Modeling the Vanishing of L-functions at the Central Point

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SMALL REU at Williams College

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Outline

- Motivation
- Background
- Main question
- Modeling vanishing
- Effective Matrix Size
  - One-level density
  - Pair-correlation
- Cutoff value
- Future directions

Conjecture (Montgomery-Dyson, 1970s)

High on the critical line, spacings between

$$\text{zeros of the Riemann zeta function} \longleftrightarrow \text{eigenvalues of Gaussian Unitary Ensemble.}$$

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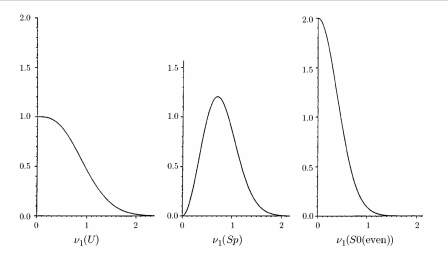
Conjecture (Katz-Sarnak, 1990s)

Katz-Sarnak conjectured the of the following distributions match:

- lowest-lying zeros at the critical point of families of L-functions,
- eigenvalues of characteristic polynomials at 1 of random matrices.



Source: N. M. Katz and P. Sarnak, *Zeros of zeta functions and symmetry*, Bulletin of the American Mathematical Society (1) **36** (1999), pages 1-26.

An Excised Orthogonal Model

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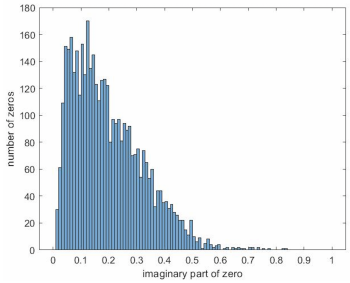

Example of repulsion from zero for a family of twists of a cuspidal newform

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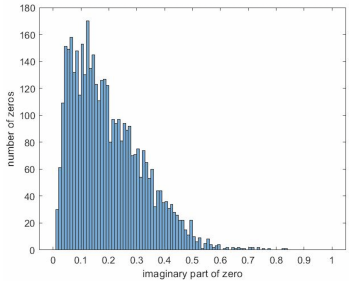

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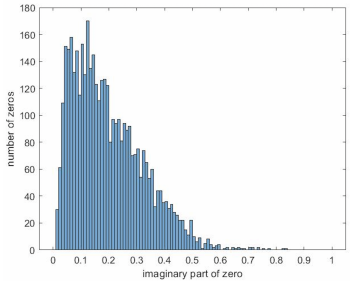

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Motivating Question

Can we generalize this model to families of a cuspidal newform L-function?

For $\mathrm{Re}(s) > 1$, the Hecke L-function attached to $f \in S_k^{\mathrm{new}}(M, \chi_f)$ of level odd prime M, weight k, and nebentypus χ_f is given by the Dirichlet series

$$f(z) \;=\; \sum_{n=1}^{\infty} a_f(n) e^{2\pi i n z}, .$$

Put $\lambda_f(n) = a_f(n)/n^{(k-1)/2}$.

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Then the Euler product is

$$L(f,s) \;=\; \prod_p \left(1 - \lambda_f(p)p^{-s} + \chi_f(p)p^{-2s}\right)^{-1}.$$

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The functional equation of the completed L-function is given by

$$\Lambda(f,s) \;=\; \chi_f \epsilon_f \Lambda(f, 1-s),$$

where ϵ_f is the root number.

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We twist its L-function $L(f,s)$ by a quadratic character ψ_d as such

$$L(f\otimes\psi_d,s) \;=\; \sum_{n=1}^{\infty}\frac{\psi_d(n)\lambda_f(n)}{n^s} \;=\; \prod_{p}\Big(1-\psi_d(p)\lambda_f(p)p^{-s}+\psi_d(p)\chi_f(p)p^{-2s}\Big)^{-1}.$$

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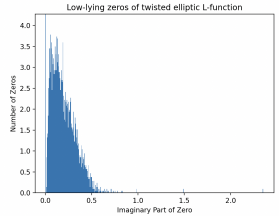
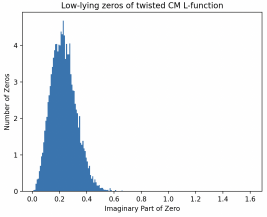
We twist the L-function $L(f,s)$ by those characters with fundamental discriminant $d\in\mathcal{D}$ that come from

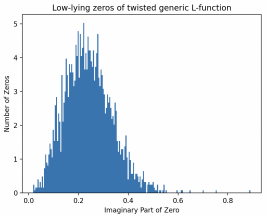
$$\mathcal{D}_f(X) \coloneqq \begin{cases} \{d\in\mathcal{D}\,|\,0<d\le X,\ \psi_d(M)\epsilon_f=+1\} & \chi_f \text{ principal, even twists,}\\ \{d\in\mathcal{D}\,|\,0<d\le X,\ \psi_d(M)=\{\pm1\}\} & \chi_f \text{ non-principal, } f=\overline{f},\\ \{d\in\mathcal{D}\,|\,0<d\le X\} & \chi_f \text{ non-principal, } f\neq\overline{f}. \end{cases}$$

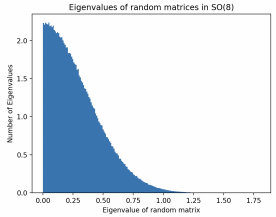
Motivating question, revisited

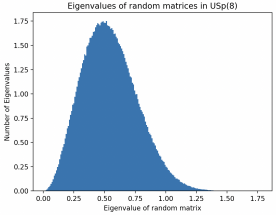
How do we model the behavior of the lowest-lying zeros of a family of quadratic twists of a given cuspidal newform L-function?

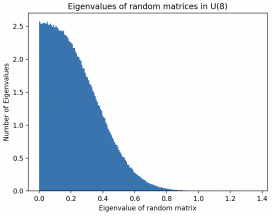












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- Step 1. Find the associated symmetry group of a given cuspidal newform twisted by a quadratic character over the family $\mathcal{D}_f(X)$.
  - Substep 1. Find one-level density of $SO(2N)$, $USp(2N)$, and $U(N)$.
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The (asymptotic scaled) one-level density of matrix groups for large N is given by

$$R_{G(N)} = \int_0^1 \varphi(\theta)(1 + K_{G(N)}(\theta) + O(N^{-3}))\, d\theta,$$

$$K_{G(N)}(\theta) = \begin{cases} \dfrac{\sin(2\pi\theta)}{2\pi\theta} - \dfrac{1+\cos(2\pi\theta)}{2N} - \dfrac{\pi\theta\sin(2\pi\theta)}{6N^2} & \mathrm{SO}(2N), \\ -\dfrac{\sin(2\pi\theta)}{2\pi\theta} + \dfrac{1-\cos(2\pi\theta)}{2N} + \dfrac{\pi\theta\sin(2\pi\theta)}{6N^2} & \mathrm{USp}(2N), \\ 0 & \mathrm{U}(N). \end{cases}$$

Assuming the ratios conjectures, the one-level density for the scaled zeros of a family of twists for a given cuspidal newform is

$$D_1(f;\varphi) \;=\; \int_{\mathbb{R}} g(\tau)(1+Q(\tau)+O(R^{-3}))\,d\tau,$$

where

$$Q(\tau)=\begin{cases} \dfrac{\sin(2\pi\tau)}{2\pi\tau}-a_1\dfrac{1+\cos(2\pi\tau)}{R}-a_2\dfrac{\pi\tau\sin(2\pi\tau)}{R^2} & \chi_f \text{ principal, even twists,}\\ -\dfrac{\sin(2\pi\tau)}{2\pi\tau}+b_1\dfrac{1-\cos(2\pi\tau)}{R}+b_2\dfrac{\pi\tau\sin(2\pi\tau)}{R^2} & \chi_f \text{ non-principal, } f=\overline{f},\\ \dfrac{c_1+c_2\cos(2\pi\tau)}{R}+d_1\dfrac{\pi\tau\sin(2\pi\tau)}{R^2} & \chi_f \text{ non-principal, } f\neq\overline{f}. \end{cases}$$

$$\begin{cases} K_{SO(2N)}(\theta) = \dfrac{\sin(2\pi\theta)}{2\pi\theta} - \dfrac{1+\cos(2\pi\theta)}{2N} - \dfrac{\pi\theta\sin(2\pi\theta)}{6N^2} & \mathrm{SO}(2N), \\ Q(\tau) = \dfrac{\sin(2\pi\tau)}{2\pi\tau} - a_1\dfrac{1+\cos(2\pi\tau)}{R} - a_2\dfrac{\pi\tau\sin(2\pi\tau)}{R^2} & \chi_f \text{ principal, even twists,} \end{cases}$$

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$$\begin{cases} K_{USp(2N)}(\theta) &= -\dfrac{\sin(2\pi\theta)}{2\pi\theta} + \dfrac{1-\cos(2\pi\theta)}{2N} + \dfrac{\pi\theta\sin(2\pi\theta)}{6N^2} \\ Q(\tau) &= -\dfrac{\sin(2\pi\tau)}{2\pi\tau} + b_1\dfrac{1-\cos(2\pi\tau)}{R} + b_2\dfrac{\pi\tau\sin(2\pi\tau)}{R^2} \end{cases} \quad \begin{array}{l} \mathrm{USp}(2N), \\ \chi_f \text{ non-principal, } f = \overline{f}, \end{array}$$

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$$\begin{cases} K_{U(N)}(\theta) &= 0 \\ Q(\tau) &= \dfrac{c_1 + c_2\cos(2\pi\tau)}{R} + d_1\dfrac{\pi\tau\sin(2\pi\tau)}{R^2} \end{cases} \quad \begin{array}{l} \mathrm{U}(N), \\ \chi_f \text{ non-principal, } f \neq \overline{f}. \end{array}$$

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We match lower order terms of the one-level densities coming from matrix groups and newforms to obtain the effective matrix size

$$N_{\text{effective}} = \begin{cases} R/a_1 & \chi_f \text{ principal, even twists,} \\ R/b_1 & \chi_f \text{ non-principal, } f = \overline{f}. \end{cases}$$

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Remark the scaled one-level density for $U(N)$ has no lower order terms while the generic case has lower order terms:

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In the limit, the scaled one-level densities match.
To find the effective matrix size for the generic case, we need another statistic.

For large T, we denote the pair-correlation of a family of twists of a given form f by

$$P(f \otimes \psi_d; \varphi) \;=\; \sum_{0<\gamma,\gamma'<T} \varphi(\gamma - \gamma'),$$

where the γ's are the imaginary part of the zeros and φ a (holomorphic) test function.

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where the γ's are the imaginary part of the zeros and φ a (holomorphic) test function. Using the ratios conjecture and analyticity, we expand the above using series expansions

$$P(f \otimes \psi_d; \varphi) \;\coloneqq\; \frac{T}{2\pi} R\Bigg[h(0) + \int_{\mathbb{R}} h(y)\Bigg(1 - \left(\frac{\sin \pi y}{\pi y}\right)^2 + \frac{e_1 - e_2 \sin^2 \pi y}{R^2} - \frac{e_3 \pi y \sin 2\pi y}{R^3} + O(R^{-4})\Bigg)\, dy\Bigg] + O(T^{\varepsilon+1/2}),$$

where

$$R \;=\; \log\left(\frac{\sqrt{M}|d|T}{2\pi e}\right). \tag{1}$$

Compare the $U(N)$ pair-correlation

$$1 - \left(\frac{\sin \pi y}{\pi y}\right)^2 - \frac{\sin^2 \pi y}{3N^2},$$

to the scaled one-level density, we compare the term

$$1 - \left(\frac{\sin \pi y}{\pi y}\right)^2 + \frac{e_1 - e_2 \sin^2 \pi y}{R^2} - e_3 \frac{\pi y \sin 2\pi y}{R^3}.$$

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Minimizing the "distance" between these two functions in the L^2 norm suggests that we choose

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Minimizing the "distance" between these two functions in the L^2 norm suggests that we choose

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Taking the expectation as we range over all fundamental discriminants in our family gives

$$N_{\text{effective}} := \frac{R}{\sqrt{3\langle e_2 \rangle - 4\langle e_1 \rangle}}.$$

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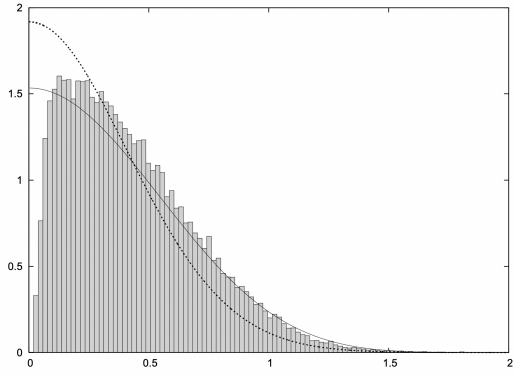
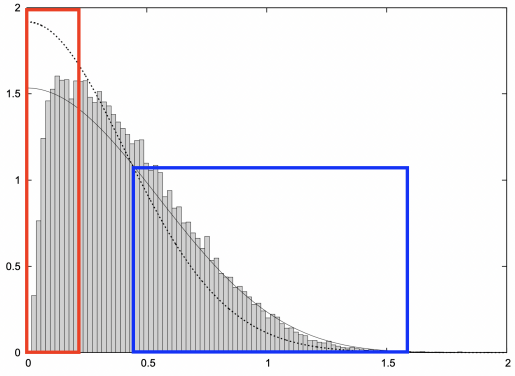
Red box highlights repulsion of zeros of cuspidal newform.
Blue box highlights need to fit tail of matrix group.





E. Dueñez, D. K. Huynh, J. P. Keating, S. J. Miller, N. C. Snaith, *A random matrix model for elliptic curve L-functions of finite conductor*, Journal of Physics A **45** (2012) (32pp).

- Compute symmetric square L-function at $s = 1$ for a cuspidal newform in PARI/GP.
- Find integral function $\xi_{\mathrm{sym}^2 f}(s)$ of symmetric square L-function and evaluate it at $s = 1$.

Acknowledgements

We would like to thank our mentor, Professor Steven J. Miller, and previous years of SMALL for their contributions.

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