# 2nd Level Low-Order Terms of Holomorphic Cusp Newforms

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## Introduction

| Key-word | Definition |
|---|---|
| **L-function** | A function that can be written both as a sum indexed by natural numbers and a product indexed by primes. |

$$L(s,f)=\sum_{n=1}^{\infty}\frac{a_f(n)}{n^s}=\prod_{p\text{ prime}}L_p(s,f)^{-1}$$

| Key-word | Definition |
|---|---|
| **Grand Riemann Hypothesis** | Assumption that non-trivial zeros lie on the critical line $\mathrm{Re}(s)=\frac{1}{2}$. |
| **n-level density** | A statistics to study how low-lying zeros of L-functions are spaced on the critical line. |

$$D_n(f;\Phi)=\sum_{\substack{j_1,\dots,j_n\\ j_i\neq\pm j_k}}\phi_1\left(\frac{\log R_f}{2\pi}\gamma_{j_1;f}\right)\cdots\phi_n\left(\frac{\log R_f}{2\pi}\gamma_{j_n;f}\right)$$

| Key-word | Definition |
|---|---|
| **Average weighted $n$-level density** | Weighted Average of n-level density over a family. |

$$\frac{1}{\sum_{f\in\mathcal{F}}w_f}\sum_{f\in\mathcal{F}}w_fD_n(f,\phi)$$

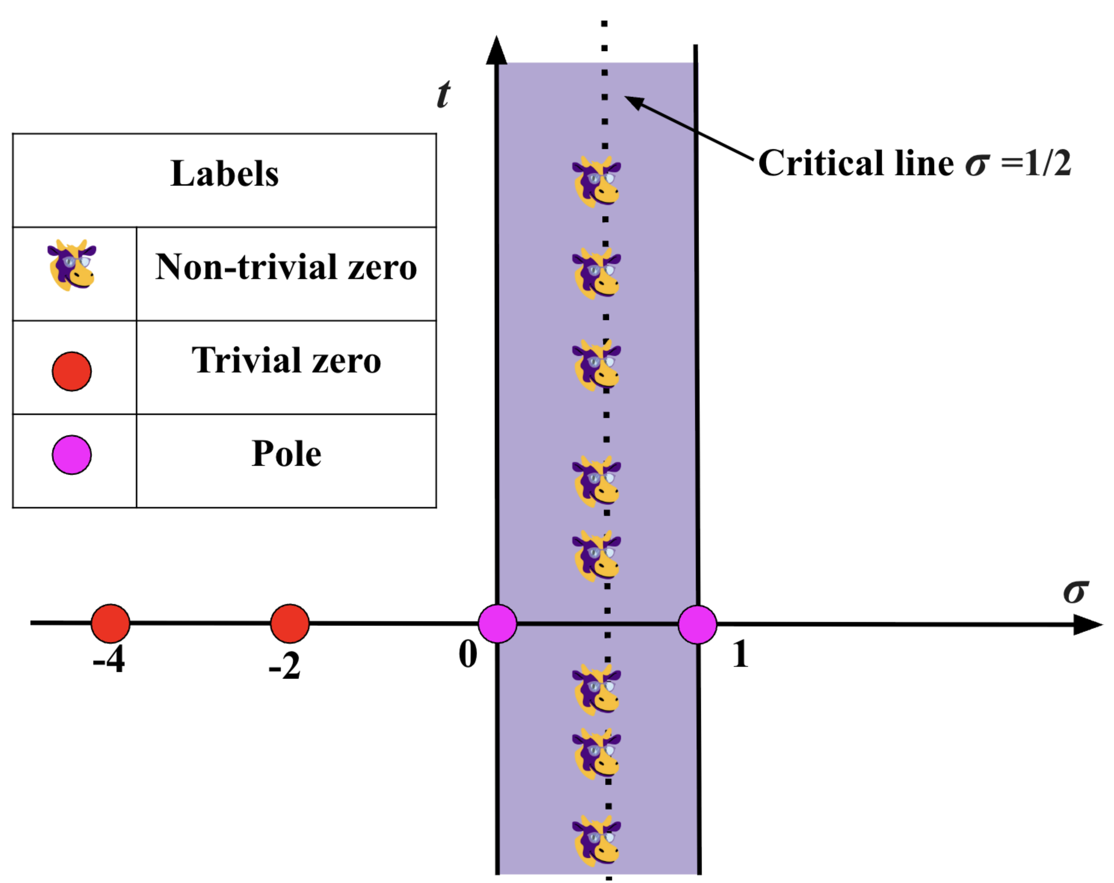


**Figure 1:** Example of critical line zero distribution

## Motivation

### Katz-Sarnak Cojecture

The **average weighted N-level density** behaves like a Gaussian distribution, agreeing to one from Random Matrix Ensemble's eigenvalue distribution.

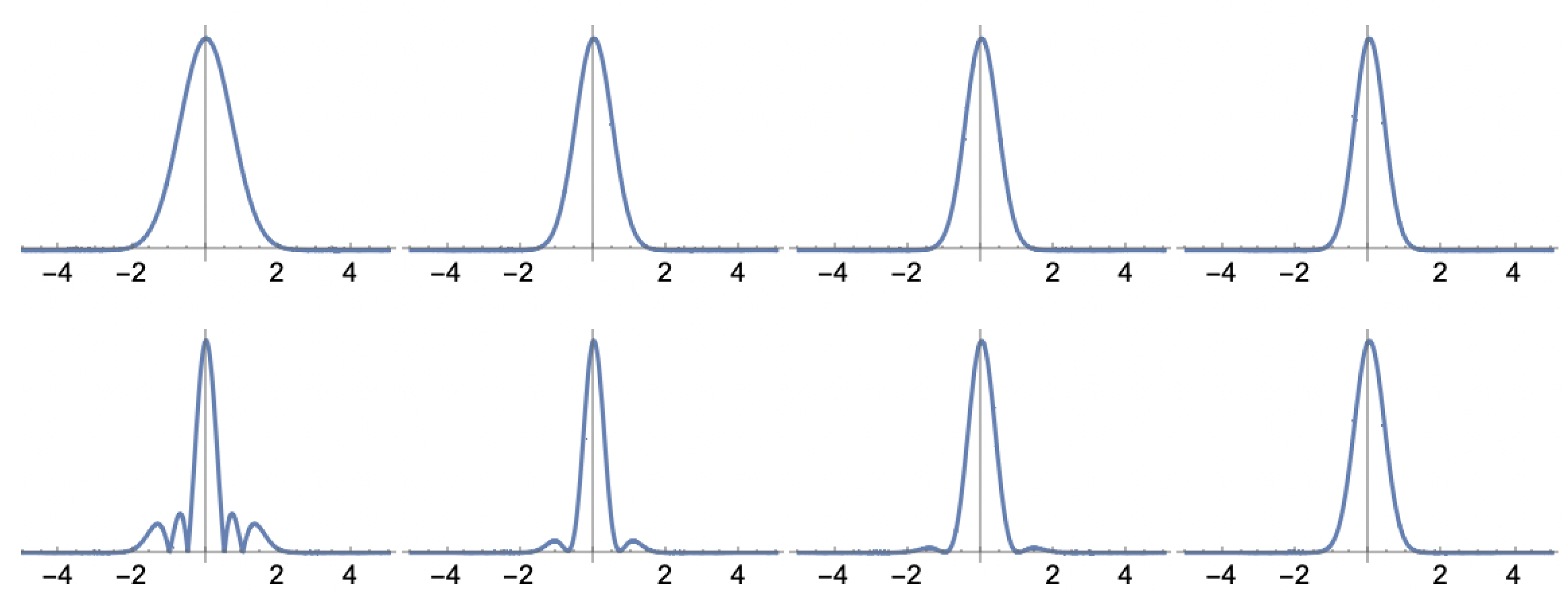


**Figure 2:** Lower Order Terms affect the speed of convergence to Gaussian.

## The Main Theorem

### Theorem of LOT

Let $\log R$ be the average log-conductor of a finite family of $L$-functions $\mathcal{F}$. For sufficient test function $\Phi(x,y)=\phi_1(x)\phi_2(y)$

$$\begin{aligned}D_2(\mathcal{F},\Phi)=&\frac{A_{k,N}(\phi_1)A_{k,N}(\phi_2)}{\log^2 R}+\left(\frac{A_{k,N}(\phi_2)}{\log R}\right)S_1(\mathcal{F},\phi_1)+\left(\frac{A_{k,N}(\phi_1)}{\log R}\right)S_1(\mathcal{F},\phi_2)\\&+S_2(\mathcal{F},\phi_1,\phi_2)-2\left(\frac{A_{k,N}(\phi_1\phi_2)}{\log R}+S_1(\mathcal{F},\phi_1\phi_2)\right)\\&+\Phi(0,0)\frac{\sum_{f\in\mathcal{F}}w_R(f)(1-\epsilon_f)}{2W_R(\mathcal{F})}\end{aligned}$$

where the sum $S_2$ decomposes as

$$S_2(\mathcal{F},\phi_1,\phi_2)=S_{B''}(\mathcal{F})+2S_{B'}(\mathcal{F})+S_B(\mathcal{F})+O\left(\frac{1}{\log^4 R}\right)$$

and the sum $S_1$ has the expansion

$$S_1(\mathcal{F},\phi)=S_{A'}(\mathcal{F})+S_0(\mathcal{F})+S_1(\mathcal{F})+S_2(\mathcal{F})+S_A(\mathcal{F})+O\left(\frac{1}{\log^4 R}\right).$$

## Using Harmonic Weights

We use the **Harmonic weights** defined by

$$w_R(f):=\frac{Z_N(1,f)}{Z(1,f)}.$$

This weight is essentially constant over the family of holomorphic cusp newforms and naturally occur during the derivation of the Peterson trace formula.

We looked at the behaivior of $D_2(\mathcal{F},\Phi)$ for

- Case 1: $N$ is prime with $N\to\infty$.
- Case 2: $N=q_1q_2$ with $q_1$ fixed and $q_2\to\infty$.
- Case 3: $N=q_1q_2$ with both $q_1$ and $q_2\to\infty$.

## Future Directions

1. Generalize to $n$-level density; we looked at 2-level density.
2. Look at different weights; we looked at harmonic weights.
3. Further generalization of $N$ squarefree to look for new behavior.
4. This is an easter egg. If you see this, ask us for chocolate.

## Selected References

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2. Steven J. Miller. "Lower order terms in the 1-level density for families of holomorphic cuspidal newforms". In: Acta Arithmetica 137.1 (2009), pp. 51–98. doi: 10.4064/aa137- 1-3. arXiv: 0704.0924 [math.NT]. url: https://arxiv.org/abs/0704.0924.

## Acknowledgements

We would like to thank our advisor, Prof. Steven J. Miller, for his guidance.
We also gratefully acknowledge the support of the National Science Foundation under Grant No. DMS-2241623, as well as Columbia University, Princeton University, the University of Michigan, the University of Washington, Williams College, and Yale University.