Upper Bounds for the Lowest First Zero in Families of Cuspidal Newforms

Glenn Bruda (University of Florida), Timothy Cheek (University of Michigan), Kareem Jaber (Princeton University), Steven J. Miller (Williams College), Vismay Sharan (Yale University), and Saad Waheed (Williams College)

glenn.bruda@ufl.edu, timcheek@umich.edu, kj5388@princeton.edu, sjm1@williams.edu, vismay.sharan@yale.edu, sw21@williams.edu

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Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the t_i?

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Examples: Spacings between

- ◇ Energy Levels of Nuclei.
- ◇ Eigenvalues of Matrices.
- ◇ Zeros of L-functions.

Sketch of proofs

In studying many statistics, often three key steps:

- ◇ Determine the correct scale for events.
- ◇ Develop an explicit formula relating what want to study to what can study.
- ◇ Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Riemann Zeta Function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

General L-functions

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

Functional Equation:

$$\Lambda(s,f) = \Lambda_\infty(s,f) L(s,f) = \Lambda(1-s,f).$$

Generalized Riemann Hypothesis (RH):

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Observation: Spacings b/w zeros appear same as b/w eigenvalues of Complex Hermitian matrices $\overline{A}^T = A$.

Distribution of zeros

- $\diamond$ $\zeta(s) \neq 0$ for $\mathfrak{Re}(s) = 1$: $\pi(x)$, $\pi_{a,q}(x)$.
- $\diamond$ GRH: error terms.
- $\diamond$ GSH: Chebyshev's bias.
- $\diamond$ Analytic rank, adjacent spacings: $h(D)$.

Explicit Formula (Contour Integration)

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

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&= \frac{d}{ds}\sum_p \log\left(1-p^{-s}\right) \\
&= \sum_p \frac{\log p\cdot p^{-s}}{1-p^{-s}} = \sum_p \frac{\log p}{p^s} + \mathrm{Good}(s).
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Contour Integration:

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\frac{x^s}{s}\,ds \quad \text{vs} \quad \sum_p \log p \int \left(\frac{x}{p}\right)^s \frac{ds}{s}.
$$

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Contour Integration:

$$\int -\frac{\zeta'(s)}{\zeta(s)}\phi(s)ds \quad \text{vs} \quad \sum_p \log p\int\phi(s)p^{-s}ds.$$

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$$

Contour Integration (see Fourier Transform arising):

$$
\int -\frac{\zeta'(s)}{\zeta(s)}\phi(s)ds \quad \text{vs} \quad \sum_p \log p \int \phi(s)e^{-\sigma\log p}e^{-it\log p}ds.
$$

Explicit Formula: Examples

Cuspidal Newforms: Let $\mathcal{F}$ be a family of cupsidal newforms (say weight k, prime level N and possibly split by sign) $L(s,f) = \sum_n \lambda_f(n)/n^s$. Then

$$\frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}}\sum_{\gamma_f}\phi\left(\frac{\log R}{2\pi}\gamma_f\right) = \widehat{\phi}(0) + \frac{1}{2}\phi(0) - \frac{1}{|\mathcal{F}|}\sum_{f\in\mathcal{F}} P(f;\phi) + O\left(\frac{\log\log R}{\log R}\right)$$

$$P(f;\phi) = \sum_{p\nmid N}\lambda_f(p)\widehat{\phi}\left(\frac{\log p}{\log R}\right)\frac{2\log p}{\sqrt{p}\log R}.$$

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

n-level correlation

$$\lim_{N\to\infty} \frac{\#\left\{\left(\alpha_{j_1}-\alpha_{j_2},\ldots,\alpha_{j_{n-1}}-\alpha_{j_n}\right)\in B, j_i \neq j_k\right\}}{N}$$

(Instead of using a box, can use a smooth test function.)

Measures of Spacings: n-Level Correlations

$\{\alpha_j\}$ increasing sequence, box $B \subset \mathbf{R}^{n-1}$.

- $\diamond$ Normalized spacings of $\zeta(s)$ starting at 10^{20} (Odlyzko).
- $\diamond$ 2 and 3-correlations of $\zeta(s)$ (Montgomery, Hejhal).
- $\diamond$ n-level correlations for all automorphic cupsidal L-functions (Rudnick-Sarnak).
- $\diamond$ n-level correlations for the classical compact groups (Katz-Sarnak).
- $\diamond$ Insensitive to any finite set of zeros.

Measures of Spacings: n-Level Density and Families

$\phi(x) := \prod_i \phi_i(x_i)$, ϕ_i even Schwartz functions whose Fourier Transforms are compactly supported.

n-level density

$$D_{n,f}(\phi) \quad = \quad \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \phi_1\left(L_f \gamma_f^{(j_1)}\right) \cdots \phi_n\left(L_f \gamma_f^{(j_n)}\right)$$

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- $\diamond$ Individual zeros contribute in limit.
- $\diamond$ Most of contribution is from low zeros.
- $\diamond$ Average over similar curves (family).

Normalization of Zeros

Local (hard, use C_f) vs Global (easier, use $\log C = |\mathcal{F}_N|^{-1} \sum_{f \in \mathcal{F}_N} \log C_f$).
Hope: ϕ a good even test function with compact support, as $|\mathcal{F}| \to \infty$,

$$\frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} D_{n,f}(\phi) = \frac{1}{|\mathcal{F}_N|} \sum_{f \in \mathcal{F}_N} \sum_{\substack{j_1, \dots, j_n \\ j_i \neq \pm j_k}} \prod_i \phi_i \left(\frac{\log C_f}{2\pi} \gamma_E^{(j_i)} \right)$$

$$\to \int \cdots \int \phi(x) W_{n, \mathcal{G}(\mathcal{F})}(x) dx.$$

Katz-Sarnak Conjecture

As $C_f \to \infty$ the behavior of zeros near $1/2$ agrees with $N \to \infty$ limit of eigenvalues of a classical compact group.

1-Level Densities

The Fourier Transforms for the 1-level densities are

$$\begin{aligned}
\widehat{W_{1,\mathrm{SO(even)}}}(u) &= \delta_0(u) + \frac{1}{2}\eta(u) \\
\widehat{W_{1,\mathrm{SO}}}(u) &= \delta_0(u) + \frac{1}{2} \\
\widehat{W_{1,\mathrm{SO(odd)}}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) + 1 \\
\widehat{W_{1,Sp}}(u) &= \delta_0(u) - \frac{1}{2}\eta(u) \\
\widehat{W_{1,U}}(u) &= \delta_0(u)
\end{aligned}$$

where $\delta_0(u)$ is the Dirac Delta functional and

$$\eta(u) = \begin{cases} 1 & \text{if } |u| < 1 \\ \frac{1}{2} & \text{if } |u| = 1 \\ 0 & \text{if } |u| > 1. \end{cases}$$

Density of low-lying zeros (Slight Notational Change)

Definition (1-level density)

Let Φ be a Schwartz function with $\mathrm{supp}(\widehat{\Phi}) \subset (-\sigma, \sigma)$. Assume GRH and write $\rho_f = 1/2 + i\gamma_f$ for the non-trivial zeros of $L(s, f)$ counted with multiplicity. Then

$$\mathscr{OD}(f; \Phi) := \sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi} \log c_f\right),$$

is the *1-level density*, where c_f is the analytic conductor of f.

- 1-level density captures density of the zeros within height $O(1/\log c_f)$ of $s = 1/2$; since gaps between zeros are approximately c_f, this is counting (morally) a small number of zeros.
- Cannot asymptotically evaluate $\mathscr{OD}(f; \Phi)$ for a single f, must perform averaging over the family ordered by analytic conductor.

n-level density

Definition

In the setting as before, define the *n-level density* as

$$\mathscr{D}_n(f;\Phi) \;:=\; \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \prod_{i=1}^{n} \Phi_i\left(\frac{\gamma_f(j_i)}{2\pi}\log c_f\right).$$

- Computing n-level density for $n > 2$ requires knowledge of distribution of signs of the functional equation of each $L(s, f)$, which is beyond current theory.
- Hughes-Rudnick (2003): introduced n-th centered moments.
  - Similar combiniatorially, but often easier to analyze

Modular Forms

Definition (Modular form of trivial nebentypus)

We write $f \in M_k(q)$ and say f is a *modular form* of level q, even weight k, and trivial nebentypus if $f : \mathbb{H} \to \mathbb{C}$ is holomorphic and

1. for each $\tau \in \Gamma_0(q) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{q} \right\}$ we have
$$f(\tau z) \; := \; f\left(\frac{az+b}{cz+d}\right) \; = \; (cz+d)^k f(z).$$
2. for $\tau \in \mathrm{SL}_2(\mathbb{Z})$, as $\mathfrak{Im}(z) \to +\infty$ we have $(cz+d)^{-k} f(\tau z) \ll 1$.

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With $\tau = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix} \right)$, $f(z) = f(z+1)$ so f is 1-periodic and thus has a Fourier expansion at ∞:

$$f(z) \ = \ \sum_{n=0}^{\infty} a_f(n) q^n, \quad q \ = \ e^{2\pi i z}.$$

Holomorphic Cuspforms

Definition (Cuspform)

If $f \in M_k(q)$ vanishes at all cusps of $\Gamma_0(q)$ we say f is a *cuspform* and denote by $\mathcal{S}_k(q) \subset M_k(q)$ the space of holomorphic cuspforms.

- By Atkin-Lehner Theory, we have the orthogonal decomposition

$$\mathcal{S}_k(q) \;=\; \mathcal{S}_k^{\mathrm{old}}(q) \oplus \mathcal{S}_k^{\mathrm{new}}(q).$$

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- A cuspform $f \in \mathcal{S}_k(q)$ is an eigenfunction of the Hecke operators T_n for $(n, q) = 1$ and $T_n f = \lambda_f(n) f$.

The Space of Cuspidal Newforms

Definition (Newform)

If f is an eigenform of *all* the Hecke operators and the Atkin-Lehner involutions $|_k W(q)$ and $|_k W(Q_p)$ for all the primes $p \mid q$, then we say that f is a *newform* and if, in addition, f is normalized so that $\psi_f(1) = 1$ we say that f is *primitive*.

- The space $\mathcal{S}_k^{\text{new}}(q)$ of newforms has an orthogonal basis $\mathcal{H}_k(q)$ of primitive newforms.
- Trivial nebentypus $\implies$ T_n's are self-adjoint $\implies$ $\lambda_f(n) \in \mathbb{R}$ for all n.

L-functions Attached to Cuspidal Newforms

Fix $f \in \mathcal{S}_k^{\text{new}}(q)$. Then for $\mathfrak{Re}(s) > 1$, we define

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi_0(p)}{p^{2s}}\right)^{-1}$$
$$= \prod_p \left(1 - \frac{\alpha_f(p)}{p^s}\right)^{-1} \left(1 - \frac{\beta_f(p)}{p^s}\right)^{-1},$$

where χ_0 is the principal character mod q. Note, $L(s, f)$ can be analytically continued to an entire function on $\mathbb{C}$. Moreover, $L(s, f) = L(s, \bar{f})$.

Katz-Sarnak Density Conjecture for Orthogonal Symmetry

The symmetry type of the family of automorphic L-functions attached to holomorphic cuspidal newforms is orthogonal. Thus, the Katz-Sarnak density conjecture predicts that for test functions Φ whose Fourier transform has arbitrary compact support, as $Q \to \infty$

$$\frac{1}{|\mathcal{H}_k(Q)|} \sum_{f \in \mathcal{H}_k(Q)} \mathcal{OD}(f;\Phi) \longrightarrow \int_{-\infty}^{\infty} \Phi(x) W(O)(x)\, dx$$

where O is the scaling limit of the group of square orthogonal matrices. It has density

$$W(O)(x) = 1 + \frac{1}{2}\delta_0(x),$$

where $\delta_0(x)$ denotes the Dirac delta function at $x = 0$.

Extending the Support

Theorem (Iwaniec-Luo-Sarnak '00)

Assume GRH. Then for Φ any even Schwartz function with $\mathrm{supp}(\widehat{\Phi}) \subset (-2, 2)$, we have that

$$\lim_{\substack{q\to\infty \\ \square-\text{free}}} \frac{1}{|\mathcal{H}_k(q)|} \sum_{f\in\mathcal{H}_k(q)} \mathcal{OD}(f;\Phi) = \int_{-\infty}^{\infty} \Phi(x) W(O)(x)\, dx,$$

where O denotes the orthogonal type, showing agreement with the Katz-Sarnak philosophy predictions.

Recent Breakthrough

Theorem (Baluyot-Chandee-Li '23)

Assume GRH. Let Φ be an even Schwartz function such that $\operatorname{supp}(\widehat{\Phi}) \subset (-4, 4)$, and let Ψ be any smooth function compactly supported on $\mathbb{R}^+$ with $\widehat{\Psi(0)} \neq 0$. Then we have that

$$\langle \mathcal{OD}(f;\Phi)\rangle_* := \lim_{Q\to\infty} \frac{1}{N(Q)} \sum_q \Psi\left(\frac{q}{Q}\right) \sum_{f\in\mathcal{H}_k(q)}^h \mathcal{OD}(f;\Phi) = \int_{-\infty}^{\infty} \Phi(x) W(O)(x)dx,$$

where $N(Q)$ is a normalizing factor, showing agreement with the Katz-Sarnak philosophy predictions.

This doubling of support uses averaging over the level q to double the support, but many of the necessary manipulations rely on this being the 1-level density.

The n-th Centered Moments of the 1-level Density

We study the n-th centered moments of the 1-level density averaged over levels $q \asymp Q$.

Definition (n-th centered moments of the 1-level density)

In the setting as above, define the *n-th centered moment of the 1-level density* to be

$$\left\langle \prod_{i=1}^{n} \left[\mathcal{OD}(f;\Phi_i) - \langle \mathcal{OD}(f;\Phi_i)\rangle_*\right] \right\rangle_*,$$

where $\langle f \rangle_*$ means averaging f over q as described previously.

Previous work occasionally split forms based on their sign $\epsilon(f) \in \{1, -1\}$; we do not.

Main Results

Theorem (Cheek-Gilman-Jaber-Miller-Tomé '24)

Assume GRH. For Ψ non-negative and Φ_i even Schwartz functions with $\operatorname{supp}(\widehat{\Phi}) \subset (-\sigma, \sigma)$ and $\sigma \leq \min\left\{\frac{3}{2(n-1)}, \frac{4}{2n-\mathbf{1}_{2\nmid n}}\right\}$ we have that

$$\left\langle \prod_{i=1}^{n} \left(\mathbf{OD}(f;\Phi_i) - \langle \mathbf{OD}(f;\Phi_i)\rangle_*\right) \right\rangle_* = \frac{\mathbf{1}_{2|n}}{(n/2)!} \sum_{\tau \in S_n} \prod_{i=1}^{n/2} \int_{-\infty}^{\infty} |u| \widehat{\Phi_{\tau(2i-1)}}(u) \widehat{\Phi_{\tau(2i)}}(u)\, du.$$

As such, our work is a generalization of the BCL '23 $n = 1, \sigma = 4$ result.

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As such, our work is a generalization of the BCL '23 $n = 1, \sigma = 4$ result.

Notably, for $n = 3$ obtain $\sigma = \sigma_i = 3/4$, greater than previous best $\sigma = \sigma_i = 2/3$.

Main results ($n = 2$)

Corollary (Cheek-Gilman-Jaber-Miller-Tomé '24)

Let $\sigma_1 = 3/2$ and $\sigma_2 = 5/6$. Then the two-level density

$$\left\langle \sum_{j_1 \neq \pm j_2} \sum \Phi_1\left(\gamma_f(j_1)\right)\Phi_2\left(\gamma_f(j_2)\right)\right\rangle_* = 2\int_{-\infty}^{\infty} |u|\widehat{\Phi_1}(u)\widehat{\Phi_2}(u)\,du + \prod_{i=1}^{2}\left(\frac{1}{2}\Phi_i(0) + \widehat{\Phi}_i(0)\right) - \Phi_1\Phi_2(0) - 2\widehat{\Phi_1\Phi_2}(0) + \mathcal{ODD}\;\Phi_1\Phi_2(0),$$

where $\mathcal{ODD} := \langle (1-\epsilon_f)/2\rangle_*$ denotes the proportion of forms with odd functional equation. This agrees with the predictions from random matrix theory.

Main results ($n = 2$)

This is the first evidence of an interesting new phenomenon: only by taking different test functions are we able to extend the range in which the Katz-Sarnak density predictions hold. In particular, $\sigma_1 + \sigma_2 = 7/3 \ > \ 2$, where $\sigma_1 + \sigma_2 = 2$ was the previously best known.

Can use $\sigma_1 \geq \sigma_2$ such that $\sigma_1 \leq 3/2$ and $\sigma_1 + 3\sigma_2 \leq 4$; above choice maximizes $\sigma_1 + \sigma_2$.

Duality Between Primes and Zeros of L-functions

Using an explicit formula relating sums over zeros to sums of prime power coefficients of $L(s, f)$, we deduce that

$$\sum_{\gamma_f} \Phi\left(\frac{\gamma_f}{2\pi}\log q\right) = \widehat{\Phi}(0) + \frac{1}{2}\Phi(0) - \frac{2}{\log q}\sum_{p \nmid q} \frac{\lambda_f(p)\log p}{\sqrt{p}}\widehat{\Phi}\left(\frac{\log p}{\log q}\right) + O\left(\frac{\log\log q}{\log q}\right).$$

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We use a combinatorial argument together with GRH for $L(s, \mathrm{sym}^2 f)$ to reduce our task to bounding sums over *distinct* primes:

$$\sum_{\substack{p_1,\dots,p_n\nmid q \\ p_i\neq p_j}} \prod_{i=1}^{n} \frac{\lambda_f(p_i)\log p_i}{\sqrt{p_i}}\widehat{\Phi}_i\left(\frac{\log p_i}{\log q}\right).$$

Averaging Over the Extended Orthogonal Family

We average over $f \in \mathcal{H}_k(q)$ with $q \asymp Q$ and study

$$\frac{1}{N(Q)} \sum_q \Psi\left(\frac{q}{Q}\right) \frac{1}{(\log q)^n} \sum_{f \in \mathcal{H}_k(q)}^h \sum_{\substack{p_1, \dots, p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^n \frac{\lambda_f(p_i) \log p_i}{\sqrt{p_i}} \widehat{\Phi}_i\left(\frac{\log p_i}{\log q}\right)$$

$$= \frac{1}{N(Q)} \sum_q \Psi\left(\frac{q}{Q}\right) \frac{1}{(\log q)^n} \sum_{\substack{p_1, \dots, p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^n \frac{\log p_i}{\sqrt{p_i}} \widehat{\Phi}_i\left(\frac{\log p_i}{\log q}\right) \sum_{f \in \mathcal{H}_k(q)}^h \lambda_f(1) \lambda_f\left(\prod_{i=1}^n p_i\right).$$

Trace formulae

- Ng's work allows us to convert sums over $\mathscr{H}_k(q)$ to a linear combination of sums over an orthogonal basis $\mathscr{B}_k(d)$ for the space $\mathcal{S}_k(d)$, $d \mid q$: Morally, if $(m, n, q) = 1$ and for A a specific arithmetic function, then

$$\sum_{f \in \mathscr{H}_k(q)}^h \lambda_f(m)\lambda_f(n) = \sum_{\substack{q = L_1 L_2 d \\ L_1 | q_1 \\ L_2 | q_2 \\ q_2 \ \square-\text{free}}} A(L_1, L_2, d) \sum_{e | L_2^\infty} \frac{1}{e} \sum_{f \in \mathscr{B}_k(d)}^h \lambda_f(e^2 m)\lambda_f(n).$$

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- Petersson trace formula, a quasi-orthogonality relation for GL_2

$$\sum_{f \in \mathcal{B}_k(d)}^{h} \lambda_f(m)\lambda_f(n) = \delta(m, n) + \sum_{c \geq 1} \frac{S(m, n; cq)}{cq} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{cq}\right).$$

The Kuznetsov Trace Formula

Let $x := \prod p_i$. We are essentially left to analyze

$$\sum_{c \geq 1} \sum_{\substack{p_1, \ldots, p_n \nmid q \\ p_i \neq p_j}} \prod_{i=1}^{n} \frac{\log p_i}{\sqrt{p_i}} V\left(\frac{p_i}{P_i}\right) \mathrm{e}\left(v_i \frac{p_i}{P_i}\right) \sum_s \frac{S(e^2, x; cL_1 rds)}{cL_1 rds} h\left(\frac{4\pi\sqrt{e^2 x}}{cL_1 rds}\right)$$

where V is smooth and compactly supported and h is essentially a smooth truncation of J_{k-1}.

We use the Kuznetsov trace formula to convert an average over $f \in \mathscr{B}_k(d)$ into spectral terms:

$$\text{Holomorphic cuspforms} + \text{Maass cuspforms} + \text{Eisenstein series.}$$

Origin of restriction on σ

To preform the above manipulations, we technically need to sum over primes $p_1, \ldots, p_n$ without restriction (i.e. not dividing q). For $n = 1$, this is only adding back when $p_1 \mid q$, which is $O(\log Q)$, but when $n > 1$, we need to add back $p_1 \mid q, p_2, \ldots, p_n \nmid q$, so this is adding back more than $Q^{n-1-\epsilon}$ many terms. This results in the $\sigma \leq \frac{3}{2(n-1)}$ restriction.

To analyze the terms from Holomorphic and Maass cuspforms, similar techniques require $\sigma \leq \frac{4}{n}$ (the expected bound; the sum of supports is 4). On the other hand, a contour shift for the Eisenstein series term no longer in general achieves any cancellation with n even and only minimal cancellation with n odd. Thus, we need $\sigma \leq \frac{4}{2n - \mathbf{1}_{2\nmid n}}$.

Results

Previous Results

Question

Assuming the GRH, how far up must we go on the critical line before we are assured that we will see the first zero?

Previous work mostly on first (lowest) zero of an L-function. Assume GRH, zeros of the form $\frac{1}{2} + i\gamma$.

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- S. D. Miller: L-functions of real archimedian type has $\gamma < 14.13$.
- J. Bober, J. B. Conrey, D. W. Farmer, A. Fujii, S. Koutsoliotas, S. Lemurell, M. Rubinstein, H. Yoshida: General L-function has $\gamma < 22.661$.

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- J. Mestre: Elliptic curves: first zero occurs by $O(1/\log\log N_E)$, where N_E is the conductor (expect order $1/\log N_E$).
- J. Goes and S. J. Miller: One-Parameter Family of Elliptic Curves of rank r: $r + \frac{1}{2}$ normalized zeros on average within the band $\approx (-\frac{0.551329}{\sigma}, \frac{0.551329}{\sigma})$.

New Results: S. J. Miller and Tang

Theorem: Upper Bound Lowest First Zero in Even Cuspidal Families

For an odd $n = 2m+1$, whenever ω satisfies this following inequality

$$-\left(\widehat{\phi_\omega}(0) + \frac{1}{2}\int_{-\sigma/n}^{\sigma/n} \widehat{\phi_\omega}(y)dy\right)^n < 1_{n \text{ even}}(n-1)!!\sigma_{\phi_\omega}^n + S(n,a;\phi_\omega),$$

at least one form with at least one normalized zero in $(-\omega,\omega)$. Can take

$$\omega > \left(-\frac{\sigma\int_0^1 h(u)^2\,du + \frac{\sigma^2}{4}\int_0^{2/\sigma}\int_{v-1}^1 h(u)h(v-u)\,du\,dv}{\frac{1}{\sigma}\int_0^1 h(u)h''(u)\,du + \frac{1}{4}\int_0^{2/\sigma}\int_{v-1}^1 h(u)h''(v-u)\,du\,dv}\right)^{-\frac{1}{2}}\pi^{-1}. \tag{1}$$

Only know for $\sigma < 2$ (under GRH).
Get $\omega_{\min}(2,h) > 0.21864$ for $h = \cos(\pi y/2)$.

New Results

Theorem: Normalized Zeros Near the Central Point

$P_{r,\rho}(\mathcal{F})$: percent of forms with at least r normalized zeros in $(-\rho, \rho)$.

For even n and $r \geq \mu(\phi, \mathcal{F})/\phi(\rho)$:

$$P_{r,\rho}(\mathcal{F}) \ \leq \ \frac{1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n, a; \phi)}{(r\phi(\rho) - \mu(\phi, \mathcal{F}))^n}.$$

New Results

Theorem: Lower Bound In Terms of Derivatives

From the same methods used to prove the original bound on the first zero for even families we obtain,

$$\omega_{\min} > \frac{1}{2\pi}\left(-\frac{g_w''(0)+\int_0^1 g_w''(x)\,dx}{\int_0^1 g_w(x)\,dx + g_w(0)}\right)^{1/2}.$$

Explicit Bounds

Number of zeros	2-level	4-level	6-level
6	N/A	10.849910	48.154279
16	N/A	0.004235	$2.83230 \cdot 10^{-4}$
26	N/A	$3.541901 \cdot 10^{-4}$	$6.716802 \cdot 10^{-6}$
28	420.045063	$2.486819 \cdot 10^{-4}$	$3.943864 \cdot 10^{-6}$
30	20.991406	$1.796948 \cdot 10^{-4}$	$2.418466 \cdot 10^{-6}$
32	6.651738	$1.330555 \cdot 10^{-4}$	$1.538761 \cdot 10^{-6}$
34	3.220871	$1.006126 \cdot 10^{-4}$	$1.010576 \cdot 10^{-6}$

Table: Upper bound on probability of forms with at least r normalized zeros within 0.8 average spacing from central point, using naive test function with support $2/n$. "N/A" means restriction in our theorem not met.

Constructions and Proofs

Preliminaries

- Convolution:

$$(A * B)(x) = \int_{-\infty}^{\infty} A(t)B(x-t)dt.$$

- Fourier Transform:

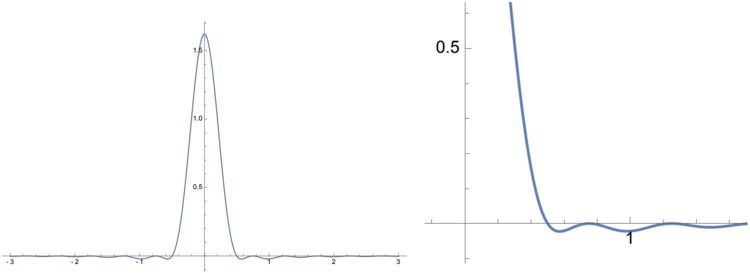
$$\widehat{A}(y) = \int_{-\infty}^{\infty} A(x)e^{-2\pi ixy}dx$$
$$\widehat{A''}(y) = -(2\pi y)^2\widehat{A}(y).$$

- Lemma: $(\widehat{A * B})(y) = \widehat{A}(y) \cdot \widehat{B}(y)$;
  in particular, $(\widehat{A * A})(y) = \widehat{A}(y)^2 \geq 0$ if A is even.

Construction of Test Function

Create compactly supported $\widehat{\phi}(y)$.

- Choose $h(y)$ even, twice continuously differentiable, supported on $(-1, 1)$, monotonically decreasing.
- $f(y) := h\left(\frac{2y}{\sigma/n}\right)$.
- $g(y) := (f * f)(y), \quad \widehat{g}(x) = \widehat{f}(x)^2 \geq 0$.
- $\widehat{\phi}_\omega(y) := g(y) + (2\pi\omega)^{-2} g''(y)$ thus $\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2)$.



Plot of $\phi_\omega(x) = \widehat{g}(x) \cdot (1 - (x/\omega)^2)$, for $h = \cos\left(\frac{\pi y}{2}\right)$, $\sigma = 2$, $\omega = .5$, and $n = 1$.

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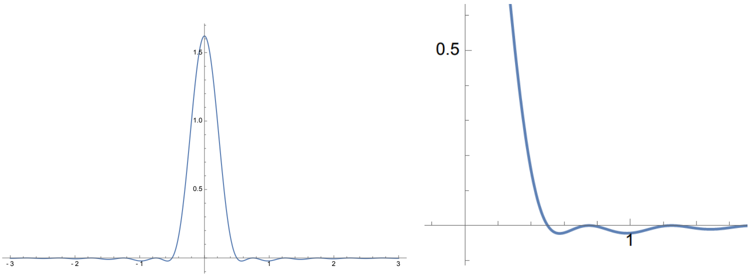

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Sketch of Proof: Key Expansion

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$$-\left(\widehat{\phi_\omega}(0) + \frac{1}{2}\int_{-\sigma/n}^{\sigma/n} \widehat{\phi_\omega}(y)dy\right)^n < 1_{n \text{ even}}(n-1)!!\sigma_{\phi_\omega}^n + S(n, a; \phi_\omega),$$

there exists at least one form with at least one normalized zero in $(-\omega, \omega)$.

Sketch of Proof: Key Expansion

Replace mean from finite N with the limit:

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \frac{1}{|\mathscr{F}_N|} \sum_{f\in\mathscr{F}_N} \left(\sum_j \phi\left(\gamma_{f,j} c_n\right) - \mu(\phi, \mathscr{F}) \right)^n$$
$$= 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n, a; \phi),$$

and main term of the mean of the 1-level density of $\mathscr{F}_N$ is

$$\mu(\phi, \mathscr{F}) \ := \ \widehat{\phi}(0) + \frac{1}{2}\int_{-\infty}^{\infty} \widehat{\phi}(y)dy.$$

Key Observation

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \left(\sum_j \phi(\widetilde{\gamma}_{f,j} c_n) - \mu(\phi,\mathcal{F}) \right)^n$$
$$= 1_{n \text{ even}}(n-1)!!\sigma_\phi^n \pm S(n,a;\phi).$$

$$\phi_\omega(x) = \widehat{g}(x)\cdot(1-(x/\omega)^2).$$

- $\phi_\omega(x) \geq 0$ when $|x| \leq \omega$, and $\phi_\omega(x) \leq 0$ when $|x| > \omega$.
- Contribution of zeroes for $|x| \geq \omega$ is non-positive.
- As n odd, doesn't decrease if drop these non-positive contributions: why we restrict to odd n.

Sketch of Proof: Proof by Contradiction

Dropping negative contributions:

$$\lim_{\substack{N\to\infty \\ N\text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} \left(\sum_{\color{red}{|\gamma_{f,j}|\leq\omega}} \phi_\omega(\gamma_{f,j} c_n) - \mu(\phi_\omega, \mathcal{F}) \right)^n \ \geq\ S(n, a; \phi_\omega).$$

Sketch of Proof: Proof by Contradiction

Assume no forms have a zero on the interval $(-\omega, \omega)$:

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} (-\mu(\phi_\omega, \mathcal{F}))^n \ \geq \ S(n, a; \phi_\omega),$$

$$(-\mu(\phi_\omega, \mathcal{F}))^n \lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} 1 \ \geq \ S(n, a; \phi_\omega).$$

As $\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} 1 = 1$, get

$$(-\mu(\phi_\omega, \mathcal{F}))^n \ \geq \ S(n, a; \phi_\omega).$$

Sketch of Proof: Continued

Because of the compact support of $\widehat{\phi}_\omega$,

$$-\left(\widehat{\phi}_\omega(0)+\frac{1}{2}\int_{-\sigma/n}^{\sigma/n}\widehat{\phi}_\omega(y)dy\right)^n \geq S(n,a;\phi_\omega).$$

Thus, if ω satisfies the following inequality

$$-\left(\widehat{\phi}_\omega(0)+\frac{1}{2}\int_{-\sigma/n}^{\sigma/n}\widehat{\phi}_\omega(y)dy\right)^n < S(n,a;\phi_\omega),$$

we get a contradiction, so at least one form has a normalized zero in $(-\omega,\omega)$.

Explicit Bound from 1-Level Density

First Zero from 1-Level

The first zero of the family of cuspidal newforms exists on the interval $(-\omega_{\min}, \omega_{\min})$, where

$$\omega_{\min} > \left(- \frac{\sigma \int_0^1 h(u)^2 \, du + \frac{\sigma^2}{4} \int_0^{2/\sigma} \int_{v-1}^1 h(u)h(v-u) \, du \, dv}{\frac{1}{\sigma} \int_0^1 h(u)h''(u) \, du + \frac{1}{4} \int_0^{2/\sigma} \int_{v-1}^1 h(u)h''(v-u) \, du \, dv} \right)^{-\frac{1}{2}} \pi^{-1}. \tag{2}$$

Number theory known only for $\sigma < 2$ (under GRH).

Get $\omega_{\min}(2, h) > 0.21864$ for $h = \cos(\pi y/2)$.

Remarks on Computation and Support σ

- Restrictions with higher level computation.
- Riemann sum approximation.
- Currently worse bounds with $\sigma = 2$ for larger n.
- Higher level yields better bounds if support large.
- Larger n better if σ larger.

Main Theorem 2

Naive Test Function

The naive test functions are the Fourier pair

$$\phi_{\text{naive}}(x) \;=\; \left(\frac{\sin(\pi\sigma_n x)}{(\pi\sigma_n x)}\right)^2 \quad , \quad \widehat{\phi}_{\text{naive}}(y) = \frac{1}{\sigma_n}\left(y - \frac{|y|}{\sigma_n}\right)$$

for $|y| \; < \; \sigma_n$ where σ_n is the support.

Theorem: Normalized Zeros Near the Central Point

$P_{r,\rho}(\mathscr{F})$: percent of forms with at least r normalized zeros in $(-\rho, \rho)$.
For even n and $r \geq \mu(\phi, \mathscr{F})/\phi(\rho)$:

$$P_{r,\rho}(\mathscr{F}) \;\leq\; \frac{1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n, a; \phi)}{(r\phi(\rho) - \mu(\phi, \mathscr{F}))^n}.$$

Sketch of Proof

Even n, dropping all with less than r zeros in $(-\rho, \rho)$ drops a non-negative sum:

$$\lim_{\substack{N\to\infty \\ N \text{prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_{N,r}^{(\rho)}} \left(\sum_{|\gamma_{f,j}|\le\rho} \phi(\gamma_{f,j} c_n) + T_f(\phi) - \mu(\phi,\mathcal{F}) \right)^n \ \le\ 1_{n \text{ even}}(n-1)!!\sigma_\phi^n + S(n,a;\phi)$$

Replace the summation of $\phi(\gamma_{f,j}c_n)$ with $r\phi(\rho)$; can drop $T_f(\phi)$ and not increase LHS if $r \ge \mu(\phi,\mathcal{F})/\phi(\rho)$:

$$\lim_{\substack{N\to\infty \\ N \text{ prime}}} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_{N,r}^{(\rho)}} (r\phi(\rho) - \mu(\phi,\mathcal{F}))^n \ \le\ \cdots.$$

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Explicit Bounds

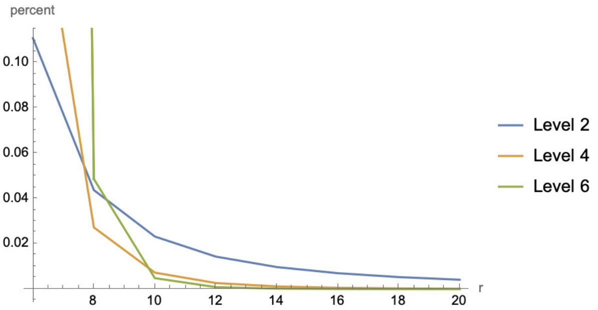

Figure: Percentage vs. number of zeros (for a fixed $\rho = .4$).

Higher levels starts above lower when r small, decrease faster and eventually gives better results as r grows.

Expanding Space for Test Functions

Main Idea

The construction of the test function requires $\widehat{g}(x)$ to decay at the rate of $\Theta(|x|^{-4})$ so it may decay faster than the term $(1-(x/\omega)^2)$.

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We can multiply $\phi(x)$ by a polynomial term of an even degree such that $\widehat{g}(x)$ decays at a rate $|x|^{-A}$, where $A > 4$.

Conditions on the Polynomial

As mentioned previously, ϕ_ω must satisfy the condition, such that $\phi_\omega(x) \geq 0$ when $|x| \leq \omega$ and $\phi_\omega \leq 0$ when $|x| > \omega$ and must be even and decay, such that $\phi_\omega \to 0$ as $x \to \infty$.

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Therefore the polynomial term must be positive and even, so we can write

$$\phi(x) = \widehat{g}(x)(1 - (x/\omega)^2)(1 + c_1 x^2 + c_2 x^4 + \ldots + c_w x^{2w}),$$

where w is the degree of differentiability of $h(x)$ at $x = 1$.

Since $\widehat{g}_w(x) = \widehat{g}(x)(1 + c_1 x^2 + c_2 x^4 + ... + c_w x^{2w})$,

$$\widehat{g}_w(x) = \widehat{g}(x) + c_1 \widehat{g}(x) x^2 + c_2 \widehat{g}(x) x^4 + \cdots + c_w \widehat{g}(x) x^{2w}).$$

We then use the properties of the Fourier transform to deduce that

$$\begin{aligned} g_w(x) &= g(x) - c_1 (2\pi)^{-2} g''(x) + \cdots + c_w (2\pi i)^{-2w} \frac{d^{2w}}{dx^{2w}} g(x) \\ &= g(x) + \sum_{k=1}^{w} c_k (-4\pi^2)^{-k} \frac{d^{2k}}{dx^{2k}} g(x). \end{aligned}$$

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$$\omega_{min} > \frac{1}{2\pi}\left(-\frac{g_w''(0)+\int_0^1 g_w''(x)\,dx}{\int_0^1 g_w(x)\,dx + g_w(0)}\right)^{1/2}.$$

Constraints on Coefficients

We can consider the constraints on the coefficients c_k of the polynomial. Consider

$$p_a(x) = \prod_{i=1}^{a} (\lambda_i x^2 - 1)^2,$$

a positive even polynomial of degree $4a$ with all real roots.

The c_k terms depend on the λ_i parameters so we write,

$$c_k \ = \ (-1)^{2a-k} \sum_{1 \leq r_1 < r_2 < \cdots < r_i \leq 2a} \lambda_{r_1} \lambda_{r_2} \cdots \lambda_{r_i}.$$

Because all the zeros are real, the coefficients c_k of p_a are minimal constants.

Since we aim to minimize $\omega_{\min}$ with respect to the c_k we use a program to minimize the $\{\lambda_i\}$ given w, h. Take

$$h(x) \ = \ (1-x^2)^{2w+1}\left(\prod_{j=1}^{s}(1-\alpha_j x^2)+\beta\right),$$

where s denotes the number of zeros this polynomial may have and $0 \leq \alpha_j \leq 1$ and $\beta \geq 0$.

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Thus, a minimization program may be able to take in the constants of σ, s, and w, while optimizing constraints for α_j and λ_i to minimize ω with respect to these parameters.

When letting the differentiability of h, $w = 1$, the support of the test function, $\sigma = 2$, and the degree of the polynomial for h,$s = 4$, a Mathematica program suited for minimization estimates $\omega_{\text{min}} = 0.218503$.

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There is a convergence of c_k independent of the of the original $h(x)$, so the zeros of an optimal g_ω may be approximated by a program

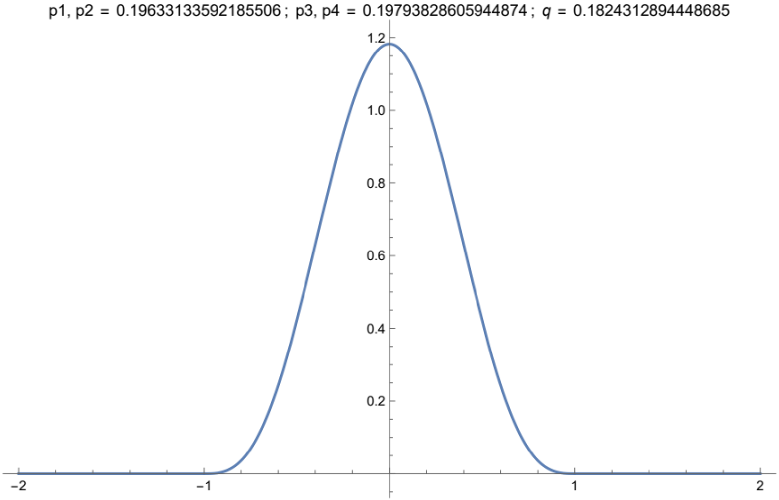

Figure: Result of a program optimizing h for $w, \sigma, s = 1, 2, 4$ respectively.

Future Works

Improving Bounds

- Optimize test function.
- Increase support of test function.
- Recent studies increased the support to 4 (Baluyot, Chandee, and Li) for a certain group of L-functions....

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