When almost all sets are difference dominated in $\mathbb{Z}/n\mathbb{Z}$

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Integers Conference
University of West Georgia
October 7, 2016

Background

Given a set $A \subset \mathbb{Z}$, define the *sumset* and *difference set*

$$A + A := \{a + b : a, b \in A\}$$
$$A - A := \{a - b : a, b \in A\}$$

Definition

If $|A + A| > |A - A|$, A is said to be *sum-dominated*.
If $|A + A| = |A - A|$, A is said to be *balanced*.
If $|A + A| < |A - A|$, A is said to be *difference-dominated*.

Background

- Addition commutes, subtraction doesn't.

"Even though there exist sets A which have more sums than differences, such sets should be rare, and it must be true with the right way of counting that the vast majority of sets satisfies $|A - A| > |A + A|$."

–Melvyn Nathanson

Known results

Theorem (Martin and O'Bryant, 2006)

A positive proportion of sets of integers are sum-dominated, in the sense that the quantity

$$\liminf_{n\to\infty} \frac{\#\text{ of sum-dominated subsets of }\{1,\ldots,n\}}{2^n}$$

is positive.

Equivalent: if we pick a subset of $\{1,\ldots,n\}$ uniformly at random, the probability of being sum-dominated is nonzero as $n \to \infty$.

Known results

What's going on?

- "Fringe" elements are most important.
  - Large numbers and small numbers have fewer representations as sums than numbers in the middle.
  - Think of rolling two dice – more ways to get 7 than 12.
- If A is big, then almost every possible sum and difference is realized.

Known results

- What if we pick random subsets in a different way?
- Construct $A \subseteq \{1, \ldots, n\} \subset \mathbb{Z}$ randomly by picking each element independently with probability $p(n)$.
    - Uniform case corresponds to $p(n) = 1/2$ constant.
    - Let $p(n)$ decay to 0 as $n \to \infty$ (smaller sets are more likely to be picked).

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Theorem (Hegarty and Miller, 2009)

Let $A \subseteq \{1, \dots, n\} \subset \mathbb{Z}$ be chosen randomly in this way where $p(n) = o(1)$. Then

$$\text{Prob}\,(A \text{ is difference-dominated}) \to 1 \text{ as } n \to \infty.$$

New setting

- Look at subsets $A \subseteq \mathbb{Z}/n\mathbb{Z}$ (i.e., take sums and differences modulo n).
  - No fringe elements!
- Construct randomly according to decaying probability $p(n)$.
  - Try to avoid sumsets and difference sets being full.

Notation

Let $X(n)$ and $Y(n)$ be random variables depending on n. We write $X(n) \sim Y(n)$ if, for every $\epsilon > 0$,

$$\text{Prob}\left(\left|\frac{X(n)}{Y(n)} - 1\right| < \epsilon\right) \to 1 \text{ as } n \to \infty.$$

Our result (full statement)

Theorem (HLM, 2016)

Let $A \subseteq \mathbb{Z}/n\mathbb{Z}$ be chosen randomly according to a binomial parameter $p(n) = o(1)$.

- (Fast decay) If $p(n) = o(n^{-1/2})$, then $|A+A| \sim \frac{1}{2}(np(n))^2$ and $|A-A| \sim (np(n))^2$.
- (Critical decay) If $p(n) = c \cdot n^{-1/2}$, then $|A+A| \sim (1-\exp(-c^2/2))n$ and $|A-A| \sim (1-\exp(-c^2))n$.
- (Slow decay) If $\sqrt{\log n} \cdot n^{-1/2} = o(p(n))$ and n is prime, then $|A+A| \sim |A-A| \sim n$.

Our result (qualitative statement)

Theorem (HLM, 2016)

Let $A \subseteq \mathbb{Z}/n\mathbb{Z}$ be chosen randomly according to a binomial parameter $p(n) = o(1)$.

- (Fast/critical decay) If $p(n) = O(n^{-1/2})$, then

$$\text{Prob}\,(A \text{ is difference-dominated}) \to 1 \text{ as } n \to \infty.$$

- (Slow decay) If $n^{-1/2}\sqrt{\log n} = o(p(n))$ and n is prime, then

$$\text{Prob}\,(A \text{ is balanced}) \to 1 \text{ as } n \to \infty.$$

Fast/critical decay ($p(n) = O(n^{-1/2})$)

- Expect $|A| \sim np(n)$.
- Control number of times a sum or difference is realized more than once.
  - Compute mean number of repeats and bound the variance.
  - Modify techniques of Hegarty and Miller.
- In slow decay case, get

$$|A+A| \sim \binom{|A|}{2} = \frac{1}{2}|A|(|A|-1) \sim \frac{1}{2}(np(n))^2$$
$$|A-A| \sim |A|(|A|-1) \sim (np(n))^2.$$

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- Critical decay case is similar, but a bit more delicate.

Slow decay ($\sqrt{\log n} \cdot n^{-1/2} = o(p(n))$)

- No control over number of repeats.
  - When $p(n) \gg n^{-1/2}$, expect $|A| \sim np(n) \gg n^{1/2}$.
  - Number of pairs $\sim |A|^2 \gg n$, but only n possible sums!

Slow decay ($\sqrt{\log n} \cdot n^{-1/2} = o(p(n))$)

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    - When $p(n) \gg n^{-1/2}$, expect $|A| \sim np(n) \gg n^{1/2}$.
    - Number of pairs $\sim |A|^2 \gg n$, but only n possible sums!
- Compute number of *missing* sums and differences instead.
    - Show they are both 0 with high probability.

Idea of proof

- S^c := number of missing sums.
- D^c := number of missing differences.
- Show $\mathbb{E}[S^c]$, $\mathbb{E}[D^c]$, $\mathrm{Var}(S^c)$, and $\mathrm{Var}(D^c)$ all tend to 0 as $n \to \infty$.
  - By Chebyshev's inequality, this implies $\mathrm{Prob}(S^c = D^c = 0) \to 1$ as $n \to \infty$.

Comparison with $\mathbb{Z}$

- In $\mathbb{Z}$, $\mathbb{E}[S^c]$ and $\mathbb{E}[D^c]$ don't tend to 0 (Hegarty & Miller).
  - Qualitatively different behavior in $\mathbb{Z}/n\mathbb{Z}$.
- In $\mathbb{Z}$, need heavy machinery from probability to prove strong concentration.
  - More elementary arguments in $\mathbb{Z}/n\mathbb{Z}$.

Computing $\mathbb{E}[S^c]$

- Write

$$\mathbb{E}[S^c] = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} \text{Prob}\left(k \notin A + A\right).$$

Computing $\mathbb{E}[S^c]$

- Each $k \in \mathbb{Z}/n\mathbb{Z}$ can be written as a sum in exactly $(n+1)/2$ *disjoint* ways.
  - This is what separates $\mathbb{Z}/n\mathbb{Z}$ from $\mathbb{Z}$.

Computing $\mathbb{E}[S^c]$

- Each $k \in \mathbb{Z}/n\mathbb{Z}$ can be written as a sum in exactly $(n+1)/2$ *disjoint* ways.
    - This is what separates $\mathbb{Z}/n\mathbb{Z}$ from $\mathbb{Z}$.
- $\text{Prob}\,(k \notin A + A) = (1 - p^2)^{(n+1)/2}$ independently of k.
- $\mathbb{E}[S^c] = n(1 - p^2)^{(n+1)/2} \sim n(1 - p^2)^{n/2}$.
    - Note: doesn't tend to 0 unless $\sqrt{\log n} \cdot n^{-1/2} = o(p(n))$.

Computing $\mathbb{E}\left[D^c\right]$

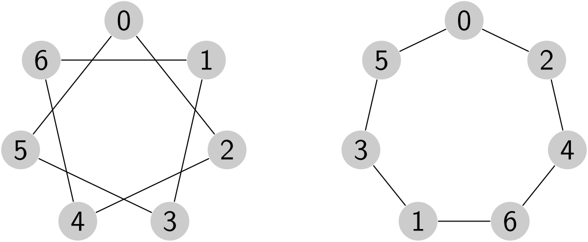
- Each $k \in \mathbb{Z}/n\mathbb{Z}$ can be written as a difference in exactly n different ways.
  - Pairs aren't disjoint, so we can't count them independently like we did for sums.

Computing $\mathbb{E}\left[D^c\right]$

- Each $k \in \mathbb{Z}/n\mathbb{Z}$ can be written as a difference in exactly n different ways.
  - Pairs aren't disjoint, so we can't count them independently like we did for sums.
- Translate to graph theory.

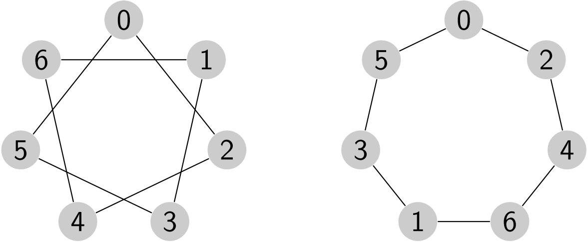
Graph theoretic framework

- Modeling $\text{Prob}\,(k \notin A - A)$.
- Each element of $\mathbb{Z}/n\mathbb{Z}$ is a vertex, connect a to b if $a - b \equiv k \pmod{n}$.
- Example $(n = 7, k = 2)$:



Computing $\mathbb{E}\left[D^c\right]$

- $\text{Prob}\left(k \notin A - A\right)$ is the same as the probability that no two adjacent vertices are in A.
- Equivalent: pick a random subset of $\{1, \ldots, n\}$, probability that it doesn't contain any consecutive elements.



Computing $\mathbb{E}\left[D^c\right]$

- Counting problem – probability is

$$\sum_{r=1}^{\lfloor n/2 \rfloor} \left[\binom{n-r+1}{r} - \binom{n-r-1}{r-2} \right] p^r (1-p)^{n-r}$$

$$\sim \sum_{r=1}^{\lfloor n/2 \rfloor} \binom{n-r}{r} p^r (1-p)^{n-r}.$$

- So

$$\mathbb{E}\left[D^c\right] \sim n \sum_{r=1}^{\lfloor n/2 \rfloor} \binom{n-r}{r} p^r (1-p)^{n-r}.$$

Computing variances

- Define indicator random variables

$$X_k \ := \begin{cases} 1 & k \notin A + A \\ 0 & k \in A + A. \end{cases}$$

- $S^c = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} X_k.$

Computing variances

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Computing variances

- Define indicator random variables

$$X_k := \begin{cases} 1 & k \notin A + A \\ 0 & k \in A + A. \end{cases}$$

- $S^c = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} X_k$.
- X_k are not independent, so

$$\mathrm{Var}\left(S^c\right) = \sum_{k \in \mathbb{Z}/n\mathbb{Z}} \mathrm{Var}\left(X_k\right) + \sum_{i \neq j \in \mathbb{Z}/n\mathbb{Z}} \mathrm{Cov}\left(X_i, X_j\right).$$

Computing variances

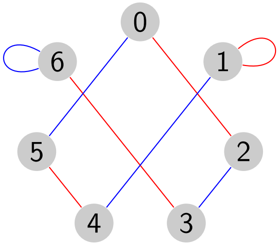
- Covariance terms rely on evaluating

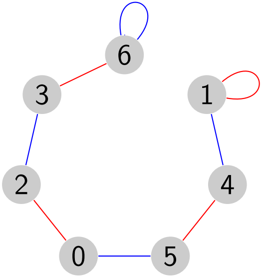
$$\mathrm{Prob}\left(i \notin A + A \text{ and } j \notin A + A\right).$$

- Graph theory works again!

Graph theoretic framework

- n, i, j fixed.
- Connect a and b with an edge if $a + b \equiv i$ or $a + b \equiv j \mod n$.
- Example ($n = 7$, $i = 2$, $j = 5$):





Computing variances

- Translate to same counting problem.
- So

$$\text{Prob}\left(i \notin A+A \text{ and } j \notin A+A\right) \sim \sum_{r=1}^{\lfloor n/2 \rfloor} \binom{n-r}{r} p^r (1-p)^{n-r}.$$

- In variance expression, this term dominates, giving

$$\text{Var}\left(S^c\right) \sim n^2 \sum_{r=1}^{\lfloor n/2 \rfloor} \binom{n-r}{r} p^r (1-p)^{n-r}.$$

- $\text{Var}\left(D^c\right)$ handled similarly.

Getting a good estimate

Key Lemma

Let

$$F(n) := \sum_{r=1}^{\lfloor n/2 \rfloor} \binom{n-r}{r} p^r (1-p)^{n-r}.$$

Then $F(n) = o(1/n^3)$.

Getting a good estimate

- By comparing to a binomial distribution and using Stirling's formula, we can get the bound

$$n^3 F(n) \;\leq\; 2n^4(e^p - pe^p)^n.$$

- Take log and use power series expansion:

$$\log(n^3 F(n)) \;\ll\; \log n - \frac{1}{2}np^2 + O(np^3).$$

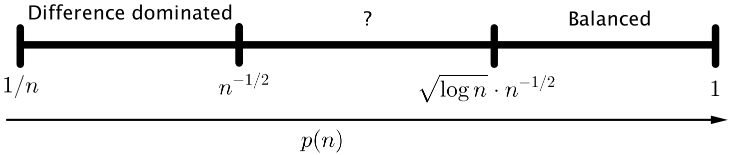
- Tends to $-\infty$ provided

$$\log n = o(np^2) \iff \sqrt{\log n} \cdot n^{-1/2} = o(p(n)).$$

"Correspondence" principle

- When $p(n)$ decays rapidly, subsets of $\mathbb{Z}/n\mathbb{Z}$ behave like subsets of $\mathbb{Z}$ (as $n \to \infty$).
- When $p(n)$ decays slowly, subsets of $\mathbb{Z}/n\mathbb{Z}$ behave as if $p(n)$ were constant (as $n \to \infty$).

Open questions



- What happens when $n^{-1/2} \ll p(n) \ll \sqrt{\log n} \cdot n^{-1/2}$?
- Can we extend slow decay analysis to non-prime n?

Thanks

- Anand Hemmady, Steven J. Miller
- University of West Georgia
- Bruce Landman, Florian Luca, Melvyn Nathanson, Jarik Nesetril, Richard Nowakowski
- Williams College
- NSF/SMALL REU
- Williams College Finnerty Fund

References

P. Hegarty and S.J. Miller, "When almost all sets are difference dominated", *Random Structures and Algorithms*, **35** (2009), no. 1, 118-136. http://arxiv.org/pdf/0707.3417.pdf

G. Martin and K. O'Bryant, "Many sets have more sums than differences" (2006), arXiv: math.NT/0608131. http://arxiv.org/pdf/math/0608131.pdf

M. Nathanson, "Problems in Additive Number Theory, 1" (2006), arXiv: math.NT/0604340. http://arxiv.org/pdf/math/0604340.pdf