Centered Moments of the Weighted One-Level Density of $GL(2)$ L-Functions

Vishal Muthuvel [1] Steven Zanetti[2]

[1]Columbia University, vm2696@columbia.edu
[2]University of Michigan, szanetti@umich.edu

Joint work with L. Dillon, X. Huang, S. Kwon, M. Laurence, S. J. Miller, L. Rowen, and P. Saldin.

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Overview

For families of modular form L-functions,

- Iwaniec-Luo-Sarnak [ILS00]: unweighted one-level densities;
- Hughes-Miller [HM07]: n^{th} centered moments of unweighted one-level densities;
- Knightly-Reno [KR18]: weighted one-level densities; and
- D— et al. [DHK$^+$25]: n^{th} centered moments of weighted one-level densities.

Introduction

Intro to Modular Forms

Definition

A modular form of weight k and level N is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ that is

1. **"periodic"** with respect to the N^{th} congruence subgroup of $SL_2(\mathbb{Z})$:

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau)$$

$$\text{for all } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}), c \equiv 0 \bmod N.$$

2. **holomorphic** at all cusps: The function f is holomorphic at all cusps of $\Gamma_0(N)$, including ∞.

Hecke Operator

- A Holomorphic cusp forms have a Fourier expansion of the form
$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}.$$

Definition

Define the **nth Hecke Operator** $T_n : S_k(N) \to S_k(N)$ to be:
$$T_n f(\tau) = \sum_{m=0}^{\infty} \left(\sum_{d \mid \gcd(m,n)} d^{k-1} a_{mn/d^2} \right) q^m.$$

Hecke eigenvalue

- If f holomorphic cuspform of level N, for every $n \in \mathbf{N}$ relatively prime to N, f is eigenfunction of T_n.

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- Hecke eigenvalues are multiplicative:

$$\lambda_f(m)\lambda_f(n) = \sum_{d \mid \gcd(m,n)} \lambda_f\left(\frac{mn}{d^2}\right).$$

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- Using Hecke eigenvalues, we can define the L-function:

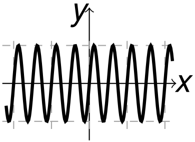
$$L(s,f) := \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p L_p(s,f)^{-1}.$$

Oldform-Newform Theory

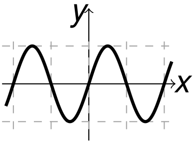
- Atkin-Lehner (1970): Possible to induce forms of level N from forms of level M when $M \mid N$:

$$f(z) \text{ has level } M \Longrightarrow f_N(z) \ := \ f\left(\frac{N}{M} z\right) \text{ has level } N$$

$y = \sin(2\pi x)$



$y = \sin\left(\frac{\pi x}{2}\right)$



- A form that cannot be induced from lower levels is called a **newform**; otherwise, an **oldform**.

Statistics of Zeros

Katz-Sarnak Conjecture

The limiting distribution of the scaled zeros near $1/2$ of any family of "naturally related" L-functions coincides with the limiting distribution of the scaled eigenvalues near 1 of one of the (five) classical compact groups.

- The (non-trivial) zeros of any L-function can be enumerated $\rho_k = \frac{1}{2} + i\gamma_k$.
- We study the local statistics of γ_k.
- The oldforms of level N are "naturally related" to the newforms of level M that induce them, having the same analytic conductor.

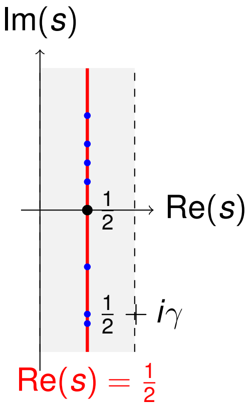
Interdisciplinary Connections

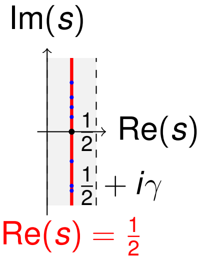
- If GRH is assumed true, the zeros of L-functions can be ordered on the critical line $\mathfrak{Re}(s) = 1/2$.
- Actuate interesting interpretations in the context of random matrix theory and nuclear physics:

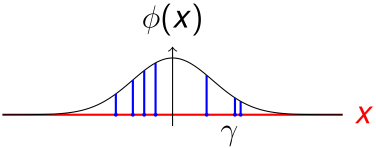
$$
\begin{aligned}
&\text{Zeros of } L\text{-Functions} \\
\longleftrightarrow\ &\text{Eigenvalues of Random Matrix Ensembles} \\
\longleftrightarrow\ &\text{Energy Levels of Heavy Nuclei}
\end{aligned}
$$

One-Level Density

- Wish to study the low-lying zeros, near the central point ($s = \frac{1}{2}$).







Definition (1-Level Density)

$L_f(s)$ the L-function associated to a modular form f; and ϕ an even Schwartz function with $\widehat{\phi}$ compactly supported:

$$D(f;\phi) \ := \ \sum_{\gamma_f} \phi\left(\frac{\log R_f}{2\pi}\gamma_f\right).$$

Explicit Formula

- For uniformity, we convert the sum over zeros to a sum over primes.

Theorem (Iwaniec, Luo, and Sarnak [ILS00])

Letting $\alpha_f(p)$ and $\beta_f(p)$ be the Satake parameters of f, and A be a sum of some digamma factors $\Gamma'(s)/\Gamma(s)$,

$$D(f;\phi) \ = \ \frac{A}{\log R} - 2\sum_p \sum_{m=1}^{\infty} \left(\frac{\alpha_f(p)^m + \beta_f(p)^m}{p^{m/2}} \right) \hat{\phi}\left(\frac{m \log p}{\log R} \right) \frac{\log p}{\log R}.$$

Weights and n^{th} Centered Moments

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  - Analytic weights in Kuznetsov trace formula [AAI+15, GK12]
- Unweighted and weighted statistics have the same distribution in most cases. Exceptions:
  - Kowalski-Saha-Tsimmerman [KST12]: $GSp(4)$ spinor L-functions
  - Knightly-Reno [KR18]: modular form L-functions.

Why does weight change convergence?

- Consider two lobster-roll competitions with different scoring schemes:

	Maine	Québec
Taste	75%	33.3%
Presentation	15%	33.3%
Creativity	10%	33.3%

Example: Contestant A and Contestant B

- Suppose we have

	Contestant A	Contestant B	Maine	Québec
Taste	10	6	75%	33.3%
Presentation	5	7.5	15%	33.3%
Creativity	5	7.5	10%	33.3%

- Who would win each contest?

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- Who would win each contest?

- Depends on the scoring scheme!
  In Maine, Contestant A would have won with 8.75/10.
  However, in Québec, Contestant B would have won better with 7/10.

Weights (Knightly Reno)

- Given a primitive real Dirichlet character χ of modulus $D \geq 1$ and $r > 0$ relatively prime to D.

$$w_f = \frac{\Lambda\left(\frac{1}{2}, f \times \chi\right) |a_f(r)|^2}{\| f \|^2}$$

for the completed L-function $\Lambda(s, f \times \chi)$

Theorem ([KR18])

For $\mathcal{F}_n = \mathcal{F}_k(N)^{new}$ ($N + k \to \infty$ as $n \to \infty$), we have

$$\lim_{n\to\infty} \frac{\sum_{f\in\mathcal{F}_n} D(f,\phi) w_f}{\sum_{f\in\mathcal{F}_n} w_f} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} \phi(x) W_{\mathrm{Sp}}(x)\, dx, & \text{if } \chi \text{ is trivial,} \\ \displaystyle\int_{-\infty}^{\infty} \phi(x) W_{\mathrm{O}}(x)\, dx, & \text{if } \chi \text{ is nontrivial.} \end{cases}$$

The n^{th} Centered Moment

Definition (n^{th} Centered Moment)

Let $\mathcal{F}_{k,N}$ be the family of holomorphic cusp newforms. Let ϕ be an even Schwartz function with compact Fourier support. Then, its **n^th^ centered moment** is given by:

$$\mathcal{A}_{\mathcal{F}_{k,N}}\left([D(\cdot,\phi) - \mathcal{A}_{\mathcal{F}_{k,N}}(D(\cdot,\phi))]^n\right)$$

where $\mathcal{A}_{\mathcal{F}_{k,N}}(Q(\cdot)) = \frac{1}{|\mathcal{F}_{k,N}|}\sum_{f\in\mathcal{F}_{k,N}} Q(f)$ for some function $Q : \mathcal{F}_{k,N} \to \mathbb{C}$.

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- Hughes and Miller computes the unweighted n^{th} centered moments for $\mathcal{F}_{k,N}$ [HM07].

Our Work

- We look at the weighted n^{th} Centered Moments of families of modular form L-functions.
- We use the same weights as Knightly:

$$w_f = \frac{\Lambda\left(\frac{1}{2}, f \times \chi\right) |a_f(r)|^2}{\| f \|^2}.$$

- We denote

$$A^{w}_{\mathcal{F}_{k,N}}(Q(\cdot)) := \lim_{N \to \infty} \frac{\sum_{f \in \mathcal{F}_{k,N}} Q(f) w_f}{\sum_{f \in \mathcal{F}_{k,N}} w_f}$$

where $Q : \mathcal{F} \to \mathbb{C}$.

Our Work

Main Theorem

Theorem (D— et al.)

Let ϕ be a Schwartz test function with $\operatorname{supp}\hat{\phi} \subset (-\frac{1}{2n}, \frac{1}{2n})$*. For real Dirichlet character χ, we have*

$$\mathcal{A}^{w}_{\mathcal{F}_{k,N}}\left[\left(D(\cdot,\phi) - \mathcal{A}^{w}_{\mathcal{F}_{k,N}}(D(\cdot,\phi))\right)^{m}\right]$$

$$= \begin{cases} (n-1)!!\ \sigma_\phi^n & \text{if } n \text{ even}, \\ 0 & \text{if } n \text{ odd}, \end{cases}$$

where $\sigma_\phi^2 = 2\int_{-\infty}^{\infty} \widehat{\phi}^2(y)|y|\,dy$

- This confirms the work of [KR18] since symplectic and orthogonal moments agree with the Gaussian on this support.

Auxiliary Lemmas

- Use explicit formula of [ILS00] to convert from sums over zeros to sums over primes.
- Generalize Jackson-Knightly's weighted trace formula [JK15] from prime powers to arbitrary integers using Hecke multiplicativity:

Lemma (D— et al.)

For any positive integer n,

$$\mathcal{A}^{w}_{\mathcal{F}_{k,N}}(\lambda_{\cdot}(n)) \;=\; n^{-\frac{1}{2}}\chi(n)\,\sigma_1((r,n)) + O\left(\frac{n^{\frac{k-1}{2}}W^k}{N^{\frac{k-1}{2}}k^{\frac{k}{2}-1}}\right),$$

where V is a constant depending on r and D, and σ_1 is the divisor sum function.

Case Work and Analysis

- For χ nontrivial,

Case	Main Term	Error Term
$m_j + n_j \geq 3$ for some j	0	$\log^{-3} R$
$(m_j, n_j) = (1, 1)$ for some j	0	$\frac{\log\log(3N)}{\log R}$
$(m_j, n_j) = (0, 2)$ for all j	$\begin{cases} 0 & t \text{ odd} \\ (t-1)!!(2\sigma_\phi^2)^{t/2} & t \text{ even} \end{cases}$	$\frac{\log\log(3N)}{\log R}$

- For χ trivial,

Case	Main Term	Error Term
$m_j + n_j \geq 3$ for some j	0	$\log^{-3} R$
$m_j + n_j \leq 2$ for all j	$\sum_{s=0}^{\lfloor t/2 \rfloor} \frac{t!}{2^s (t-s)!} \binom{t-s}{s} \left(\frac{\phi(0)}{2}\right)^{t-2s} \left(\frac{\sigma_\phi^2}{2}\right)^s$	$\frac{\log\log(3N)}{\log R}$

Combinatorial Sum

In the case $m_j + n_j \leq 2$ for all j, the contribution to the main term is given by the combinatorial sum:

$$\sum_{t=0}^{n} \binom{n}{t} (-2)^t \sum_{s=0}^{\lfloor t/2 \rfloor} \frac{t!}{2^s (t-s)!} \binom{t-s}{s} \left(\frac{\phi(0)}{2} \right)^{t-2s} \left(\frac{\sigma_\phi^2}{4} \right)^s$$

$$= \phi(0)^n \sum_{t=0}^{n} \binom{n}{t} (-1)^t \sum_{s=0}^{\lfloor t/2 \rfloor} \frac{t!}{(t-s)!} \binom{t-s}{s} \left(\frac{\sigma_\phi^2}{2\phi(0)^2} \right)^s .$$

Combinatorial Sum

In the case $m_j + n_j \leq 2$ for all j, the overall contribution to the main term is given by the combinatorial sum:

$$\phi(0)^n \sum_{t=0}^{n} \binom{n}{t} (-1)^t \sum_{s=0}^{\lfloor t/2 \rfloor} \frac{t!}{(t-s)!} \binom{t-s}{s} \left(\frac{\sigma_\phi^2}{2\phi(0)^2} \right)^s$$

$$= \phi(0)^n \sum_{t=0}^{n} \binom{n}{t} (-1)^t \sum_{s=0}^{\lfloor t/2 \rfloor} \binom{t}{2s} \frac{(2s)!}{s!} \left(\frac{\sigma_\phi^2}{2\phi(0)^2} \right)^s .$$

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$$= \phi(0)^n \sum_{t=0}^{n} \binom{n}{t} (-1)^t \sum_{s=0}^{\lfloor t/2 \rfloor} \binom{t}{2s} (2s-1)!! \left(\frac{\sigma_\phi^2}{\phi(0)^2} \right)^s .$$

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$$= \phi(0)^n \sum_{t=0}^{n} \binom{n}{t} (-1)^t \sum_{s=0}^{\lfloor t/2 \rfloor} \binom{t}{2s} \mathbb{E}\left[X^{2s}\right],$$

where X is a Gaussian random variable with mean 0 and variance $\sigma_\phi^2 / \phi(0)^2$.

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$$= \begin{cases} 0 & n \text{ odd}, \\ \phi(0)^n (n-1)!! \left(\frac{\sigma_\phi^2}{\phi(0)^2}\right)^{n/2} = (n-1)!!\ \sigma_\phi^n & n \text{ even}, \end{cases} .$$

Future Work

Future work

- Studying the n^{th} centered moment of the one-level density with the other set of weights considered by Knightly-Reno:

$$w_f = \frac{\Lambda\left(\frac{1}{2}, f \times \chi\right)\Lambda\left(\frac{1}{2}, f\right)}{\|f\|^2}.$$

- Extending the support of the test function from $\left(-\frac{1}{2n}, \frac{1}{2n}\right)$:
  - Hughes-Miller: RMT distributions are no longer Gaussian when the support is beyond $[-\frac{1}{n}, \frac{1}{n}]$.
  - Plancherel's theorem: The orthogonal and symplectic distributions are distinguishable beyond $(-\frac{1}{n}, \frac{1}{n})$.

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