Limiting Behavior in Missing Sums of Sumsets

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Introduction

Given $A \subseteq \mathbb{Z}$, define its sumset

- $A + A := \{a_1 + a_2 \mid a_1, a_2 \in A\}$.

Setting

- Fix $N \geq 0$. Fix $p \in (0, 1)$, and let $q := 1 - p$.
- Select $A \subseteq [0, N]$ by a Bernoulli process: for each $k \in [0, N]$, independently include k in A with probability p.

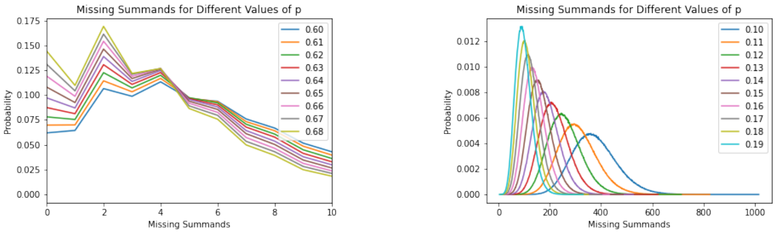
Setting

- Fix $N \geq 0$. Fix $p \in (0, 1)$, and let $q := 1 - p$.
- Select $A \subseteq [0, N]$ by a Bernoulli process: for each $k \in [0, N]$, independently include k in A with probability p.
- Recent research in $|A + A|$ as a random variable.
- Martin and O'Bryant's seminal paper [MO] compared $|A + A|$ to $|A - A|$ when $p = 1/2$.

Why study sumsets?

- Prove patterns seen from Monte Carlo simulations.
- Might potentially aid other number-theoretic work.

Observed: Divots and Concentration



(a) Large p **(b)** Small p

Figure: Point distribution function $\mathbb{P}\left(|(A+A)^c| = m\right)$ for several values of p, for N very large.

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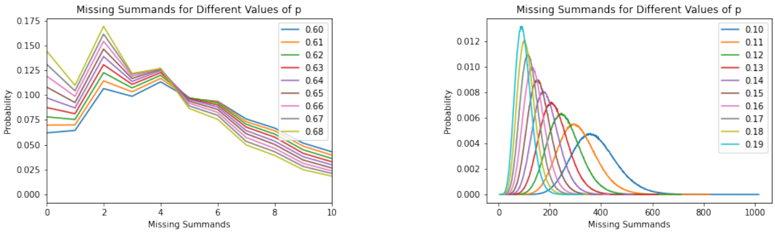

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Figure: Point distribution function $\mathbb{P}\left(|(A+A)^c| = m\right)$ for several values of p, for N very large.

- For large p, missing an even number appears more likely.
- For small p, we see concentration around the mean.

Observed: Exponential Decay

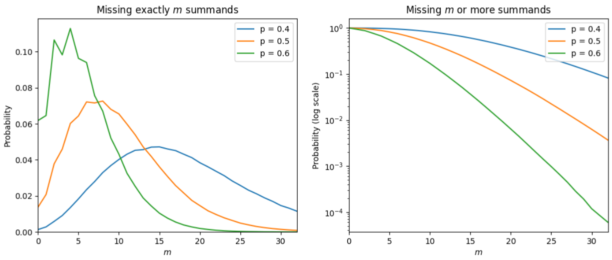

Figure: Point distribution function $\mathbb{P}(|(A+A)^c| = m)$ and cumulative distribution function $\mathbb{P}(|(A+A)^c| \geq m)$ for several values of p, for N very large.

- CDF appears to decay exponentially.

Prior Work: Mean and Variance

Theorem (Martin and O'Bryant '06 [MO])

If $p = \frac{1}{2}$, *then* $\mathbb{E}[|(A+A)^c|] = 10 + O((3/4)^{N/2})$.

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If $p = \frac{1}{2}$, then $\mathbb{E}[|(A+A)^c|] = 10 + O((3/4)^{N/2})$.

Theorem (Lazarev, Miller, and O'Bryant '13 [LMO])

If $p = \frac{1}{2}$, then for $i < j \leq N$ with i, j odd,

$$\mathbb{P}(i \text{ and } j \notin A+A) = \frac{1}{2^{j+1}} F_{q+2}^{r} F_{q+4}^{r'}$$

for q, r, r' depending on i and j, and similar formulations hold for the other 3 parity cases.

Prior Work: Exponential Decay

Theorem (Lazarev, Miller, and O'Bryant '13 [LMO])

If $p = \frac{1}{2}$, then

$$m\,(3/4)^{m/2} \ll \mathbb{P}\left(|(A+A)^c| = m\right) \ll (\phi/2)^{m/2} \qquad (1)$$

Prior Work

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- Chu, King, Luntzlara, Martinez, Miller, Shao, Sun, and Xu [CKLMMSSX] study sumsets for generic p.
- [CKLMMSSX] and [LMO] both use graph-theoretic approaches, particularly the notion of a *condition graph*.

Prior Work

Theorem (King, Martinez, Miller, Sun '19)

For $p \in [0,1]$ and $q := 1 - p$,

$$\mathbb{E}[|A+A|] = \sum_{r=0}^{n} p^r q^{n-r} \binom{n}{r} \left(2 \sum_{k=0}^{n-1} \left(1 - \frac{f(k)}{\binom{n}{r}} \right) - \left(1 - \frac{f(n-1)}{\binom{n}{r}} \right) \right),$$

where $n = N + 1$ and

$$f(k) = \begin{cases} \sum_{i=\frac{k+1}{2}}^{k+1} 2^{k+1-i} \binom{\frac{k+1}{2}}{i-\frac{k+1}{2}} \binom{n-k-1}{r-i} & \text{for } k \text{ odd} \\ \sum_{i=\frac{k}{2}}^{k} 2^{k-i} \binom{\frac{k}{2}}{i-\frac{k}{2}} \binom{n-k-1}{r-1-i} & \text{for } k \text{ even.} \end{cases}$$

In particular, where the LHS holds for $p > \frac{1}{2}$,

$$2n - 1 - 2\,\frac{1}{1-\sqrt{2q}} - (2q)^{\frac{n-1}{2}} \leq \mathbb{E}[|A+A|] \leq 2n - 1 - 2\,\frac{1 - q^{\frac{n-1}{2}}}{1-\sqrt{q}}.$$

Prior Work

Theorem (King, Martinez, Miller, Sun '19)

For $p \in (0,1)$ *and* $q := 1-p$*,*

$$\begin{aligned}\mathrm{Var}(|A+A|) &= \sum_{r=0}^{n}\binom{n}{r}p^r q^{n-r} \\ &\times\left(2\sum_{0\le i<j\le 2n-2} 1-P_r(i,j)+\sum_{0\le i\le 2n-2} 1-P_r(i)\right) \\ &-\mathbb{E}[|A+A|]^2,\end{aligned}$$

where $n = N+1$*,*

$$P_r(i) = \mathbb{P}(i \notin A+A \mid |A| = r),$$

and

$$P_r(i,j) = \mathbb{P}(i \text{ and } j \notin A+A \mid |A| = r).$$

Our Results

- Calculated the mean of $\mathbb{P}\left(|(A+A)^c| = m\right)$ exactly for generic p.
- Calculated the second moment of $\mathbb{P}\left(|(A+A)^c| = m\right)$ to leading order in $1/p$.

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This is all in the limit $N \to \infty$.

Our Setup

Setup

- Instead of considering $A \subseteq [0, N]$ for some natural number N, consider $\mathbb{A} \subseteq \mathbb{Z}_{\geq 0}$ chosen randomly via a Bernouli process.
- For any $k \in \mathbb{Z}_{\geq 0}$, include k in $\mathbb{A}$ with probability p.

Setup

- Instead of considering $A \subseteq [0, N]$ for some natural number N, consider $\mathbb{A} \subseteq \mathbb{Z}_{\geq 0}$ chosen randomly via a Bernouli process.
- For any $k \in \mathbb{Z}_{\geq 0}$, include k in $\mathbb{A}$ with probability p.
- With probability 1, $\mathbb{A}$ and $\mathbb{A}^c$ both include infinitely many elements.
- How does $\mathbb{A} + \mathbb{A}$ behave?

Motivation for $A \subseteq \mathbb{Z}_{\geq 0}$

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- Only one fringe to worry about.
- Infinite sums are nice to evaluate.
- Easy to convert to the original "finite case."
- To check if $n \in \mathbb{A} + \mathbb{A}$, only need to know about the first $n + 1$ elements: $\{0, 1, 2, \ldots, n\}$.

Mean and variance

Probability of Missing a Specific Summand

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- For each $i \geq 0$, let $\mathbb{X}_i$ be the indicator variable for $i \notin \mathbb{A} + \mathbb{A}$:

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- Then

$$\mathbb{Y} = \sum_{i=0}^{\infty} \mathbb{X}_i.$$

- To calculate $\mathbb{E}(\mathbb{Y})$, need $\mathbb{E}(\mathbb{X}_i) = \mathbb{P}(i \notin \mathbb{A} + \mathbb{A})$.

Probability of Missing a Specific Summand

Like [LMO], for odd n,

$$\{n \notin \mathbb{A}+\mathbb{A}\} \;=\; \{(0 \notin \mathbb{A} \text{ or } n \notin \mathbb{A}) \text{ and } \cdots \text{ and } \left(\frac{n-1}{2} \notin \mathbb{A} \text{ or } \frac{n+1}{2} \notin \mathbb{A}\right)\}$$

and for even n,

$$\{n \notin \mathbb{A} + \mathbb{A}\} \;=\; \{(0 \notin \mathbb{A} \text{ or } n \notin \mathbb{A}) \text{ and } \cdots \text{ and } n/2 \notin \mathbb{A}\}.$$

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Hence,

$$\mathbb{P}(n \notin \mathbb{A}+\mathbb{A}) = \begin{cases} (1-p^2)^{\frac{n+1}{2}} & n \text{ odd} \\ (1-p)(1-p^2)^{\frac{n}{2}} & n \text{ even.} \end{cases}$$

Calculating $\mathbb{E}(\mathbb{Y})$

- By the Monotone Convergence Theorem,

$$\mathbb{E}(\mathbb{Y}) = \sum_{n=0}^{\infty} \mathbb{E}(\mathbb{X}_n) = \sum_{n \text{ odd}} (1-p^2)^{(n+1)/2} + \sum_{n \text{ even}} (1-p)(1-p^2)^{n/2}.$$

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Proposition

For $p \in (0,1)$,

$$\mathbb{E}(\mathbb{Y}) = \frac{2}{p^2} - \frac{1}{p} - 1.$$

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- Let $n < m \leq N$.

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If m, n, l are all odd,

$$\mathbb{P}\,(m, n \notin A + A) \;=\; (a_{2l+2})^{\frac{(m+1)-l(m-n)}{2}} (a_{2l})^{\frac{l(m-n)-(n+1)}{2}}.$$

Similar formulas hold for other parities.

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Here, $a_1 = 1$, $a_2 = 1 - p^2$, and

$$a_k \;=\; (1-p)a_{k-1} + p(1-p)a_{k-2}. \tag{2}$$

$\mathbb{E}\left(\mathbb{Y}^2\right)$ as an infinite sum

Proposition

$$\mathbb{E}\left(\mathbb{Y}^2\right) = -\left(\frac{2}{p^2} - \frac{1}{p} - 1\right) + \\ + 2\sum_{l=1}^{\infty} \frac{a_{2l} + (1-p)a_{l-1} + (1-p)a_l a_{2l} + (1-p)^2 a_l a_{l-1}}{(1-a_{2l+2})(1-a_{2l})}. \tag{3}$$

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Asymptotics of the second moment

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$$\mathbb{E}\left(\mathbb{Y}^2\right) = \frac{4}{p^4} + o(p^{-4}).$$

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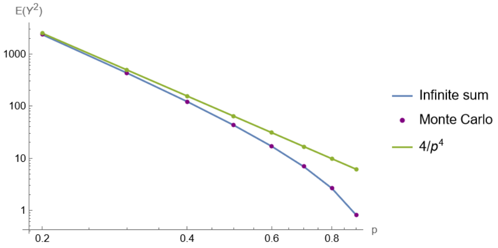


Figure: Exact values and asymptotic estimate for $\mathbb{E}\left(\mathbb{Y}^2\right)$.

Proving concentration

- Recall:

Proposition

For $p \in (0,1)$,

$$\mathbb{E}\left(\mathbb{Y}^2\right) = \frac{4}{p^4} + o(p^{-4}).$$

Proposition

For $p \in (0,1)$,

$$\mathbb{E}\left(\mathbb{Y}\right) = \frac{2}{p^2} - \frac{1}{p} - 1.$$

Proving concentration

- Therefore, the standard deviation σ,

$$\sigma = \sqrt{\mathbb{E}\left(\mathbb{Y}^2\right) - \mathbb{E}\left(\mathbb{Y}\right)^2} = o(p^{-2}) \qquad (4)$$

grows asymptotically slower than $\mathbb{E}\left(Y\right) \sim 2/p^2$.

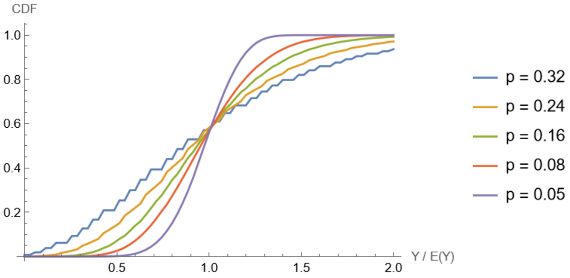


Figure: The cumulative distribution function of Y, normalized by $\mathbb{E}\left(Y\right)$, for $N = 800$ and $p = 0.05, 0.08, 0.16, 0.24, 0.32$. (Monte Carlo simulation.)

Higher Moments

A Problem with Dependencies

- To calculate $\mathbb{E}\left(\mathbb{Y}^2\right)$, need $\mathbb{P}\left(i, j \notin \mathbb{A}+\mathbb{A}\right)$.
- Unlike $\mathbb{P}\left(i \notin \mathbb{A}+\mathbb{A}\right)$, $\mathbb{P}\left(i, j \notin \mathbb{A}+\mathbb{A}\right)$ is laden with dependencies.
- Example: $\mathbb{P}\left(0 \notin \mathbb{A}+\mathbb{A}\right)=1-p$ and $\mathbb{P}\left(1 \notin \mathbb{A}+\mathbb{A}\right)=1-p^2$, but $\mathbb{P}\left(0, 1 \notin \mathbb{A}+\mathbb{A}\right)=1-p^2$.
- For higher moments, $\mathbb{E}\left(\mathbb{Y}^k\right)$, even more dependency.

A Workaround

- Instead of an exact expression, we find a bound:

$$\begin{aligned}\mathbb{E}\left(\mathbb{Y}^k\right) &= \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \mathbb{P}(n_1, \dots, n_k \notin \mathbb{A}+\mathbb{A}) \\ &\leq \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} \mathbb{P}(\max\{n_1, \dots, n_k\} \notin \mathbb{A}+\mathbb{A}).\end{aligned}$$

- We know the probability of $n \notin \mathbb{A}+\mathbb{A}$:

$$\mathbb{E}\left(\mathbb{Y}^k\right) \leq \sum_{n_1=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} (1-p^2)^{(\max\{n_1,\dots,n_k\}+1)/2}.$$

- Intuitively may not be too much loss; if $\max\{n_1, \dots, n_k\} \notin \mathbb{A}+\mathbb{A}$, many elements are missing from $\mathbb{A}$, so other values are probably also missing from $\mathbb{A}+\mathbb{A}$.

The bound

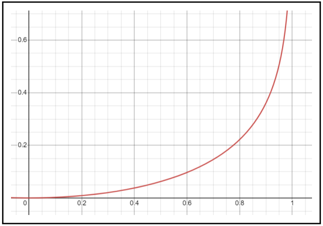
- Evaluating the “almost-geometric” sum yields

$$\mathbb{E}\left(\mathbb{Y}^k\right) \leq \left(1 + \frac{\alpha}{\sqrt{2\pi}}\right)\frac{k!}{\alpha^k},$$

where

$$\alpha := \log \frac{1}{\sqrt{1-p^2}} = \left|\log \sqrt{1-p^2}\right|.$$

- $O(k!/\alpha^k)$ moments correspond to $f(x) = e^{-\alpha x}$.



Exponential Decay

Proving exponential decay

- Since $\mathbb{E}\left(\mathbb{Y}^k\right) = O(k!/\alpha^k)$, Chernoff's inequality yields

$$\mathbb{P}\left(\mathbb{Y} \geq n\right) \;=\; O\left(n\left(1-p^2\right)^{n/2}\right).$$

- If $0, \ldots, n/2$ are missing from $\mathbb{A}$, then $0, \ldots, n$ are missing from $\mathbb{A}+\mathbb{A}$. Therefore,

$$\mathbb{P}\left(\mathbb{Y} \geq n\right) \geq (1-p)^{n/2+1}.$$

- Bounded above and below by exponential functions, $\mathbb{P}\left(\mathbb{Y} \geq n\right)$ is "approximately exponential."

Back to the Finite Case

Review of the finite case

- $A \subseteq [0, N]$ selected at random such that $\mathbb{P}(i \in A) = p$ for all i independently.
- Define $Y := 2N + 1 - |A + A|$ and $X_i := [i \notin A + A]$.
- Object of interest: random variable $Y_{N\to\infty}$,

$$\mathbb{P}(Y_{N\to\infty} = n) := \lim_{N\to\infty} \mathbb{P}(Y = n).$$

- What we will compute: the k-th moment

$$\mathbb{E}\left(Y_{N\to\infty}^k\right) = \lim_{N\to\infty} \mathbb{E}\left(Y^k\right).$$

The k-th moment of Y as a corner sum

- $\mathbb{E}\left(Y^k\right) = \sum_{i_1,\ldots,i_k=0}^{2N} \mathbb{E}\left(X_{i_1}\ldots X_{i_k}\right)$ is a sum over a k-dimensional hypercube.
- Observation: $A + A$ is “almost full” in the middle.
- Conclusion: To compute $\mathbb{E}\left(Y^k\right)$, we just need to sum over the corners of the hypercube.

Summing over the corners

- Observation: When $j - i > N$, events $i \notin A + A$ and $j \notin A + A$ are independent. Therefore, the corners are independent.
- Result of calculations: the k-th moment of $Y_{N \to \infty}$ is

$$\lim_{N\to\infty} \mathbb{E}\left(Y^k\right) = \sum_{s=0}^{k} \binom{k}{s} \mathbb{E}\left(\mathbb{Y}^s\right) \mathbb{E}\left(\mathbb{Y}^{k-s}\right).$$

The finite case, reduced

- Observation: The moments $\lim_{N\to\infty} \mathbb{E}\left(Y^k\right)$ are the same as those of $\mathbb{Y} + \mathbb{Y}'$. Apply Carleman's condition.

Theorem

The probability distribution of $Y_{N\to\infty}$ is the same as that of $\mathbb{Y} + \mathbb{Y}'$, where $\mathbb{Y}'$ is a copy of $\mathbb{Y}$ independent of it.

- Intuition: Summands can be missing from the left and right fringes, and these are independent for large N.

Future Work

- Use Euler's identity to calculate the even-odd disparity: $\mathbb{P}(\mathbb{Y} \text{ even}) - \mathbb{P}(\mathbb{Y} \text{ odd}) = \mathbb{E}\left(e^{i\pi\mathbb{Y}}\right)$.
- Get tighter bounds on the asymptotic decay rate of $\mathbb{P}(\mathbb{Y} \geq n)$.
- Investigate A^{+k}, the k-th additive power of A, as well as $A^{+\infty} = \{0\} \cup A \cup A^{+2} \ldots$, the set of all possible sums resulting from A.

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Bibliography

- O. Lazarev, S. J. Miller, K. O'Bryant, *Distribution of Missing Sums in Sumsets* (2013), Experimental Mathematics **22**, no. 2, 132–156.
- G. Martin and K. O'Bryant, *Many sets have more sums than differences*, in Additive Combinatorics, CRM Proc. Lecture Notes, vol. 43, Amer. Math. Soc., Providence, RI, 2007, pp. 287–305.
- H. V. Chu, D. King, N. Luntzlara, T. Martinez, S. J. Miller, L. Shao, C. Sun, and V. Xu, *Generalizing the distribution of missing sums in sumsets*, Journal of Number Theory **239** (2022), 402-444