Centered Moments of the Weighted One-Level Density of GL(2) L-Functions

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Introduction

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- Birch and Swinnerton-Dyer conjecture.
- Connections with random matrix theory and nuclear physics.
- Analytically study arithmetic objects.

What is an L-function?

- The Riemann zeta function with Euler Product:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \text{Re}(s) > 1.$$

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$$\xi(s) = \Gamma\left(\frac{s}{2}\right)\pi^{-\frac{s}{2}}\zeta(s) = \xi(1-s).$$

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- Riemann Hypothesis (RH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

What is a general L-function?

- A General L-function with Euler Product

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_f(n)}{n^s} = \prod_{p \text{ prime}} L_p(s,f)^{-1}, \quad \mathrm{Re}(s) > 1.$$

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$$\Lambda(s,f) = \Lambda_\infty(s,f) L(s,f) = \Lambda(1-s,f).$$

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- Functional Equation:

$$\Lambda(s,f) \;=\; \Lambda_\infty(s,f) L(s,f) \;=\; \Lambda(1-s,f).$$

- Grand Riemann Hypothesis (GRH):

All non-trivial zeros have $\mathrm{Re}(s) = \frac{1}{2}$; can write zeros as $\frac{1}{2} + i\gamma$.

Statistics of Zeros

- Assuming GRH, non-trivial zeros of L-functions can be written as $\frac{1}{2} + i\gamma_i$; possible to investigate statistics of zeros γ_i.

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Conjecture (Katz-Sarnak)

(In the limit) Scaled distribution of zeros near central point agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

Intro to Modular Forms

- *L*-functions arise throughout number theory. We care about associating *L*-functions to certain *modular forms*.

Definition

Define the the N^{th} congruence subgroup

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \mod N \right\}.$$

Intro to Modular Forms

- Now for an important definition:

Definition

A modular form of weight k and level N is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

1. **Modular Transformation Property:**

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau) \quad \text{for all} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N).$$

2. **Holomorphic at the Cusps:** The function f is holomorphic at all cusps of $\Gamma_0(N)$, including ∞.

Hecke Eigenforms

- A modular form has a Fourier expansion of the form $f(\tau) = \sum_{n=0}^{\infty} a_n e^{2\pi i n\tau}$. If $a_0 = 0$, f is called a *cusp form*.

Definition

Define the n^{th} Hecke operator as acting on the Fourier expansion as

$$T_n f(\tau) = \sum_{m=0}^{\infty} \left(\sum_{d|\gcd(m,n)} d^{k-1} a_{mn/d^2} \right) q^m.$$

Explicit formula for 1-level

- $R_f > 0$: analytic conductor of $L(s, f) = \sum_n \lambda_f(n)/n^s$.
- By GRH the non-trivial zeros are $\frac{1}{2} + i\gamma_{f,j}$.
- Satake params $\alpha_f(p), \beta_f(p)$; $\lambda_f(p^\nu) = \alpha_f(p)^\nu + \beta_f(p)^\nu$.
- This gives

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p \left(1 - \alpha_f(p)p^{-s}\right)^{-1} \left(1 - \beta_f(p)p^{-s}\right)^{-1}.$$

Explicit formula for 1-level

- We wish to find a way to study the 0's near the central point ($s = \frac{1}{2}$). Thus, we associate a **1-level density** to a modular form.

Definition (1-Level Density)

Let $L_f(s)$ be an L-function associated to a modular form f. Let ϕ be an even Schwartz function whose Fourier transform has compact support. Then, its **1-level density** is given by:

$$D_1(f;\phi) = \sum_{\gamma_f} \phi\left(\frac{\log R_f}{2\pi}\gamma_f\right).$$

Example with $\zeta(s)$

$$-\frac{\zeta'(s)}{\zeta(s)} = -\frac{d}{ds}\log\zeta(s) = -\frac{d}{ds}\log\prod_p\left(1-p^{-s}\right)^{-1}$$

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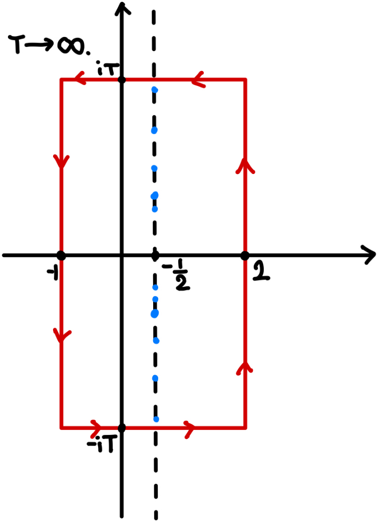
Example with $L(s) = \zeta(s)$; Contour Integral



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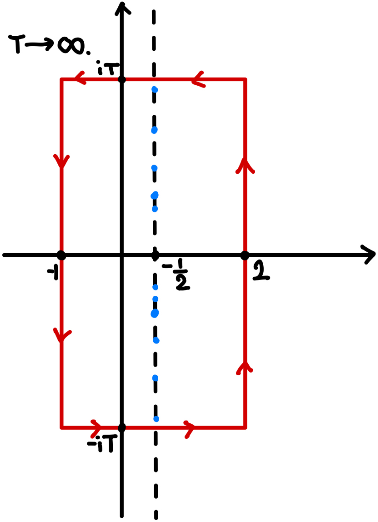

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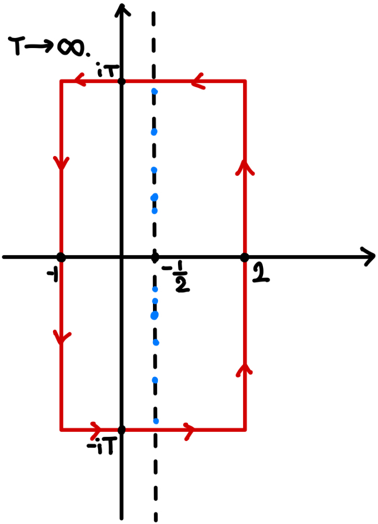

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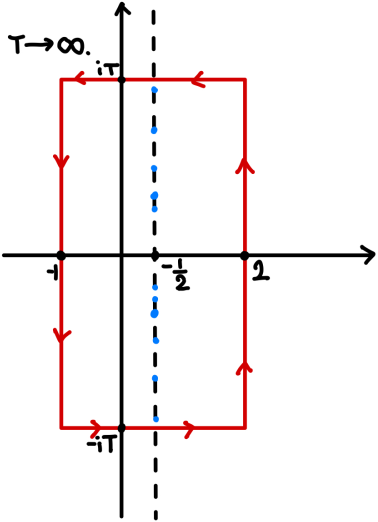

- Define $\Phi(s) = \phi(\frac{\log(R)}{2\pi i}(s - \frac{1}{2}))$.
- Integrate $\frac{\zeta'(s)}{\zeta(s)}\Phi(s)$ along the **contour** .
- By the Argument Principle, the integral is related to the values of Φ at **zeros** of $\zeta(s)$.
- Because $\frac{\zeta'(s)}{\zeta(s)} \approx -\sum_p \frac{\log(p)}{p^s} = -\sum_p \log(p)e^{-s\log(p)}$, above integral is related also to the **Fourier coefficients** of Φ and hence ϕ.

Explicit Formula (Contour Integration)

- This narrative gives us to the explicit formula for the density for generic L-function.

Theorem (Iwaniec, Luo, and Sarnak [ILS00])

Given the same conditions,

$$D_1(f;\phi) = \frac{A}{\log R} - 2\sum_{p}\sum_{m=1}^{\infty}\left(\frac{\alpha_f(p)^m + \beta_f(p)^m}{p^{m/2}}\right)\hat{\phi}\left(m\frac{\log p}{\log R}\right)\frac{\log p}{\log R}$$

where A represents a sum of digamma $(\Gamma'(s)/\Gamma(s))$ factors.

Our Work: Preliminaries

n-Level Density

We can generalize our 1-level density to n-**level density**.

Definition (n-Level Density)

Let $L_f(s)$ be an L-function associated to a modular form f. Let $\Phi(x_1, \dots, x_n) = \phi_1(x_1) \cdots \phi_n(x_n)$ where each ϕ_i are even Schwartz function with compact supported fourier transforms. Then, we define n-Level Density of $L_f(s)$ to be:

$$D_n(f;\Phi) = \sum_{\substack{j_1,\dots,j_n \\ j_i \neq \pm j_k}} \phi_1\left(\frac{\log R_f}{2\pi}\gamma_{j_1;f}\right) \cdots \phi_n\left(\frac{\log R_f}{2\pi}\gamma_{j_n;f}\right).$$

Katz-Sarnak Conjecture

As we average over $\mathcal{F} = \cup \mathcal{F}_N$, a family of L-functions ordered by conductors N, and take $N \to \infty$, n-level density converges to a scaled distribution of eigenvalues near 1 of a classical compact group, i.e.

$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} D_n(f, \phi) = \int \Phi(\vec{x}) W_{n,G(\mathcal{F})}(\vec{x}) d\vec{x}.$$

Conjecture (Katz-Sarnak)

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Conjecture (Katz-Sarnak)

(In the limit) Average n-th level density agrees with scaled distribution of eigenvalues near 1 of a classical compact group.

Introducing Weights

- Weights $\{w_f\}_{\mathcal{F}_N}$ are often used to simplify calculations.

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$$\lim_{N\to\infty} \frac{1}{|\mathcal{F}_N|} \sum_{f\in\mathcal{F}_N} D_1(f,\phi) = \lim_{N\to\infty} \frac{1}{\left(\sum_{f\in\mathcal{F}_N} 1\right)} \sum_{f\in\mathcal{F}_N} 1 \cdot D_1(f,\phi)$$

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vs.

$$\lim_{N\to\infty} \frac{1}{\left(\sum_{f\in\mathcal{F}_N} w_f\right)} \sum_{f\in\mathcal{F}_N} w_f D_1(f,\phi).$$

Does weight change convergence?

We need to be careful here!
Consider two classes with different grading schemes:

	Class N	**Class W**
Psets	25%	0%
Midterm 1	25%	0%
Midterm 2	25%	0%
Final	25%	100%

Example: Steve and Luke

- Suppose we have

	Steve	Luke
Psets	100%	0%
Midterm 1	95%	10%
Midterm 2	95%	15%
Final	15%	95%

- Who would have gotten a better grade in the class?

Example: Steve and Luke

- Suppose we have

	Steve	**Luke**
Psets	100%	0%
Midterm 1	95%	10%
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Final	15%	95%

- Who would have gotten a better grade in the class?

- Depends on the grading scheme!
  In Class N, Steve would have done better with a 76.25% and Luke with 30%.
  However, in Class W, Luke would have done better with 95% and Steve with 15%.

Weights change Convergence

- Kowalski, Saha and Tsimmerman [KST12] found that for GSp(4) spinor L-functions, adding weights yielded symplectic distribution instead of the expected orthogonal.
- Knightly and Reno [KR18] also found that weights affect the convergence.

Weights (Knightly Reno)

- Given a primitive real Dirichlet character χ of modulus $D \geq 1$ and $r > 0$ relatively prime to D. For a holomorphic newform, define the weight

$$w_f = \frac{\Lambda\left(\frac{1}{2}, f \times \chi\right) |a_f(r)|^2}{||f||^2}$$

for the completed L-function $\Lambda(s, f \times \chi)$

Theorem ([KR18])

For $\mathcal{F}_n = \mathcal{F}_k(N)^{new}$ ($N + k \to \infty$ as $n \to \infty$), we have

$$\lim_{n\to\infty} \frac{\sum_{f\in\mathcal{F}_n} D_1(f,\phi) w_f}{\sum_{f\in\mathcal{F}_n} w_f} = \begin{cases} \int_{-\infty}^{\infty} \phi(x) W_{\mathrm{Sp}}(x)\, dx, & \text{if } \chi \text{ is trivial,} \\ \int_{-\infty}^{\infty} \phi(x) W_{\mathrm{O}}(x)\, dx, & \text{if } \chi \text{ is nontrivial.} \end{cases}$$

Our Work

- For ease of notation, for a function $A : \mathcal{F} \to \mathbb{C}$, let $\mathcal{E}_w(A) = \lim_{n\to\infty} \frac{\sum_{f\in\mathcal{F}_n} A(f) w_f}{\sum_{f\in\mathcal{F}_n} w_f}$

Our Work

- For ease of notation, for a function $A : \mathcal{F} \to \mathbb{C}$, let $\mathcal{E}_w(A) = \lim_{n\to\infty} \frac{\sum_{f\in\mathcal{F}_n} A(f)w_f}{\sum_{f\in\mathcal{F}_n} w_f}$
- We generalize this work, computing the n^{th} centered moments of this one level density.

$$\mathcal{E}_w\left[(D(f,\phi) - \mathcal{E}_w(D(f,\phi)))^m\right]$$

Our Work

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Theorem (SMALL 2025)

Let ϕ be a Schwartz test function with $\operatorname{supp}\hat{\phi} \subset (-\frac{1}{2n}, \frac{1}{2n})$. For real Dirichlet character χ, we have

$$\mathcal{E}_w\left[(D(f,\phi) - \mathcal{E}_w(D(f,\phi)))^m\right] = \begin{cases} (2m-1)!!\left(\int_{-\infty}^{\infty} \hat{\phi}^2(y)|y|\,dy\right)^{m/2} & \text{if } m \text{ even}, \\ 0 & \text{if } m \text{ odd} \end{cases}$$

- This is in line with [KR18] since Gaussian and symplectic moments agree on this support.

Our Work

Proving the above theorem reduces to computing sums of the following form:

$$\sum_{\substack{(p_1,\dots p_t)\\ p_i \nmid N}} \left(\prod_{i=1}^{t} \frac{\log p_i}{p_i^{1/2} \log R} \hat{\phi}\left(\frac{\log p_i}{\log R} \right) \right) \mathcal{E}_w \left[\prod_{j=1}^{\ell} \lambda_\cdot(p_i) \right]$$

$$= \sum_{\substack{(p_1,\dots p_t)\\ p_i \nmid N}} \left(\prod_{i=1}^{t} \frac{\log p_i}{p_i^{1/2} \log R} \hat{\phi}\left(\frac{\log p_i}{\log R} \right) \right) \sum_{\substack{m_1 \equiv n_1 (2)\\ \cdots \\ m_\ell \equiv n_\ell (2)}} \left(\prod_{j=1}^{\ell} c_{m_j, n_j} \right) \mathcal{E}_w \left[\prod_{j=1}^{\ell} \lambda_\cdot(q_j^{m_j}) \right]$$

where $\prod_{i=1}^{t} p_i = \prod_{j=1}^{\ell} q_j^{n_j}$. The first step to analyzing this sum is finding a closed form for $\mathcal{E}_w [\lambda_\cdot(n)]$.

Our Work

Lemma (SMALL 2025)

For any positive integer $n = \prod_{j=1}^{\ell} q_j^{m_j}$,

$$\mathcal{E}_w\left[\lambda_\cdot(n)\right] = \frac{\chi(n)\sigma_1\left(\gcd\left(r,n\right)\right)}{\sqrt{n}} + O\left(\frac{n^{\frac{k-1}{2}}V^k}{N^{\frac{k-1}{2}}k^{\frac{k}{2}-1}}\right),$$

where V *is a constant depending on* r *and* D, *and* σ_1 *is the divisor sum function.*

This was done using Prop 3.1 of Knightly and Reno ([KR18]) and using a similar proof method.

Our Work

Through a combinatorial argument and swapping sums and products, we find we have a product of sums with factors of the form

$$\sum_{q \nmid N} \hat{\phi}\left(\frac{\log q}{\log R}\right)^{n_j} \frac{\chi(q)^{m_j} \log^{n_j} q}{q^{(n_j+m_j)/2} \log^{n_j} R} c_{m_j,n_j} \cdot \sigma_1\left(\gcd\left(r, q^{m_j}\right)\right)$$

for specific cases of n_j and m_j.

- We split into the nontrivial and trivial character case
- Main contribution comes from case $(m_j, n_j) = (0, 2)\ \forall j$ for both nontrivial and trivial
- In the trivial case, another combinatorial argument and the binomial theorem are needed

Nontrivial Character Analysis

Case	Main Term	Error Term
$m_j + n_j \geq 3$ for some j	0	$\log^{-3} R$
$(m_j, n_j) = (1, 1)$ for some j	0	$\frac{\log\log(3N)}{\log R}$
$(m_j, n_j) = (0, 2)$ for all j	$\begin{cases} 0 & t \text{ odd} \\ (t-1)!!(2\sigma_\phi^2)^{t/2} & t \text{ even} \end{cases}$	$\frac{\log\log(3N)}{\log R}$

Trivial Character Analysis

Case	Main Term	Error Term
$m_j + n_j \geq 3$ for some j	0	$\log^{-3} R$
$m_j + n_j \leq 2$ for all j	$\sum_{s=0}^{\lfloor t/2 \rfloor} \frac{t!}{2^s (t-s)!} \binom{t-s}{s} \left(\frac{\phi(0)}{2}\right)^{t-2s} \left(\frac{\sigma_\phi^2}{2}\right)^s$	$\frac{\log\log(3N)}{\log R}$

Closing

Future work

- Verify Gaussian behavior for Knightly and Reno's other weight

$$w_f = \frac{\Lambda\left(1/2, f \times \chi\right)\Lambda\left(\frac{1}{2}, f\right)}{||f||^2}$$

- Extend support of test function used from $\left(-\frac{1}{2n}, \frac{1}{2n}\right)$ to $\left(-\frac{1}{n}, \frac{1}{n}\right)$

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