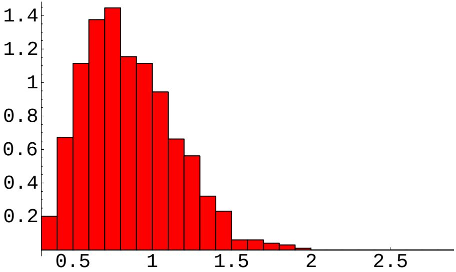
$\log(\text{cond}) \in [16, 16.5]$, median $= 1.81$, mean $= 1.82$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0



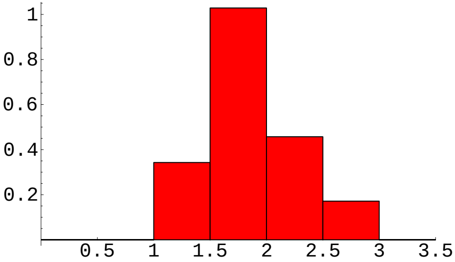
209 rank 0 curves from 14 rank 0 families,
$\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0



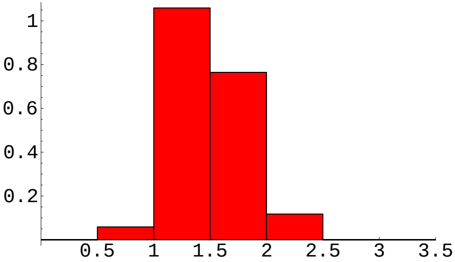
996 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$)
1st Normalized Zero above Central Point



35 curves, $\log(\text{cond}) \in [7.8, 16.1]$, $\widetilde{\mu} = 1.85$, $\mu = 1.92$, $\sigma_\mu = .41$

Rank 2 Curves from $y^2 = x^3 - T^2x + T^2$ (Rank 2 over $\mathbb{Q}(T)$) 1st Normalized Zero above Central Point



34 curves, $\log(\text{cond}) \in [16.2, 23.3]$, $\widetilde{\mu} = 1.37$, $\mu = 1.47$, $\sigma_\mu = .34$

Summary of Data

- The repulsion of the low-lying zeros increased with increasing rank, and was present even for rank 0 curves.
- As the conductors increased, the repulsion decreased.
- Statistical tests failed to reject the hypothesis that, on average, the first three zeros were all repelled equally (i.e., shifted by the same amount).

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- z_j = imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	701 Rank 2 Curves	t-Statistic
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	23 Rank 4 Curves	t-Statistic
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	64 Rank 2 Curves	t-Statistic
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

New Model for Finite Conductors

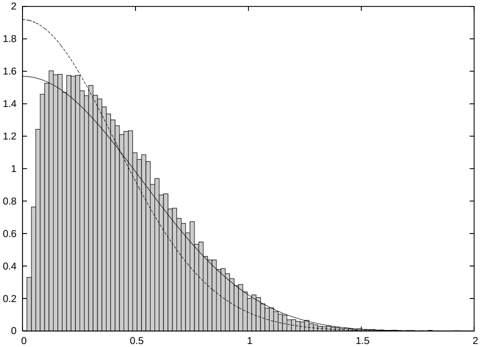
- Replace conductor N with $N_{\text{effective}}$.
  - ◇ Arithmetic info, predict with L-function Ratios Conj.
  - ◇ Do the number theory computation.

- Excised Orthogonal Ensembles.
  - ◇ $L(1/2, E)$ discretized.
  - ◇ Char. polys $\Lambda_A(\theta) = \det(I - e^{i\theta}A^{-1})$ model $L(1/2 + it, E)$.
  - ◇ Study matrices in $\mathrm{SO}(2N_{\mathit{eff}})$ with $|\Lambda_A(0)| \geq ce^N$.

- Painlevé VI differential equation solver.
  - ◇ Use explicit formulas for densities of Jacobi ensembles.
  - ◇ Key input: Selberg-Aomoto integral for initial conditions.

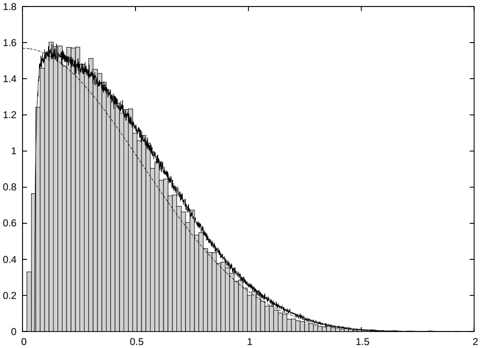
Generalizing with Owen Barrett.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with $N_{\rm eff}$ (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): N_{eff} = 2 (solid) with discretisation, and $N_{\text{eff}} = 2.32$ (dashed) without discretisation.

The lowest eigenvalue of Jacobi Random Matrix Ensembles and Painlevé VI, (with E. Dueñez, D. K. Huynh, J. Keating and N. Snaith), Journal of Physics A: Mathematical and Theoretical **43** (2010) 405204 (27pp). http://arxiv.org/pdf/1005.1298.
Models for zeros at the central point in families of elliptic curves (with E. Dueñez, D. K. Huynh, J. Keating and N. Snaith), J. Phys. A: Math. Theor. **45** (2012) 115207 (32pp). http://arxiv.org/pdf/1107.4426.

Bias Conjecture for Moments of Fourier Coefficients of Elliptic Curve L-functions

Joint with Megumi Asada, Ryan Chen, Eva Fourakis, Yujin Kim, Andrew Kwon, Jared Lichtman, Blake Mackall, Eric Winsor, Karl Winsor, Roger Weng, Michelle Wu, Jianing Yang and Kevin Yang

Families and Moments

A *one-parameter family* of elliptic curves is given by

$$\mathscr{E} : y^2 \ = \ x^3 + A(T)x + B(T)$$

where $A(T), B(T)$ are polynomials in $\mathbb{Z}[T]$.

- Each specialization of T to an integer t gives an elliptic curve $\mathscr{E}(t)$ over $\mathbb{Q}$.
- The r^{th} *moment* of the Fourier coefficients is

$$A_{r,\mathscr{E}}(p) \ = \ \sum_{t \mod p} a_{\mathscr{E}(t)}(p)^r.$$

Tate's Conjecture

Tate's Conjecture for Elliptic Surfaces

Let $\mathscr{E}/\mathbb{Q}$ be an elliptic surface and $L_2(\mathscr{E}, s)$ be the L-series attached to $H^2_{\text{ét}}(\mathscr{E}/\overline{\mathbb{Q}}, \mathbb{Q}_l)$. Then $L_2(\mathscr{E}, s)$ has a meromorphic continuation to $\mathbb{C}$ and satisfies

$$-\mathrm{ord}_{s=2} L_2(\mathscr{E}, s) \ = \ \mathrm{rank}\ NS(\mathscr{E}/\mathbb{Q}),$$

where $NS(\mathscr{E}/\mathbb{Q})$ is the $\mathbb{Q}$-rational part of the Néron-Severi group of $\mathscr{E}$. Further, $L_2(\mathscr{E}, s)$ does not vanish on the line $\mathrm{Re}(s) = 2$.

Tate's conjecture is known for rational surfaces: An elliptic surface $y^2 = x^3 + A(T)x + B(T)$ is rational iff one of the following is true:

- $0 < \max\{3\deg A, 2\deg B\} < 12$;
- $3\deg A = 2\deg B = 12$ and $\mathrm{ord}_{T=0} T^{12}\Delta(T^{-1}) = 0$.

Negative Bias in the First Moment

$A_{1,\mathscr{E}}(p)$ and Family Rank (Rosen-Silverman)

If Tate's Conjecture holds for $\mathscr{E}$ then

$$\lim_{X\to\infty}\frac{1}{X}\sum_{p\le X}\frac{A_{1,\mathscr{E}}(p)\log p}{p} \;=\; -\mathrm{rank}(\mathscr{E}/\mathbb{Q}).$$

- By the Prime Number Theorem, $A_{1,\mathscr{E}}(p)=-rp+O(1)$ implies $\mathrm{rank}(\mathscr{E}/\mathbb{Q})=r$.

Bias Conjecture

Second Moment Asymptotic (Michel)

For families $\mathscr{E}$ with $j(T)$ non-constant, the second moment is

$$A_{2,\mathscr{E}}(p) \ = \ p^2 + O(p^{3/2}).$$

- The lower order terms are of sizes $p^{3/2}$, p, $p^{1/2}$, and 1.

Bias Conjecture

Second Moment Asymptotic (Michel)

For families $\mathscr{E}$ with $j(T)$ non-constant, the second moment is

$$A_{2,\mathscr{E}}(p) \;=\; p^2 + O(p^{3/2}).$$

- The lower order terms are of sizes $p^{3/2}$, p, $p^{1/2}$, and 1.

In every family we have studied, we have observed:

Bias Conjecture

The largest lower term in the second moment expansion which does not average to 0 is on average **negative**.

Preliminary Evidence and Patterns

Let $n_{3,2,p}$ equal the number of cube roots of 2 modulo p, and set $c_0(p) = \left[\left(\frac{-3}{p}\right) + \left(\frac{3}{p}\right)\right] p$, $c_1(p) = \left[\sum_{x \bmod p} \left(\frac{x^3-x}{p}\right)\right]^2$, $c_{3/2}(p) = p\sum_{x(p)} \left(\frac{4x^3+1}{p}\right)$.

Family	$A_{1,\mathscr{E}}(p)$	$A_{2,\mathscr{E}}(p)$
$y^2 = x^3 + Sx + T$	0	$p^3 - p^2$
$y^2 = x^3 + 2^4(-3)^3(9T+1)^2$	0	$\begin{cases} 2p^2-2p & p\equiv 2 \bmod 3 \\ 0 & p\equiv 1 \bmod 3 \end{cases}$
$y^2 = x^3 \pm 4(4T+2)x$	0	$\begin{cases} 2p^2-2p & p\equiv 1 \bmod 4 \\ 0 & p\equiv 3 \bmod 4 \end{cases}$
$y^2 = x^3 + (T+1)x^2 + Tx$	0	$p^2 - 2p - 1$
$y^2 = x^3 + x^2 + 2T + 1$	0	$p^2 - 2p - \left(\frac{-3}{p}\right)$
$y^2 = x^3 + Tx^2 + 1$	$-p$	$p^2 - n_{3,2,p}p - 1 + c_{3/2}(p)$
$y^2 = x^3 - T^2x + T^2$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 - T^2x + T^4$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 + Tx^2 - (T+3)x + 1$	$-2c_{p,1;4}p$	$p^2 - 4c_{p,1;6}p - 1$

where $c_{p,a;m} = 1$ if $p \equiv a \bmod m$ and otherwise is 0.

Lower order terms and average rank

The main term of the first and second moments of the $a_t(p)$ give $r\phi(0)$ and $-\frac{1}{2}\phi(0)$.

Assume the second moment of $a_t(p)^2$ is $p^2 - m_{\mathcal{E}}p + O(1)$, $m_{\mathcal{E}} > 0$.

We have already handled the contribution from p^2, and $-m_{\mathcal{E}}p$ contributes

$$\begin{aligned} S_2 &\sim \frac{-2}{N}\sum_p \frac{\log p}{\log R}\widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{1}{p^2}\frac{N}{p}(-m_{\mathcal{E}}p) \\ &= \frac{2m_{\mathcal{E}}}{\log R}\sum_p \widehat{\phi}\left(2\frac{\log p}{\log R}\right)\frac{\log p}{p^2}. \end{aligned}$$

Thus there is a contribution of size $\frac{1}{\log R}$.

Lower order terms and average rank

Let r_t denote the number of zeros of E_t at the central point (i.e., the analytic rank). Then up to our $O\left(\frac{\log\log R}{\log R}\right)$ errors (which we think should be smaller), we have

$$\begin{aligned} \frac{1}{N}\sum_{t=N}^{2N} r_t\phi(0) &\leq \frac{\phi(0)}{\sigma} + \left(r + \frac{1}{2}\right)\phi(0) + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}\phi(0) \\ \text{Ave Rank}_{[N,2N]}(\mathcal{E}) &\leq \frac{1}{\sigma} + r + \frac{1}{2} + \left(\frac{.986}{\sigma} - \frac{2.966}{\sigma^2\log R}\right)\frac{m_{\mathcal{E}}}{\log R}. \end{aligned}$$

$\sigma = 1$, $m_{\mathcal{E}} = 1$: for conductors of size 10^{12}, the average rank is bounded by $1 + r + \frac{1}{2} + .03 = r + \frac{1}{2} + 1.03$. This is significantly higher than Fermigier's observed $r + \frac{1}{2} + .40$.

$\sigma = 2$: lower order correction contributes .02 for conductors of size 10^{12}, the average rank bounded by $\frac{1}{2} + r + \frac{1}{2} + .02 = r + \frac{1}{2} + .52$. Now in the ballpark of Fermigier's bound (already there without the potential correction term!).

Recent Results

- Proved for many small rank families where can compute closed form expressions (linear / quadratic Legendre sums).

- Recent work of M. Kazalicki and B. Naskrecki:
  $\diamond$ *Second moments and the bias conjecture for the family of cubic pencils* (2020), preprint, https://arxiv.org/pdf/2012.11306.
  $\diamond$*Diophantine triples and K3 surfaces* (2021), preprint, https://arxiv.org/pdf/2101.11705.
- Numerics for moderate rank families indicate might be a positive bias.

- Interpretation similar to Berry-Essen theorem.

Optimal Test Functions

Joint with Elżbieta Bołdyriew, Fangu Chen, Charles Devlin VI, Jason Zhao

The optimization problem

Quantities of interest

- $\mathrm{AveRank}(\mathscr{F}(Q))$, the average order of vanishing at the central point $s = \frac{1}{2}$ for $\mathscr{F}(Q)$.
- $\mathrm{WeightedAveRank}(\mathscr{F}(Q))$, the weighted average order of vanishing at the central point for $\mathscr{F}(Q)$.

The optimization problem

Quantities of interest

- AveRank($\mathscr{F}(Q)$), the average order of vanishing at the central point $s = \frac{1}{2}$ for $\mathscr{F}(Q)$.
- WeightedAveRank($\mathscr{F}(Q)$), the weighted average order of vanishing at the central point for $\mathscr{F}(Q)$.

Bounds and an optimization problem

$$\lim_{Q\to\infty} \mathrm{AveRank}(\mathscr{F}(Q)) \leq \frac{\int_{\mathbb{R}} \phi(x) W_{1,G}(x)\, \mathscr{D}_f x}{\phi(0)}$$

$$\lim_{Q\to\infty} \mathrm{WeightedAveRank}(\mathscr{F}(Q)) \leq \frac{\int_{\mathbb{R}^n} \Phi(x) W_{n,G}(x)\, \mathscr{D}_f x}{\Phi(0)}$$

We want to minimize the right hand sides.

Old results: 1-level densities

Main Idea

Reduce the optimization problem to a differential equations problem via functional analysis.

1-level results: Iwaniec-Luo-Sarnak, Freeman-Miller.

New results: 2-level densities

Main Idea

Restrict domain to only those test functions which are products of single variable test functions $\Phi(x, y) = \phi_1(x)\phi_2(y)$ for fixed admissible $\phi_1(x)$.

Can view the problem as a one-variable integration of ϕ_2 against a function of the form $\delta + m$, i.e., analogous to 1-level case.

Fixed test function and resulting estimates

Let

$$\phi_1(x) = \left(\frac{\sin(2\pi x)}{2\pi x}\right)^2.$$

Obtain naive estimates choosing

$$\Phi(x,y) = \phi_1(x)\phi_1(y).$$

Family	Naive	Optimal
SO(even)	$\frac{5}{12} \approx 0.4166$	$\frac{1}{96}\left(54\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right) - 5\right) \approx 0.3784$
SO(odd)	$\frac{13}{12} \approx 1.0833$	$\frac{1}{32}\left(33 + 2\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right)\right) \approx 1.079$
O	$\frac{3}{4} \approx 0.75$	$\frac{1}{24}\left(13 + 6\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right)\right) \approx 0.733$
U	$\frac{1}{2} \approx 0.5$	$\frac{1}{12}(4 + 3\cot(1)) \approx 0.4939$
Sp	$\frac{1}{12} \approx 0.0833$	$\frac{1}{32}(3 + 2\cot(2)) \approx 0.06515$

Applications to Order of Vanishing

$\Pr(N) :=$ probability that $L(s,f)$ has zero of order N at $s = 1/2$.

$$\sum_{N=2}^{\infty} 2\Pr(N) \le \sum_{N=0}^{\infty} N(N-1)\Pr(N) \le \frac{\int_{\mathbb{R}^2} \Phi(x,y) W_{2,\mathscr{G}}(x,y)\, \mathscr{D}_f x \mathscr{D}_f y}{\Phi(0,0)}$$

$$\Pr(0)+\Pr(1) \ge \begin{cases} 1 - \frac{13+6\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right)}{48} \approx 0.633493 & W_{2,\mathrm{O}} \\ 1 - \frac{54\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right)-5}{192} \approx 0.810776 & W_{2,\mathrm{SO(Even)}} \\ 1 - \frac{33+2\sqrt{3}\cot\left(\frac{2}{\sqrt{3}}\right)}{64} \approx 0.460457 & W_{2,\mathrm{SO(Odd)}} \\ 1 - \frac{4+3\cot(1)}{24} \approx 0.753072 & W_{2,\mathrm{U}} \\ 1 - \frac{3+2\cot(2)}{64} \approx 0.967427 & W_{2,\mathrm{Sp}} \end{cases}$$

Key observations

Observation 1

Ahiezer's Theorem and the Paley-Wiener theorem give a correspondence between test functions and L^2

$$\phi \text{ test function with } \operatorname{supp}(\hat{\phi}) \subseteq [-2\sigma, 2\sigma]$$

$$\Updownarrow$$

$$\widehat{\phi}(\xi) = (g * \breve{g})(\xi) \text{ for } g \in L^2[-\sigma, \sigma]$$

where $\breve{g}(\xi) = \overline{g(-\xi)}$.

Key observations

Observation 2

By Plancharel's theorem

$$\frac{\int_{\mathbb{R}} \phi(x) W_{1,G}(x)\, \mathscr{D}_f x}{\phi(0)} = \frac{\int_{\mathbb{R}} \hat{\phi}(\xi) \widehat{W_{1,G}}(\xi)\, \mathscr{D}_f \xi}{\int_{\mathbb{R}} \hat{\phi}(\xi)\, \mathscr{D}_f \xi}.$$

Observation 3

The Fourier transforms of the 1-level distributions $W_{1,G}$ take the form

$$\widehat{W_{1,G}}(\xi) = \delta(\xi) + m_G(\xi)$$

where m_G is a real-valued even step function.

Step 1: Convert to minimization over L^2

Define compact, self-adjoint linear operator $K : L^2[-\sigma, \sigma] \to L^2[-\sigma, \sigma]$

$$(Kg)(x) := \int_{-\sigma}^{\sigma} m(x-y) g(y) \, \mathscr{D}_f y.$$

Let $\operatorname{supp} \widehat{\phi} \subseteq [-2\sigma, 2\sigma]$, then

$$\frac{\int_{\mathbb{R}} \hat{\phi}(\xi) \widehat{W_{1,G}}(\xi) \, \mathscr{D}_f \xi}{\int_{\mathbb{R}} \hat{\phi}(\xi) \, \mathscr{D}_f \xi} = \frac{\int_{\mathbb{R}} (g * \breve{g})(\xi)(\delta(\xi) + m(\xi)) \, \mathscr{D}_f \xi}{\int_{\mathbb{R}} (g * \breve{g})(\xi) \, \mathscr{D}_f \xi}$$

Step 1: Convert to minimization over L^2

$$= \frac{\langle g, g\rangle_{L^2} + \int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} m(\xi - y) g(y) \,\mathscr{D}_f y \overline{g(\xi)}\,\mathscr{D}_f \xi}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}$$

$$= \frac{\langle g, g\rangle_{L^2} + \langle Kg, g\rangle_{L^2}}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}$$

$$= \frac{\langle (I+K)g, g\rangle_{L^2}}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}.$$

Equivalent optimization problem

Minimize the functional $R : L^2[-\frac{1}{2}, \frac{1}{2}] \to \mathbb{R}$ given by

$$R(g) := \frac{\langle (I+K)g, g\rangle_{L^2}}{|\langle \mathbf{1}, g\rangle_{L^2}|^2}.$$

Step 2: Fredholm theory

Some observations:

- $R(g) \geq \lim_{Q\to\infty} \text{AveRank}(\mathscr{F}(Q)) \geq 0.$
- Spectral Theorem $\implies$ orthonormal basis of eigenvectors of K, eigenvalues λ_j.
- $\lambda_j \geq -1$.

Case 1: Eigenvalue (-1)

Let $f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ not orthogonal to 1 and $(I + K)f_0 = 0$,

$$R(f_0) = \frac{\langle (I + K)f_0, f_0\rangle_{L^2}}{|\langle \mathbf{1}, f_0\rangle_{L^2}|^2} = 0.$$

Step 2: Minimization

Case 2: $\lambda_j > -1$ for all j

More functional analysis!

- $\ker(I + K) = \{0\}$ (all eigenvalues > -1).
- By Fredholm Theory, exists unique $f_0 \in L^2[-\frac{1}{2}, \frac{1}{2}]$ satisfying $(I + K)f_0 = \mathbf{1}$.
- $A := \langle \mathbf{1}, f_0 \rangle = \langle (I + K)f_0, f_0 \rangle_{L^2} > 0$.

For $g = f_0 + h \in L^2[-\frac{1}{2}, \frac{1}{2}]$ with $\langle \mathbf{1}, g \rangle_{L^2} \neq 0$, WLOG $\langle \mathbf{1}, g \rangle_{L^2} = A$. Then $\langle \mathbf{1}, h \rangle_{L^2} = 0$, so

$$R(g) = \frac{\langle \mathbf{1}, f_0 \rangle_{L^2} + \langle (I + K)h, h \rangle_{L^2} + \langle \mathbf{1}, h \rangle_{L^2} + \langle h, \mathbf{1} \rangle_{L^2}}{|A|^2}$$

$$= \frac{A + \langle (I + K)h, h \rangle_{L^2} + 0 + 0}{|A|^2} \geq \frac{1}{A} = R(f_0)$$

Challenges

Larger support and higher level densities give better estimates on the average order of vanishing. Two main obstructions:

1. Small support does not detect non-constant kernels, e.g. $\widehat{W_{1,\mathrm{Sp}}}(x) = \delta(x) - \frac{1}{2}\mathbf{1}_{[-1,1]}(x)$.

Challenges

Larger support and higher level densities give better estimates on the average order of vanishing. Two main obstructions:

1. Small support does not detect non-constant kernels, e.g. $\widehat{W_{1,\mathrm{Sp}}}(x) = \delta(x) - \frac{1}{2}\mathbf{1}_{[-1,1]}(x)$.
2. $\widehat{W_{n,G}}$ more complicated and higher dimensional integral operators not as well-understood.

References and Appendix on Optimal Test Function Details

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Simplifications

A restricted optimization problem

Fix a test function ϕ_1 with $\operatorname{supp} \widehat{\phi_1} \subseteq [-1, 1]$, we want to minimize

$$\frac{\int_{[-1,1]^2} \hat{\phi}_1(\xi_1)\hat{\phi}_2(\xi_2)\widehat{W_{2,G}}(\xi)\, \mathscr{D}_f\xi_1\, \mathscr{D}_f\xi_2}{\phi_1(0)\phi_2(0)}$$

over test functions ϕ_2 with $\operatorname{supp}(\hat{\phi}_2) \subset [-1, 1]$.

Example: Unitary

The 2-level distributions:

$$W_{2,\mathrm{U}}(x) = 1 - \frac{\sin^2(\pi(x_1 - x_2))}{\pi^2(x_1 - x_2)^2},$$

$$\widehat{W_{2,\mathrm{U}}}(\xi) = \delta(\xi_1)\delta(\xi_2) + \delta(\xi_1 + \xi_2)(|\xi_1| - 1)\mathbf{1}(\xi_1).$$

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$$\widehat{W_{2,\mathrm{U}}}(\xi) = \delta(\xi_1)\delta(\xi_2) + \delta(\xi_1 + \xi_2)(|\xi_1| - 1)\mathbf{1}(\xi_1).$$

For ϕ_1 arbitrary,

$$\begin{aligned}\tilde{V}_{\phi_1,\mathrm{U}}(\xi_2) &= \frac{1}{\phi_1(0)} \int_{\mathbb{R}} \hat{\phi}_1(\xi_2)\widehat{W_{2,\mathrm{U}}}(\xi)\, \mathscr{D}_f\xi_1 \\ &= \frac{\hat{\phi}_1(0)}{\phi_1(0)}\delta(\xi_2) + \frac{\hat{\phi}_1(-\xi_2)}{\phi_1(0)}(|\xi_2| - 1)\mathbf{1}(\xi_2) \\ &= c_{\phi_1,\mathrm{U}}\delta(\xi_2) + \tilde{m}_{\phi_1,\mathrm{U}}(\xi_2)\end{aligned}$$

Recovering the 1-level set-up

For each classical compact group G,

$$V_{\phi_1,G}(\xi_2) := \delta(\xi_2) + m_{\phi_1,G}(\xi_2), \qquad m_{\phi_1,G}(\xi_2) := \frac{\tilde{m}_{\phi_1,G}(\xi_2)}{c_{\phi_1,G}}.$$

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Optimization problem rehashed

Minimize

$$\frac{\int_{[-1,1]} \hat{\phi}_2(\xi_2) V_{\phi_1,G}(\xi_2) \mathscr{D}_f \xi_2}{\phi_2(0)}$$

over test functions ϕ_2 with supp $\widehat{\phi_2} \subseteq [-1,1]$.

Collect the 1-level goodies

There exists a unique $g_{\phi_1,G} \in L^2[-1/2, 1/2]$ satisfying

$$1 = g_{\phi_1,G}(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} m_{\phi_1,G}(x-y) g_{\phi_1,G}(y)\, \mathscr{D}_f y.$$

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$$1 = g_{\phi_1,G}(x) + \int_{-\frac{1}{2}}^{\frac{1}{2}} m_{\phi_1,G}(x-y) g_{\phi_1,G}(y)\, \mathscr{D}_f y.$$

Moreover,

$$\frac{c_{\phi_1,G}}{\langle \mathbf{1}, g_{\phi_1,G}\rangle_{L^2}} = \inf_{\phi_2} \frac{\int_{[-1,1]} \hat{\phi}_2(\xi_2) V_{\phi_1,G}(\xi_2) \mathscr{D}_f \xi_2}{\phi_2(0)}.$$

Choosing ϕ_1

Natural choice of test function is the Fourier pair

$$\phi_1(x) = \left(\frac{\sin(2\pi x)}{2\pi x}\right)^2, \qquad \widehat{\phi}_1(\xi) = (1 - |\xi|)\mathbf{1}_{[-1,1]}(\xi)$$

Key observation

Kernels take the form of quadratic polynomials in $|x|$ on $[-1, 1]$, i.e.

$$m_{\phi_1,G}(x) = (a + b|x| + c|x|^2)\mathbf{1}_{[-1,1]}(x).$$

Differentiation under the integral

Exercise for the reader, order one term:

$$\frac{d}{dx}\int_{-1/2}^{1/2}|x-y|g(y)\mathscr{D}_f y = \int_{-1/2}^{x} g(y)\mathscr{D}_f y - \int_{x}^{1/2} g(y)\mathscr{D}_f y,$$

$$\frac{d^2}{dx^2}\int_{-1/2}^{1/2}|x-y|g(y)\mathscr{D}_f y = 2g(x)$$

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$$\frac{d^2}{dx^2}\int_{-1/2}^{1/2}|x-y|g(y)\mathscr{D}_f y = 2g(x)$$

Order two term:

$$\frac{d}{dx}\int_{-1/2}^{1/2}(x-y)^2 g(y)\mathscr{D}_f y = \int_{-1/2}^{1/2}(2x-2y)g(y)\mathscr{D}_f y,$$
$$\frac{d^2}{dx^2}\int_{-1/2}^{1/2}(x-y)^2 g(y)\mathscr{D}_f y = 2\int_{-1/2}^{1/2} g(y)\mathscr{D}_f y.$$

Example: $W_{2,\mathrm{U}}$

We want to find $g \in L^2[-1/2, 1/2]$ obeying

$$1 = g(x) - \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 - |x - y|)^2 g(y) \mathscr{D}_f y.$$

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We want to find $g \in L^2[-1/2, 1/2]$ obeying

$$1 = g(x) - \int_{-\frac{1}{2}}^{\frac{1}{2}} (1 - |x - y|)^2 g(y) \mathscr{D}_f y.$$

$$0 = g''(x) - 4g(x) + 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} g(y) \mathscr{D}_f y$$

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$$0 = g'''(x) - 4g'(x)$$

Assuming evenness, solutions take the form

$$g(x) = A\cos(2x) + C.$$

Optimal $g_{\phi_1,G}$ for supp $\widehat{\phi_1}$, supp $\widehat{\phi_2} \subseteq [-1,1]$

$$g_{\phi_1,\mathrm{SO(Even)}}(x) = \frac{216\cos(4x/\sqrt{3}) + 36\sqrt{3}\sin(2/\sqrt{3})}{162\cos(2/\sqrt{3}) - 5\sqrt{3}\sin(2/\sqrt{3})},$$

$$g_{\phi_1,\mathrm{SO(Odd)}}(x) = \frac{8\cos(4x/\sqrt{3}) + 12\sqrt{3}\sin(2/\sqrt{3})}{11\sqrt{3}\sin(2/\sqrt{3}) + 2\cos(2/\sqrt{3})},$$

$$g_{\phi_1,\mathrm{U}}(x) = \frac{6\cos(2x) + 6\sin(1)}{3\cos(1) + 4\sin(1)},$$

$$g_{\phi_1,\mathrm{O}}(x) = \frac{36\cos(4x/\sqrt{3}) + 18\sqrt{3}\sin(2/\sqrt{3})}{18\cos(2/\sqrt{3}) + 13\sqrt{3}\sin(2/\sqrt{3})},$$

$$g_{\phi_1,\mathrm{Sp}}(x) = \frac{8\cos(4x) + 12\sin(2)}{2\cos(2) + 3\sin(2)}$$