Limiting Distributions in the Bulk and Blips of Matrix Ensembles Under Anticommutator Operators

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Introduction

Fundamental Problem: Spacing Between Events

General Formulation: Studying system, observe values at $t_1, t_2, t_3, \dots$.

Question: What rules govern the spacings between the t_i?

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Examples: Spacings between

- Energy Levels of Nuclei.
- Eigenvalues of Matrices.
- Zeros of L-functions.
- Summands in Zeckendorf Decompositions.
- Primes.
- $n^k\alpha \bmod 1$.

Sketch of proofs

In studying many statistics, often three key steps:

1. Determine correct scale for events.
2. Develop an explicit formula relating what we want to study to something we understand.
3. Use an averaging formula to analyze the quantities above.

It is not always trivial to figure out what is the correct statistic to study!

Classical
Random Matrix Theory

Origins of Random Matrix Theory

Classical Mechanics: 3 Body Problem Intractable.

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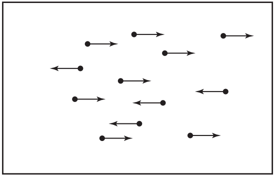
Fundamental Equation:

$$H\psi_n \;=\; E_n\psi_n$$

H : matrix, entries depend on system
E_n : energy levels
ψ_n : energy eigenfunctions

Origins (continued)



- Statistical Mechanics: for each configuration, calculate quantity (say pressure).
- Average over all configurations – most configurations close to system average.
- Nuclear physics: choose matrix at random, calculate eigenvalues, average over matrices (real Symmetric $A = A^T$, complex Hermitian $\overline{A}^T = A$).

Random Matrix Ensembles

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1N} \\ a_{12} & a_{22} & a_{23} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1N} & a_{2N} & a_{3N} & \cdots & a_{NN} \end{pmatrix} = A^T, \quad a_{ij} = a_{ji}$$

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Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

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Fix p, define

$$\text{Prob}(A) = \prod_{1 \le i \le j \le N} p(a_{ij}).$$

This means

$$\text{Prob}\left(A : a_{ij} \in [\alpha_{ij}, \beta_{ij}]\right) = \prod_{1 \le i \le j \le N} \int_{x_{ij}=\alpha_{ij}}^{\beta_{ij}} p(x_{ij}) dx_{ij}.$$

Eigenvalue Trace Lemma

Want to understand the eigenvalues of A, but it is the matrix elements that are chosen randomly and independently.

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Eigenvalue Trace Lemma

Let A be an $N \times N$ matrix with eigenvalues $\lambda_i(A)$. Then

$$\mathrm{Trace}(A^k) \;=\; \sum_{n=1}^{N} \lambda_i(A)^k,$$

where

$$\mathrm{Trace}(A^k) \;=\; \sum_{i_1=1}^{N} \cdots \sum_{i_k=1}^{N} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_N i_1}.$$

Eigenvalue Distribution

$\delta(x - x_0)$ is a unit point mass at x_0:
$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)dx = f(x_0)$.

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To each A, attach a probability measure:

$$\mu_{A,N}(x) = \frac{1}{N}\sum_{i=1}^{N}\delta\left(x - \frac{\lambda_i(A)}{2\sqrt{N}}\right)$$

$$\int_a^b \mu_{A,N}(x)dx = \frac{\#\left\{\lambda_i : \frac{\lambda_i(A)}{2\sqrt{N}} \in [a,b]\right\}}{N}$$

$$\text{k}^{\text{th}}\text{ moment} = \frac{\sum_{i=1}^{N}\lambda_i(A)^k}{2^k N^{\frac{k}{2}+1}} = \frac{\text{Trace}(A^k)}{2^k N^{\frac{k}{2}+1}}.$$

Density of States

Wigner's Semi-Circle Law

Wigner's Semi-Circle Law

$N \times N$ real symmetric matrices, entries i.i.d.r.v. from a fixed $p(x)$ with mean 0, variance 1, and other moments finite. Then for almost all A, as $N \to \infty$

$$\mu_{A,N}(x) \longrightarrow \begin{cases} \frac{2}{\pi}\sqrt{1-x^2} & \text{if } |x| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

SKETCH OF PROOF: Correct Scale

$$\text{Trace}(A^2) \;=\; \sum_{i=1}^{N} \lambda_i(A)^2.$$

By the Central Limit Theorem:

$$\text{Trace}(A^2) \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}a_{ji} \;=\; \sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \;\sim\; N^2$$

$$\sum_{i=1}^{N} \lambda_i(A)^2 \;\sim\; N^2$$

Gives $N\text{Ave}(\lambda_i(A)^2) \sim N^2$ or $\text{Ave}(\lambda_i(A)) \sim \sqrt{N}$.

SKETCH OF PROOF: Averaging Formula

Recall k-th moment of $\mu_{A,N}(x)$ is $\mathrm{Trace}(A^k)/2^k N^{k/2+1}$.

Average k-th moment is

$$\int \cdots \int \frac{\mathrm{Trace}(A^k)}{2^k N^{k/2+1}} \prod_{i \le j} p(a_{ij}) da_{ij}.$$

Proof by method of moments: Two steps

- Show average of k-th moments converge to moments of semi-circle as $N \to \infty$;
- Control variance (show it tends to zero as $N \to \infty$).

SKETCH OF PROOF: Averaging Formula for Second Moment

Substituting into expansion gives

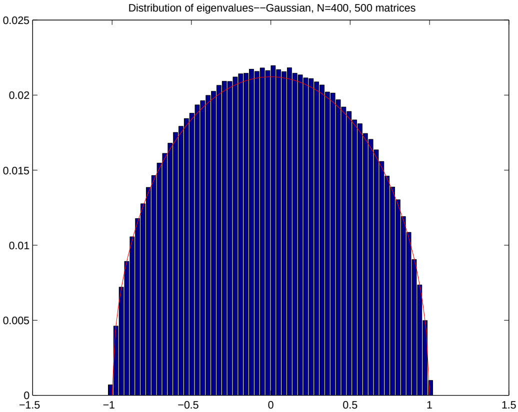
$$\frac{1}{2^2 N^2}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sum_{i=1}^{N}\sum_{j=1}^{N} a_{ij}^2 \cdot p(a_{11})da_{11}\cdots p(a_{NN})da_{NN}$$

Integration factors as

$$\int_{a_{ij}=-\infty}^{\infty} a_{ij}^2 p(a_{ij})da_{ij} \quad \cdot \prod_{\substack{(k,l)\neq(i,j)\\ k<l}} \int_{a_{kl}=-\infty}^{\infty} p(a_{kl})da_{kl} \;=\; 1.$$

Higher moments involve more advanced combinatorics (Catalan numbers).

Numerical example: Gaussian density



500 Matrices: Gaussian 400×400

$$p(x) \;=\; \frac{1}{\sqrt{2\pi}}\; e^{-x^2/2}$$

Real Symmetric Toeplitz Matrices

Chris Hammond and Steven J. Miller

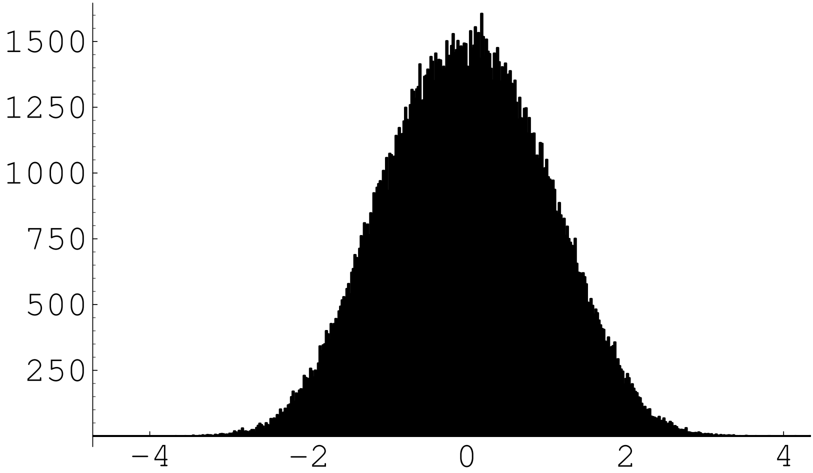
Toeplitz Ensembles

Toeplitz matrix is of the form

$$\begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} \\ b_{-1} & b_0 & b_1 & \cdots & b_{N-2} \\ b_{-2} & b_{-1} & b_0 & \cdots & b_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{1-N} & b_{2-N} & b_{3-N} & \cdots & b_0 \end{pmatrix}$$

- Will consider Real Symmetric Toeplitz matrices.
- Main diagonal zero, $N-1$ independent parameters.
- Normalize Eigenvalues by $\sqrt{N}$.

Numerical Observations: Thoughts?



Eigenvalue Density Measure

$$\mu_{A,N}(x)dx \;=\; \frac{1}{N}\sum_{i=1}^{N}\delta\left(x-\frac{\lambda_i(A)}{\sqrt{N}}\right)dx.$$

The k^{th} moment of $\mu_{A,N}(x)$ is

$$M_k(A,N) \;=\; \frac{1}{N^{\frac{k}{2}+1}}\sum_{i=1}^{N}\lambda_i^k(A) \;=\; \frac{\mathrm{Trace}(A^k)}{N^{\frac{k}{2}+1}}.$$

Let

$$M_k \;=\; \lim_{N\to\infty}\mathbb{E}_A\left[M_k(A,N)\right];$$

have $M_2 = 1$ and $M_{2k+1} = 0$.

Even Moments

$$M_{2k}(N) \;=\; \frac{1}{N^{k+1}} \sum_{1 \le i_1, \cdots, i_{2k} \le N} \mathbb{E}(b_{|i_1-i_2|} b_{|i_2-i_3|} \cdots b_{|i_{2k}-i_1|}).$$

Main Term: b_j's matched in pairs, say

$$b_{|i_m - i_{m+1}|} \;=\; b_{|i_n - i_{n+1}|}, \quad x_m \;=\; |i_m - i_{m+1}| \;=\; |i_n - i_{n+1}|.$$

Two possibilities:

$$i_m - i_{m+1} \;=\; i_n - i_{n+1} \quad \text{or} \quad i_m - i_{m+1} \;=\; -(i_n - i_{n+1}).$$

$(2k-1)!!$ ways to pair, 2^k choices of sign.

Main Term: All Signs Negative (else lower order contribution)

$$M_{2k}(N) = \frac{1}{N^{k+1}} \sum_{1 \le i_1, \cdots, i_{2k} \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} \cdots b_{|i_{2k} - i_1|}).$$

Let $x_1, \dots, x_k$ be the values of the $|i_j - i_{j+1}|$'s, $\epsilon_1, \dots, \epsilon_k$ the choices of sign. Define $\widetilde{x}_1 = i_1 - i_2$, $\widetilde{x}_2 = i_2 - i_3, \dots$.

$$\begin{aligned} i_2 &= i_1 - \widetilde{x}_1 \\ i_3 &= i_1 - \widetilde{x}_1 - \widetilde{x}_2 \\ &\vdots \\ i_1 &= i_1 - \widetilde{x}_1 - \cdots - \widetilde{x}_{2k} \\ \widetilde{x}_1 + \cdots + \widetilde{x}_{2k} &= \sum_{j=1}^{k} (1 + \epsilon_j) \eta_j x_j \ = \ 0, \quad \eta_j \ = \ \pm 1. \end{aligned}$$

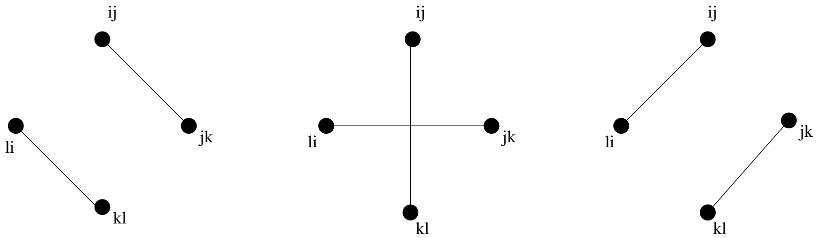
Even Moments: Summary

Main Term: paired, all signs negative.

$$M_{2k}(N) \ \leq \ (2k-1)!! + O_k\left(\frac{1}{N}x\right).$$

Bounded by Gaussian.

The Fourth Moment



$$M_4(N) = \frac{1}{N^3} \sum_{1 \le i_1, i_2, i_3, i_4 \le N} \mathbb{E}(b_{|i_1 - i_2|} b_{|i_2 - i_3|} b_{|i_3 - i_4|} b_{|i_4 - i_1|})$$

Let $x_j = |i_j - i_{j+1}|$.

The Fourth Moment

Case One: $x_1 = x_2, x_3 = x_4$:

$$i_1 - i_2 = -(i_2 - i_3) \quad \text{and} \quad i_3 - i_4 = -(i_4 - i_1).$$

Implies

$$i_1 = i_3, \quad i_2 \text{ and } i_4 \text{ arbitrary.}$$

Left with $\mathbb{E}[b_{x_1}^2 b_{x_3}^2]$:

$$N^3 - N \text{ times get } 1, \; N \text{ times get } p_4 = \mathbb{E}[b_{x_1}^4].$$

Contributes 1 in the limit.

The Fourth Moment

$$M_4(N) = \frac{1}{N^3} \sum_{1 \le i_1, i_2, i_3, i_4 \le N} \mathbb{E}(b_{|i_1-i_2|} b_{|i_2-i_3|} b_{|i_3-i_4|} b_{|i_4-i_1|})$$

Case Two: Diophantine Obstruction: $x_1 = x_3$ and $x_2 = x_4$.

$$i_1 - i_2 \ = \ -(i_3 - i_4) \quad \text{and} \quad i_2 - i_3 \ = \ -(i_4 - i_1).$$

This yields

$$i_1 \ = \ i_2 + i_4 - i_3, \quad i_1, i_2, i_3, i_4 \in \{1, \dots, N\}.$$

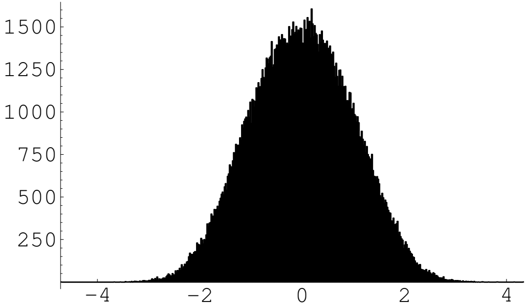
If $i_2, i_4 \ge \frac{2N}{3}$ and $i_3 < \frac{N}{3}$, $i_1 > N$: at most $(1 - \frac{1}{27})N^3$ valid choices.

The Fourth Moment

Theorem: Fourth Moment: Let p_4 be the fourth moment of p. Then

$$M_4(N) = 2\frac{2}{3} + O_{p_4}\left(\frac{1}{N}\right).$$

500 Toeplitz Matrices, 400×400.



Main Result

Theorem: HM '05

For real symmetric Toeplitz matrices, the limiting spectral measure converges in probability to a unique measure of unbounded support which is not the Gaussian. If p is even have strong convergence).

Massey, Miller and Sinsheimer '07 proved that if first row is a palindrome converges to a Gaussian.

Block Circulant Ensemble

With Murat Koloğlu, Gene Kopp, Fred Strauch and Wentao Xiong.

The Ensemble of *m*-Block Circulant Matrices

Symmetric matrices periodic with period m on wrapped diagonals, i.e., symmetric block circulant matrices.

8-by-8 real symmetric 2-block circulant matrix:

$$\left(\begin{array}{cc|cc|cc|cc}
c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\
c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\
\hline
c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\
c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\
\hline
c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\
d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\
\hline
c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0
\end{array}\right).$$

Choose distinct entries i.i.d.r.v.

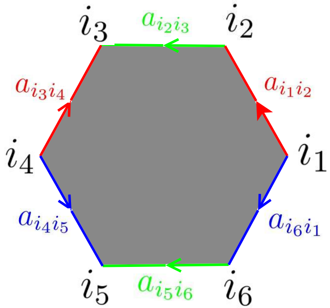
Oriented Matchings and Dualization

Compute moments of eigenvalue distribution (as m stays fixed and $N \to \infty$) using the combinatorics of pairings. Rewrite:

$$\begin{aligned} M_n(N) &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{1 \le i_1, \dots, i_n \le N} \mathbb{E}(a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_n i_1}) \\ &= \frac{1}{N^{\frac{n}{2}+1}} \sum_{\sim} \eta(\sim) m_{d_1(\sim)} \cdots m_{d_l(\sim)}. \end{aligned}$$

where the sum is over oriented matchings on the edges $\{(1,2),(2,3),\dots,(n,1)\}$ of a regular n-gon.

Oriented Matchings and Dualization

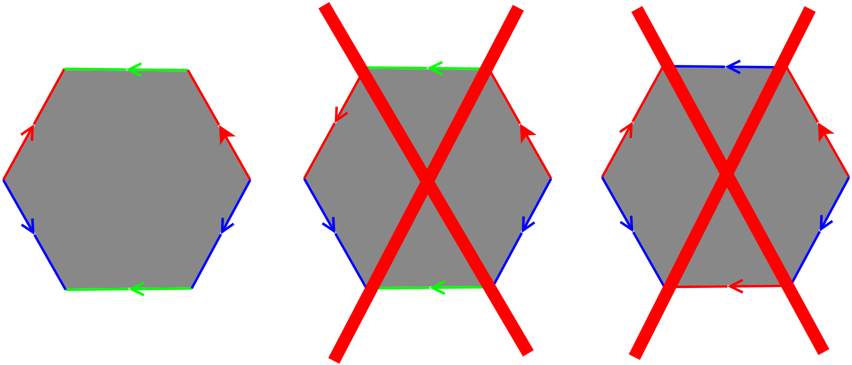


$$\left(\begin{array}{cc|cc|cc|cc}
c_0 & c_1 & c_2 & c_3 & c_4 & d_3 & c_2 & d_1 \\
c_1 & d_0 & d_1 & d_2 & d_3 & d_4 & c_3 & d_2 \\
\hline
c_2 & d_1 & c_0 & c_1 & c_2 & c_3 & c_4 & d_3 \\
c_3 & d_2 & c_1 & d_0 & d_1 & d_2 & d_3 & d_4 \\
\hline
c_4 & d_3 & c_2 & d_1 & c_0 & c_1 & c_2 & c_3 \\
d_3 & d_4 & c_3 & d_2 & c_1 & d_0 & d_1 & d_2 \\
\hline
c_2 & c_3 & c_4 & d_3 & c_2 & d_1 & c_0 & c_1 \\
d_1 & d_2 & d_3 & d_4 & c_3 & d_2 & c_1 & d_0
\end{array}\right)$$

Figure: An oriented matching in the expansion for $M_n(N) = M_6(8)$.

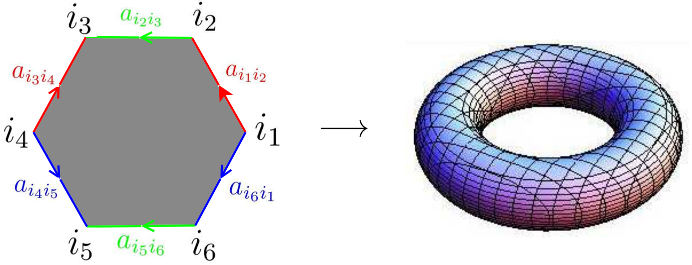
Contributing Terms

As $N \to \infty$, the only terms that contribute to this sum are those in which the entries are matched in pairs and with opposite orientation.



Only Topology Matters

Think of pairings as topological identifications; the contributing ones give rise to orientable surfaces.



Contribution from such a pairing is m^{-2g}, where g is the genus (number of holes) of the surface. Proof: combinatorial argument involving Euler characteristic.

Computing the Even Moments

Theorem: Even Moment Formula

$$M_{2k} = \sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) m^{-2g} + O_k\left(\frac{1}{N}\right),$$

with $\varepsilon_g(k)$ the number of pairings of the edges of a $(2k)$-gon giving rise to a genus g surface.

J. Harer and D. Zagier (1986) gave generating functions for the $\varepsilon_g(k)$.

Harer and Zagier

$$\sum_{g=0}^{\lfloor k/2 \rfloor} \varepsilon_g(k) r^{k+1-2g} \;=\; (2k-1)!!\; c(k,r)$$

where

$$1 + 2\sum_{k=0}^{\infty} c(k,r) x^{k+1} \;=\; \left(\frac{1+x}{1-x}\right)^r.$$

Thus, we write

$$M_{2k} \;=\; m^{-(k+1)} (2k-1)!!\, c(k,m).$$

A multiplicative convolution and Cauchy's residue formula yield the characteristic function of the distribution.

$$\begin{aligned}\phi(t) &= \sum_{k=0}^{\infty} \frac{(it)^{2k} M_{2k}}{(2k)!} = \frac{1}{m}\sum_{k=0}^{\infty} \frac{(-t^2/2m)^k}{k!} c(k,m) \\ &= \frac{1}{2\pi i m} \oint_{|z|=2} \frac{1}{2z^{-1}} \left(\left(\frac{1+z^{-1}}{1-z^{-1}} \right)^m - 1 \right) e^{-t^2 z/2m} \frac{dz}{z} \\ &= \frac{1}{m} e^{\frac{-t^2}{2m}} \sum_{\ell=1}^{m} \binom{m}{\ell} \frac{1}{(\ell-1)!} \left(\frac{-t^2}{m} \right)^{\ell-1}.\end{aligned}$$

Results

Fourier transform and algebra yields

Theorem: Koloğlu, Kopp and Miller

The limiting spectral density function $f_m(x)$ of the real symmetric m-block circulant ensemble is given by the formula

$$f_m(x) = \frac{e^{-\frac{mx^2}{2}}}{\sqrt{2\pi m}} \sum_{r=0}^{m} \frac{1}{(2r)!} \sum_{s=0}^{m-r} \binom{m}{r+s+1} \frac{(2r+2s)!}{(r+s)!s!} \left(-\frac{1}{2}\right)^s (mx^2)^r.$$

As $m \to \infty$, the limiting spectral densities approach the semicircle distribution.

Results (continued)

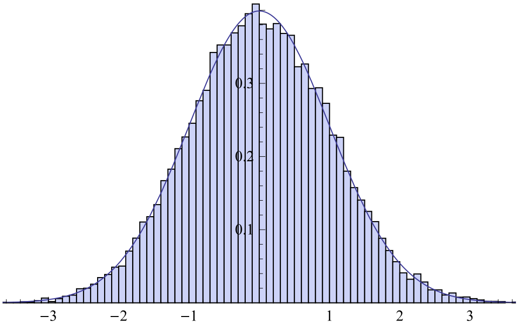


Figure: Plot for f_1 and histogram of eigenvalues of 100 circulant matrices of size 400×400.

Results (continued)

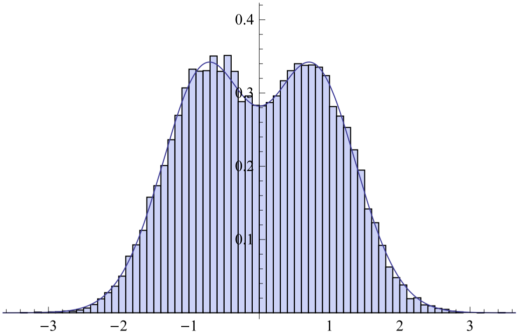


Figure: Plot for f_2 and histogram of eigenvalues of 100 2-block circulant matrices of size 400×400.

Results (continued)

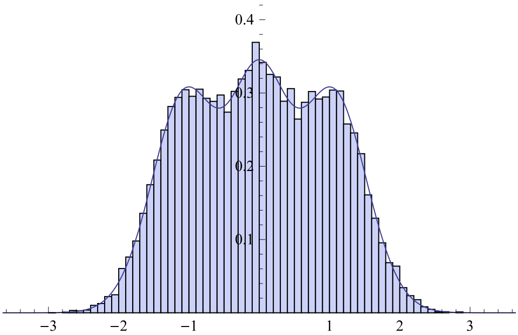


Figure: Plot for f_3 and histogram of eigenvalues of 100 3-block circulant matrices of size 402×402.

Results (continued)

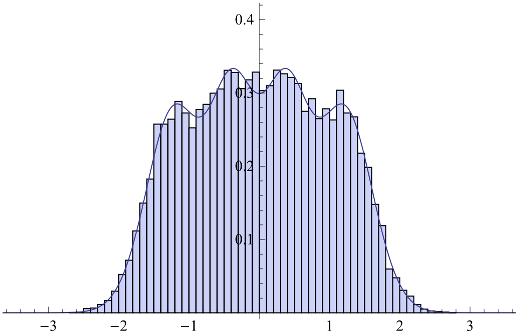


Figure: Plot for f_4 and histogram of eigenvalues of 100 4-block circulant matrices of size 400×400.

Results (continued)

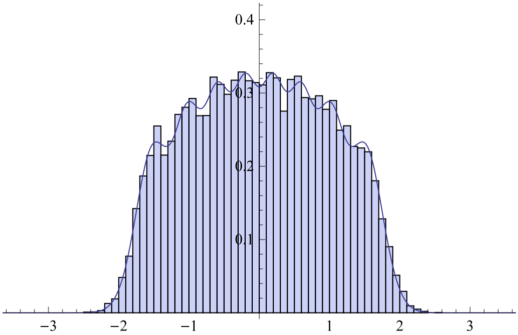


Figure: Plot for f_8 and histogram of eigenvalues of 100 8-block circulant matrices of size 400×400.

Results (continued)

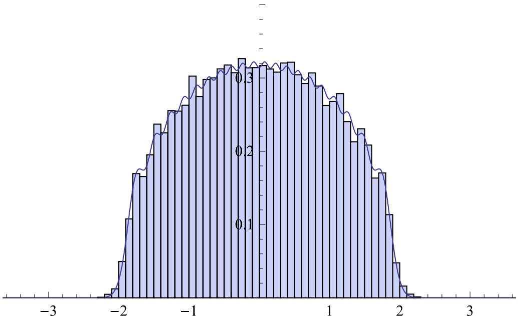


Figure: Plot for f_{20} and histogram of eigenvalues of 100 20-block circulant matrices of size 400×400.

Results (continued)

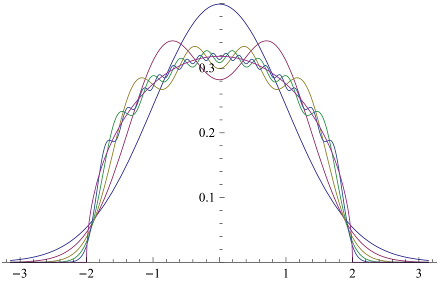


Figure: Plot of convergence to the semi-circle.

The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices (with Murat Koloğlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), Journal of Theoretical Probability **26** (2013), no. 4, 1020–1060. http://arxiv.org/abs/1008.4812

Checkerboard Matrices

- First paper with Paula Burkhardt, Peter Cohen, Jonathan Dewitt, Max Hlavacek, Carsten Sprunger, Yen Nhi Truong Vu, Roger Van Peski, and Kevin Yang, and an appendix joint with Manuel Fernandez and Nicholas Sieger.
- Second paper with Ryan Chen, Yujin Kim, Jared Lichtman, Shannon Sweitzer, and Eric Winsor.
- Third paper with Fangu Chen, Yuxin Lin and Jiahui Yu.

Checkerboard Matrices: $N \times N$ (k, w)-checkerboard ensemble

Matrices $M = (m_{ij}) = M^T$ with a_{ij} iidrv, mean 0, variance 1, finite higher moments, w fixed and

$$m_{ij} \;=\; \begin{cases} a_{ij} & \text{if } i \not\equiv j \text{ mod } k \\ w & \text{if } i \equiv j \text{ mod } k. \end{cases}$$

Example: $(3, w)$-checkerboard matrix:

$$\begin{pmatrix} w & a_{0,1} & a_{0,2} & w & a_{0,4} & \cdots & a_{0,N-1} \\ a_{1,0} & w & a_{1,2} & a_{1,3} & w & \cdots & a_{1,N-1} \\ a_{2,0} & a_{2,1} & w & a_{2,3} & a_{2,4} & \cdots & w \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{0,N-1} & a_{1,N-1} & w & a_{3,N-1} & a_{4,N-1} & \cdots & w \end{pmatrix}$$

Split Eigenvalue Distribution

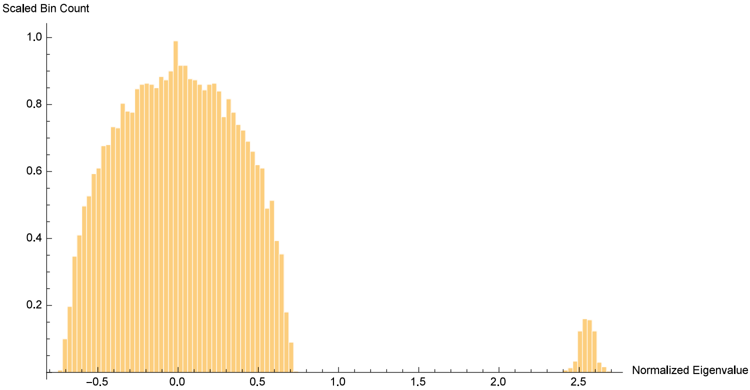


Figure: Histogram of normalized eigenvalues: 2-checkerboard 100×100 matrices, 100 trials.

Split Eigenvalue Distribution

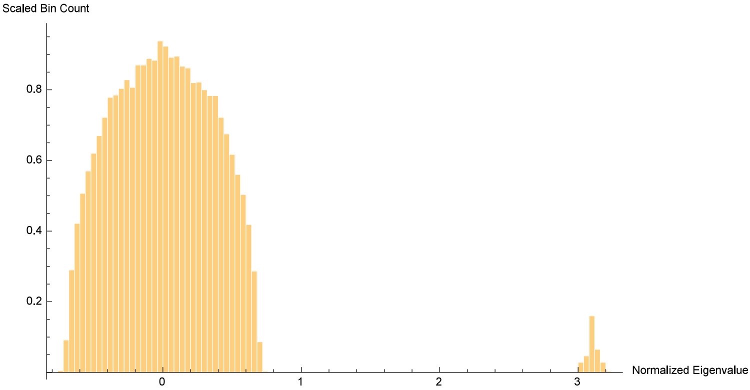


Figure: Histogram of normalized eigenvalues: 2-checkerboard 150×150 matrices, 100 trials.

Split Eigenvalue Distribution

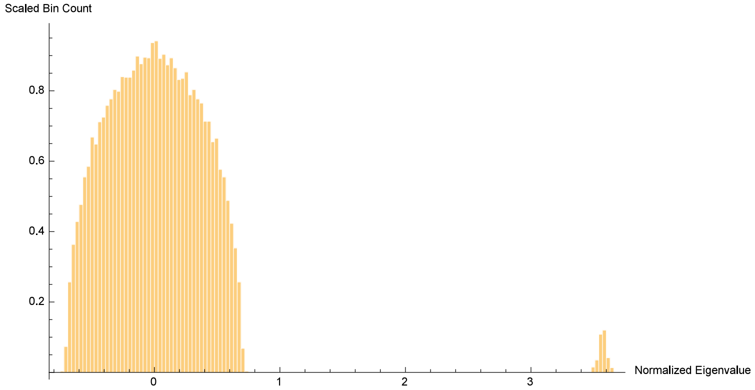


Figure: Histogram of normalized eigenvalues: 2-checkerboard 200×200 matrices, 100 trials.

Split Eigenvalue Distribution

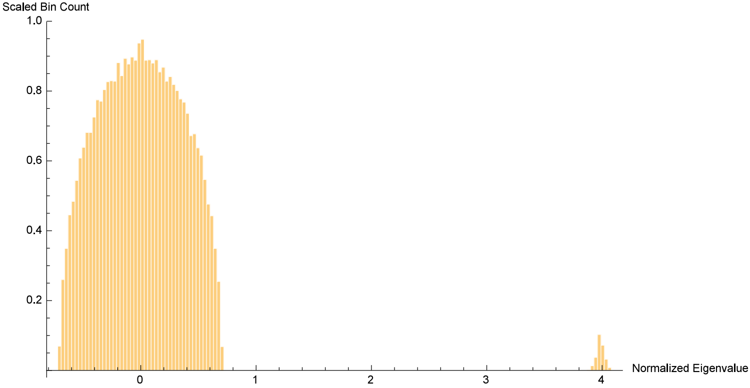


Figure: Histogram of normalized eigenvalues: 2-checkerboard 250×250 matrices, 100 trials.

Split Eigenvalue Distribution

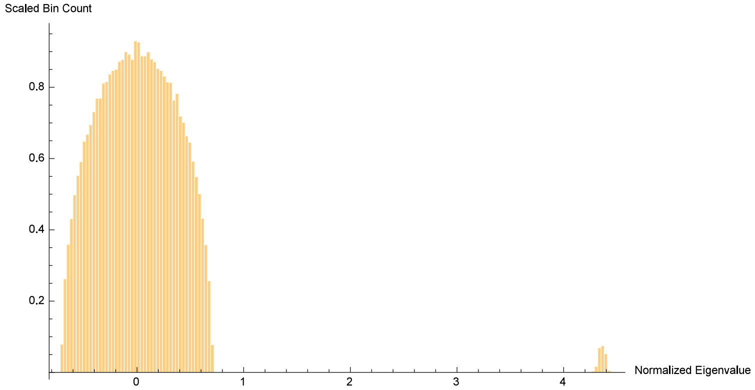


Figure: Histogram of normalized eigenvalues: 2-checkerboard 300×300 matrices, 100 trials.

Split Eigenvalue Distribution

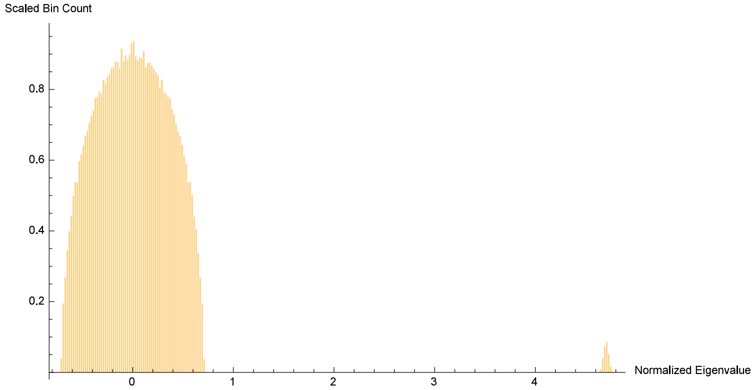


Figure: Histogram of normalized eigenvalues: 2-checkerboard 350×350 matrices, 100 trials.

The Weighting Function

Use weighting function $f_n(x) = x^{2n}(x-2)^{2n}$.

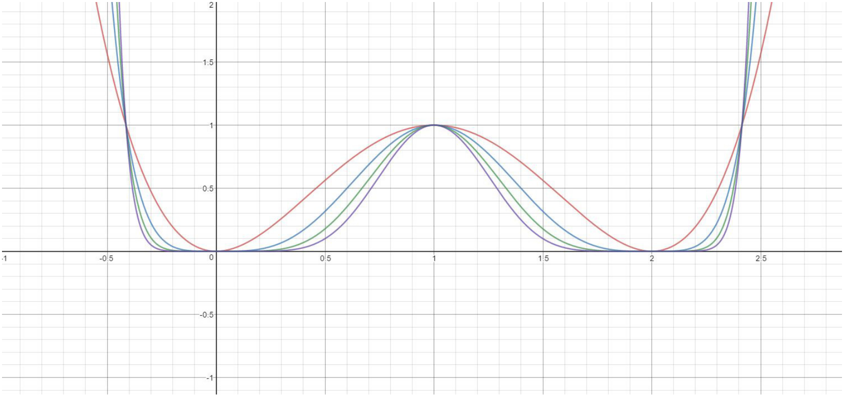


Figure: $f_n(x)$ plotted for $n \in \{1, 2, 3, 4\}$.

The Weighting Function

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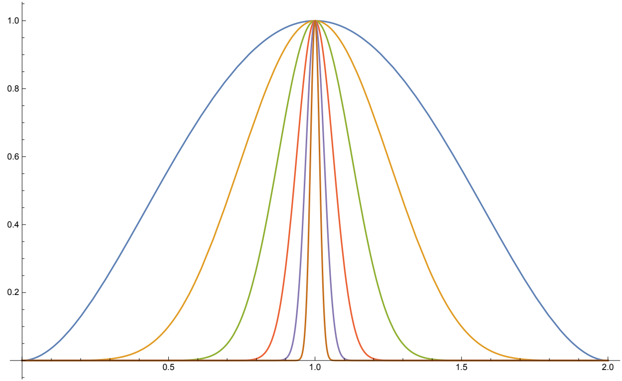

Figure: $f_n(x)$ plotted for $n = 4^m, m \in \{0, 1, \ldots, 5\}$.

Spectral distribution of hollow GOE

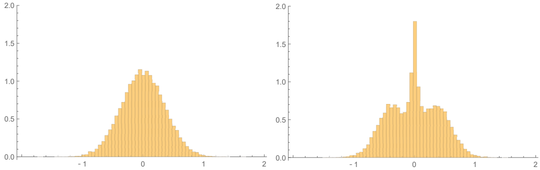

Figure: Hist. of eigenvals of 32000 (Left) 2×2 hollow GOE matrices, (Right) 3×3 hollow GOE matrices.

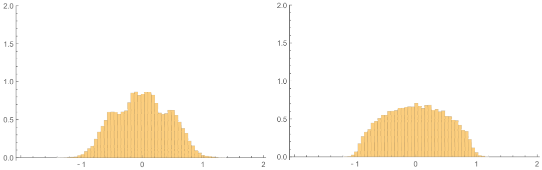

Figure: Hist. of eigenvals of 32000 (Left) 4×4 hollow GOE matrices, (Right) 16×16 hollow GOE matrices.

Anti-Commutator and Checkerboard

Joint work with Glenn Bruda, Bruce Fang, Raul Marquez, Beni Prapashtica, Vismay Sharan, Daeyoung Son, Saad Waheed, and Janine Wang.

Combining two RMT Ensembles

Question

Is there a natural way to combine two random matrix ensembles such that

1. *All the eigenvalues are real;*
2. *The combination is symmetric.*

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Definition

Consider the Anticommutator product, namely

$$\{A, B\} \ := \ AB + BA.$$

Anticommutator of two RMT Ensembles

1. GOE;
2. Palindromic Toeplitz;
3. *k*-checkerboard.

Spectral Density of the Anticommutator

Definition (Spectral Density of the Anticommutator)

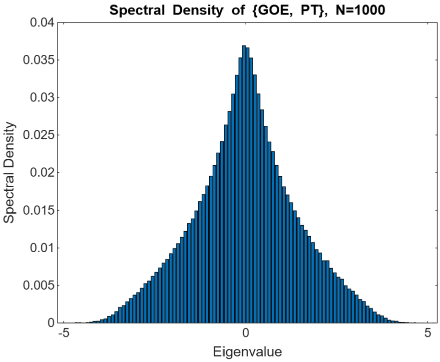
$$\mu_N(x) := \lim_{N\to\infty} \frac{1}{N} \sum_{\lambda\in\Lambda} \delta\left(x - \frac{\lambda}{N}\right).$$

Theorem (Moments of Spectral Density)

$$M_{N,k}(X_N Z_N + Z_N X_N) = \mathbb{E}\left[\mathrm{Tr}\left(\left[\frac{1}{N}(X_N Z_N + Z_N X_N)\right]^k\right)\right].$$

Moments of $\{\mathrm{GOE}, \mathrm{PTE}\}$

Can we compute the moments of the Spectral distribution of the anticommutator of two ensembles GOE, PTE?




```
2th moment: 2.005185
3th moment: -0.000116
4th moment: 12.220592
5th moment: -0.059222
6th moment: 110.056541
7th moment: 2.953869
8th moment: 1177.779577
```

Figure: Moments of $\{GOE, PTE\}$

Anticommutator of Checkerboards

Question

What is the limiting spectral distribution of the anticommutator of k-checkerboard and j-checkerboard?

Multiple Regimes

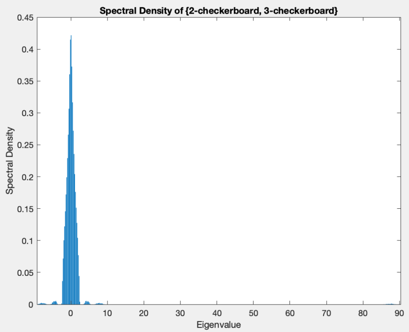


Figure: Multiple Regimes

One bulk and five smaller regimes (blip regimes).

A Closer Look

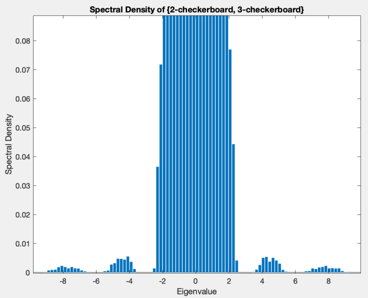


Figure: Intermediary Blips

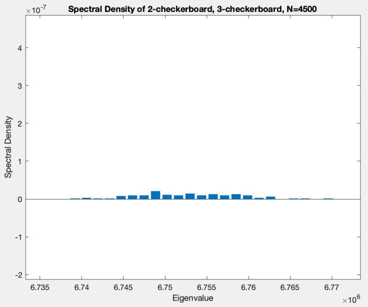


Figure: Largest Blip

Limiting Spectral Distribution

Observation

Numerical simulation tells us location of 5 blip regimes:

1. $\frac{N^2}{kj} + \Theta(N)$ *(1 blip eigenvalue);*
2. $\pm\frac{1}{k}\sqrt{1-\frac{1}{j}}N^{3/2} + \Theta(N)$ *($k-1$ blip eigevalues);*
3. $\pm\frac{1}{j}\sqrt{1-\frac{1}{k}}N^{3/2} + \Theta(N)$ *($j-1$ blip eigenvalues).*

Remark

Standard techniques fail to find centered distribution $\rightarrow$ construction of weight functions.

Definitions

We focus on the spectral distribution of the largest blip.

Definition

The **empirical largest blip spectral measure** of $\{A_N, B_N\}$:

$$\mu_{\{A_N,B_N\}}(x) = \sum_{\lambda \text{ eigenvalues}} g_0^{2n}\left(\frac{jk\lambda}{2N^2}\right)\delta\left(x - \left(\frac{\lambda - \frac{2}{jk}N^2}{N}\right)\right),$$

where $g_0^{2n}(x) = x^{2n}(2-x)^{2n}$, $n(N) = \log\log(N)$.

Weight Function for Largest Blip Regime

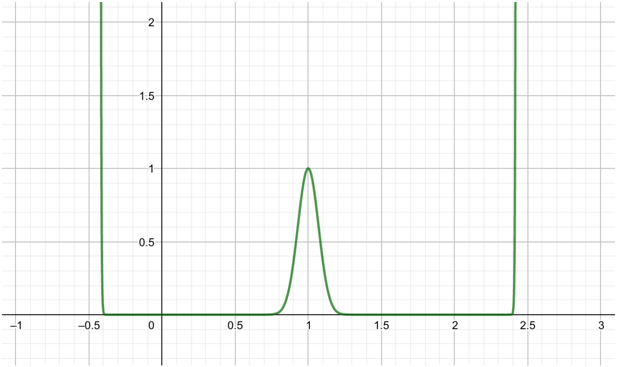


Figure: $g_0(x)^{100} = x^{100}(2-x)^{100}$

Moments of the Empirical Largest Blip Spectral Measure

Theorem

The m^{th} moment of the largest blip spectral measure is

$$\mathbb{E}\left[\mu^{(m)}_{\{A_N,B_N\}}\right] = \sum_{\substack{m_{1a}+m_{1b}+m_{2a}+m_{2b}=m; \\ m_{1a},m_{1b}\text{ even}}} C(m, m_{1a}, m_{2a}, m_{1b}, m_{2b}) \left(k\sqrt{1-\frac{1}{k}}\right)^{m_{1a}+2m_{2a}} \left(j\sqrt{1-\frac{1}{j}}\right)^{m_{1b}+2m_{2b}},$$

where $C(m, m_{1a}, m_{2a}, m_{1b}, m_{2b}) :=$

$$m!\left(\frac{2}{jk}\right)^m \frac{2^{\frac{m_{1a}+m_{1b}}{2}-2(m_{2a}+m_{2b})} m_{1a}!!\, m_{1b}!!}{m_{1a}!\, m_{1b}!\, m_{2a}!\, m_{2b}!}.$$

THANK YOU!

Publications: Random Matrix Theory

1. *Distribution of eigenvalues for the ensemble of real symmetric Toeplitz matrices* (with Christopher Hammond), Journal of Theoretical Probability **18** (2005), no. 3, 537–566. http://arxiv.org/abs/math/0312215

2. *Distribution of eigenvalues of real symmetric palindromic Toeplitz matrices and circulant matrices* (with Adam Massey and John Sinsheimer), Journal of Theoretical Probability **20** (2007), no. 3, 637–662. http://arxiv.org/abs/math/0512146

3. *Nuclei, Primes and the Random Matrix Connection* (with Frank W. K. Firk), Symmetry **1** (2009), 64–105; doi:10.3390/sym1010064. http://arxiv.org/abs/0909.4914

4. *Distribution of eigenvalues for highly palindromic real symmetric Toeplitz matrices* (with Steven Jackson and Thuy Pham), Journal of Theoretical Probability **25** (2012), 464–495. http://arxiv.org/abs/1003.2010

5. *The Limiting Spectral Measure for Ensembles of Symmetric Block Circulant Matrices* (with Murat Koloğlu, Gene S. Kopp, Frederick W. Strauch and Wentao Xiong), Journal of Theoretical Probability **26** (2013), no. 4, 1020–1060. http://arxiv.org/abs/1008.4812

6. *From Quantum Systems to L-Functions: Pair Correlation Statistics and Beyond* (with Owen Barrett, Frank W. K. Firk and Caroline Turnage-Butterbaugh), in Open Problems in Mathematics (editors John Nash Jr. and Michael Th. Rassias), Springer-Verlag, 2016, pages 123–171. https://arxiv.org/pdf/1505.07481

7. *Random Matrix Ensembles with Split Limiting Behavior* (with Paula Burkhardt, Peter Cohen, Jonathan Dewitt, Max Hlavacek, Carsten Sprunger, Yen Nhi Truong Vu, Roger Van Peski, and Kevin Yang, and an appendix joint with Manuel Fernandez and Nicholas Sieger), Random Matrices: Theory and Applications **7** (2018), no. 3, 1850006 (30 pages), DOI: 10.1142/S2010326318500065.